A vertex-centered dual discontinuous Galerkin method

Motivation
- Discontinuous Galerkin (DG): a rational higher-order strategy to discretize equations with a dominating hyperbolic character
- DG reduces to a cell-centered finite-volume method at lowest order
- Unfortunately, many modern codes (for CFD e.g.) use vertex-centered finite-volume schemes

Objective
- To design a discontinuous Galerkin method that reduces to a common edge-based, vertex-centered finite-volume scheme at lowest order
- Can thus be implemented within the framework of existing software
- No need to code from scratch!

Properties
- The new discontinuous Galerkin method operates on a dual mesh
- Surrounding each primal node point is a group of elementary elements on the dual mesh
- Approximations: continuous, piecewise polynomials on each group of dual elements
- Approximations discontinuous between groups

For this mesh, there are 6 dual triangles surrounding each primal mesh point.
**A vertex-centered dual discontinuous Galerkin method**

**Model problem**

\[ \beta \cdot \nabla u + au = f \quad \text{in } \Omega, \]
\[ u = g \quad \text{on } \Gamma_-, \]

**DG approximation**

Find \( u_h \in V_h \) such that

\[ a(u_h, v_h) = L(v_h) \quad \forall v_h \in V_h \]

where \( a(u_h, v_h) = \sum_{i=1}^{M} a_i(u_h, v_h), \)

\[ L(v_h) = \sum_{i=1}^{M} L_i(v_h), \]

and

\[ a_i(u_h, v_h) = \int_{K_i} v_h \beta \cdot \nabla u_h \, dx - \int_{\partial K_i \cap \Gamma_-} \beta \cdot \nu v_h (u_h^+ - u_h^-) \, ds \]
\[ - \int_{\partial K_i \cap \Gamma_-} \beta \cdot \nu v_h u_h^+ \, ds + \int_{K_i} a v_h u_h \, dx, \]

\[ L_i(v_h) = - \int_{\partial K_i \cap \Gamma_-} \beta \cdot \nu v_h g \, ds + \int_{K_i} v_h f \, dx, \]

**Table**

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**Errors and convergence rates**

For a smooth solution \( u \), assuming \( \| u - u_h \|_0 \sim Ch^s \)

**Graph**

- Exact
- Piecewise constants (finite volume scheme)

**Graph**

- Piecewise linears
- Coarse mesh (48 nodes)
- Finer mesh (2449 nodes)