Variational and Geometric Aspects of Compatible Discretizations

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Why are we here?

We conclude...that exterior calculus is here to stay, that it will gradually replace tensor methods in numerous situations where it is the more natural tool, that it will find more and more applications because of its inner simplicity. Physicists are beginning to realize its usefulness; perhaps it will soon make its way into engineering.

H. Flanders, 1963

There’s generally a time lag of some fifty years between mathematical theories and their applications...

1950 + 50 = 2000

It’s about time!
Acknowledgments and Sources

• **Variational methods**
  - A. Aziz, et al. (1972)
  - G. Strang and G. Fix, (1973)
  - F. Brezzi, RAIRO B-R2 (1974)
  - G. Fix, M. Gunzburger, R. Nicolaides, CMA 5, (1979)
  - F. Brezzi, C. Bathe, CMAME 82, (1990)

• **Direct/geometric methods**
  - J. Dodzuik, (1976)
  - M. Hyman, J. Scovel, LAUR (1988-92)
  - M. Hyman, M. Shashkov, S. Stenberg (1995-98)
  - R. Nicolaides, SINUM 29 (1992)

• **Connections**
  - C. Mattiussi, JCP (1997)

• **Thanks to**
  - D. Arnold (IMA)
  - M. Gunzburger (FSU)
  - R. Lehoucq (SNL)
  - R. Nicolaides (CMU)
  - A. Robinson (SNL)
  - M. Shashkov (LANL)
  - C. Scovel (LANL)
  - K. Trapp (CMU)
How different people discretize

Discretization is a model reduction that replaces a physical process by a parametrized family of algebraic equations.
What do we want to know?

1. **Is the sequence of algebraic equations well-behaved?**
   - Are all problems **uniquely** and **stably** (in $h$) solvable?
   - Do solutions **converge** to the exact solutions as $h \to 0$?

2. **Are physical and discrete models compatible?**
   - Are solutions **physically** meaningful?
   - Do they **mimic**, e.g., invariants, symmetries of actual states?

3. **How to make a compatible & accurate discretization?**
   - How to **choose** the variables and where to place them;
   - How to avoid **spurious** solutions.

We revisit earlier discussion with a particular focus on how
   - **variational** compatibility (Arnold)
   - **geometric** compatibility (Nicolaides, Shashkov)

can be used to answer these questions.
A sequence of linear systems vs. a single linear system

\[ Ku = F \]

\[ Ku = 0 \implies u \equiv 0 \]

- **Solvability**

\[ K_h u_h = F_h \]

\[ K_h u_h = 0 \implies u_h \equiv 0 \]

- **Stability**

\[ \frac{\| \Delta u \|}{\| u \|} \leq \| K \| K^{-1} \| \frac{\| \Delta F \|}{\| F \|} \]

Stability of linear systems arising from PDEs cannot be assessed by standard condition number:

\[ \| K \| K^{-1} = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \quad \Rightarrow \quad O(h^{-2}) \]

\[ \| u_h \|_X^2 = u_h^T S_h u_h \]

\[ \| K_h \| = \sup_{v_h} \frac{\| K_h v_h \|_*}{\| v_h \|_X} \]

\[ \| K_h \| K^{-1} \| \leq ??? \]
Stability of a sequence

\[ \|K_h\| \leq \alpha \quad \& \quad \|K_h^{-1}\| \leq \frac{1}{\text{glb}(K_h)} \quad \Rightarrow \quad \|K_h\| \|K_h^{-1}\| \leq \frac{\alpha}{\text{glb}(K_h)} \]

glb suggested by G. Golub

\[ \text{glb}(K_h) = \inf_{u_h} \|K_h u_h\|_X = \inf_{v_h} \sup_{u_h} \frac{v_h^T K_h u_h}{\|v_h\|_X \|u_h\|_X} \]

\[ \inf_{u_h} \sup_{v_h} \frac{v_h^T K_h u_h}{\|v_h\|_X \|u_h\|_X} \geq \gamma > 0 \quad \Rightarrow \quad \|K_h\| \|K_h^{-1}\| \leq \frac{\alpha}{\gamma} \]

stability = \alpha and \gamma are independent of h

\[ \min \max_{u_h, v_h} \frac{v_h^T K_h u_h}{\left( v_h^T S_h v_h \right)^{1/2} \left( u_h^T S_h u_h \right)^{1/2}} \geq \gamma \]

The smallest generalized singular value of \(K_h\) must be bounded away from zero, independent of \(h\).
Variational Methods

Galerkin approximation of operator equations

\[ D(A) \subseteq X \]
\[ D(A) \supseteq X_h \]
\[ X \quad Au = f \quad f \]
\[ \downarrow \quad \downarrow \quad Q_h \]
\[ X_h \quad Q_h Au_h = f_h \quad f_h \]
\[ f \in R(A) \subseteq Y \]
\[ f_h \in Y_h \not\subseteq R(A) \]

Galerkin theorem

solvability

\[ Q_h y \in R(Q_h AP_h) \]

stability

\[ \|Q_h Au_h\|_y \geq \gamma \|u_h\|_X \]

\[ P_h u \to u \quad \& \quad Q_h y \to y \]

\[ \|Q_h Au - Q_h AP_h u\|_y \to 0 \]

Variational compatibility

\\[ \downarrow \]

Unique solvability and quasi-optimal convergence

\[ \|u - u_h\|_X \leq \|P_h u - u\|_X + \frac{1}{\gamma} \|Q_h Au - Q_h AP_h u\|_y \]
Variational settings for FEM

**Optimization**

\[ \min_{v \in X} \frac{1}{2} \langle Av, v \rangle - \langle f, v \rangle \]

subject to \( Bv = 0 \)

seek \( u \in X \) s.t. \( K(u, v) = F(v) \) \( \forall v \in Y \)

**No optimization**

\[ X \ni u \perp_K Y \]

\[ K(u - u, v) = 0 \ \forall v \in Y \]

**FEM = variational principle + piecewise polynomial subspaces**

seek \( u_h \in X_h \) s.t. \( K(u_h, v_h) = F(v_h) \) \( \forall v_h \in Y_h \)

\[
\begin{align*}
\left((u - u_h, v_h)\right)_K &= 0 \\
\forall v_h &\in X_h \\
(a(u_h, v_h) + b(p_h, v_h) &= (f, v_h) \ \forall v_h \in V_h \\
b(q_h, u_h) &= 0 \ \forall q_h \in P_h \\
K(u - u_h, v_h) &= 0 \ \forall v_h \in Y_h
\end{align*}
\]

**Projection**

**Quasi-projection**
Examples

No Optimization
- Advection-Diffusion-Reaction models
- Navier-Stokes equations

Constrained Optimization

Kelvin principle:
- the solenoidal velocity field that minimizes kinetic energy is irrotational

Dirichlet principle:
- the irrotational velocity field that minimizes kinetic energy is solenoidal

Unconstrained Optimization
- Poisson equation
No Optimization

Variational problem

\[ K(u,v) = F(v) \quad \forall v \in Y \]

Unique solvability & stability

\[ K(u,v) \leq \alpha \|u\|_x \|v\|_y \quad \text{continuity} \]

\[ \sup_{v \in Y} \frac{K(u,v)}{\|v\|_y} \geq \gamma \|u\|_x \quad \forall u \in X \quad \text{Inf-sup (I)} \]

\[ \sup_{u \in X} \frac{K(u,v)}{\|u\|_X} \geq 0 \quad \forall v \in Y \quad \text{Inf-sup (II)} \]

\[ \|u\|_x \leq C \|F\| \]
Compatibility

Discrete problem

\[ K(u_h, v_h) = F(v_h) \quad \forall v_h \in Y_h \quad K_h u_h = F_h \]

Variational compatibility

**conformity:** \( X_h \subset X \); \( Y_h \subset Y \) \( \Rightarrow \) **continuity**

*Necessary but insufficient!*

**Inf-sup (I)**

\[ \sup_{v_h \in Y_h} \frac{K(u_h, v_h)}{\|v_h\|_{Y_h}} \geq \gamma_h \|u_h\|_X \quad \forall u_h \in X_h \]

**Inf-sup (II)**

\[ \sup_{u_h \in X_h} \frac{K(u_h, v_h)}{\|u_h\|_{X_h}} \geq 0 \quad \forall v_h \in Y_h \]

\[ \|u_h\|_X \leq C\|F\| \quad \|u - u_h\|_X \leq \left(1 + \frac{1}{\gamma_h}\right) \inf_{v_h \in Y_h} \|u - v_h\|_X \]
Constrained Optimization

**Variational problem**  \( X = Y = V \times S \)

\[
\min_{v \in V} \max_{q \in S} \frac{1}{2} \langle Av, v \rangle - \langle f, v \rangle - \langle Bv, q \rangle
\]

\[
a(u, v) + b(p, v) = (f, v) \quad \forall v \in V
\]

\[
b(q, u) = 0 \quad \forall q \in S
\]

**Unique solvability & stability**

**continuity of \( a \) and \( b \)**

\[
a(v, v) \geq C_a \|v\|_V \quad \forall v \in Z
\]

coercivity on \( Z \)

\[
\sup_{v \in V} \frac{b(p, v)}{\|v\|_V} \geq \gamma_b \|p\|_S \quad \forall p \in S
\]

inf-sup for \( B \)

\[
\|u\|_V + \|p\|_S \leq C \|f\|_V
\]
Computational mathematics and algorithms

Compatibility

Discrete Problem

\[ Au + B^* p = F \]
\[ Bu = 0 \]

\[
\begin{pmatrix}
A_h & B_h^T \\
B_h & 0
\end{pmatrix}
\begin{pmatrix}
v_h \\
p_h
\end{pmatrix}
=
\begin{pmatrix}
f_h \\
0
\end{pmatrix}
\]

Variational compatibility

conformity \hspace{1cm} V_h \subset V; \hspace{0.5cm} S_h \subset S \Rightarrow \hspace{0.5cm} \text{continuity}

\[
\forall p_h \in S_h \hspace{0.5cm} \exists \nu \in V \hspace{0.5cm} \text{s.t. } b(p_h, \nu) \geq \gamma \| p_h \|_S \| \nu \|_V
\]

Necessary but insufficient:

\[
\exists \nu_h \in V_h \hspace{0.5cm} \text{s.t. } b(p_h, \nu_h) \geq \gamma \| p_h \|_S \| \nu_h \|_V
\]

\[ Z_h \neq \emptyset? \]

\[ Z_h \neq \emptyset: \hspace{0.5cm} Z_h \not\subset Z \Rightarrow \hspace{0.5cm} a(\nu_h, \nu_h) \geq C_a \| \nu_h \|_V \hspace{0.5cm} \forall \nu_h \in Z_h ?? \]
Variational compatibility

**conformity**
\[ V_h \subset V; \quad S_h \subset S \]

**coercivity on** \( Z_h \)
\[ a(v_h, v_h) \geq C_a \| v_h \|_V \quad \forall v_h \in Z_h; \]

**inf-sup for** \( B_h \)
\[ \sup_{v_h \in V_h} \frac{b(p_h, v_h)}{\| v_h \|_V} \geq \gamma_h \| p_h \|_S \quad \forall p_h \in S_h \]

\[ Z_h \neq \emptyset; \quad \forall p_h \in S_h \quad \exists v_h \in V_h \text{ s.t. } b(p_h, v_h) \geq \gamma \| p_h \|_S \| v_h \|_V \]

\[ \| u_h \|_V + \| p_h \|_S \leq C \| f \|_V, \quad \| u - u_h \|_V \leq C_1 \inf_{v_h} \| u - v_h \|_V + C_2 \Theta(Z, Z_h) \inf_{q_h} \| p - q_h \|_S \]
\[ \| p - p_h \|_S \leq C_3 \inf_{v_h} \| u - v_h \|_V + C_4 \inf_{q_h} \| p - q_h \|_S \]
Unconstrained Optimization

Variational problem  \( X = Y \)

\[
\min_{v \in X} \frac{1}{2} \langle Av, v \rangle - \langle f, v \rangle \quad K(u, v) = F(v) \quad \forall v \in X
\]

Unique solvability & stability

\[
K(u, v) \leq C_b \|u\|_X \|v\|_X \quad \text{continuity}
\]

\[
K(v, v) \geq C_a \|v\|_X^2 \quad \forall v \in X \quad \text{coercivity}
\]

Discrete problem

\[
K(u_h, v_h) = F(v_h) \quad \forall v_h \in Y_h \quad K_h u_h = F_h \quad K(v_h, v_h) \geq C_a \|v_h\|_X^2 \quad \forall v_h \in X_h
\]

Variational compatibility

\[
\text{conformity:} \quad X_h \subset X \quad \Rightarrow \quad \text{continuity} \& \text{coercivity}!
\]

\[
\|u_h\|_X \leq C \|F\| \quad \|u - u_h\|_X \leq \frac{1}{C_a} \inf_{v_h \in X_h} \|u - v_h\|_X
\]
# A summary of variational settings for FEM

<table>
<thead>
<tr>
<th>Features</th>
<th>Variational setting</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Optimization type</td>
</tr>
<tr>
<td></td>
<td>Unconstrained</td>
</tr>
<tr>
<td>Unique solvability</td>
<td>Continuity</td>
</tr>
<tr>
<td></td>
<td>Continuity</td>
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<td></td>
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<tr>
<td>Variational compatibility</td>
<td>Conformity</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Algebraic problem type</td>
<td>Symmetric</td>
</tr>
<tr>
<td></td>
<td>positive definite</td>
</tr>
</tbody>
</table>
What does variational compatibility buy you

Sequence stability is equivalent to variational compatibility

\[
\inf_u \sup_v \frac{v^T K_h u_h}{\|v_h\|_X \|u_h\|_X} \geq \gamma \iff \inf_u \sup_v \frac{K(u_h, v_h)}{\|v_h\|_X \|u_h\|_X} \geq \gamma
\]

Allows to assert powerful results about the asymptotic behavior

- quasi-optimal error estimates
- unique solvability for any \( h \)
- stability of discrete solutions (uniform invertibility)

This answers the 1st question:

1. Is the family of algebraic equations well-behaved?
   - are all problems uniquely and stably (in \( h \)) solvable?
   - do solutions converge to the exact solutions as \( h \to 0 \)?
What does variational compatibility say about the other issues?

Not much

Variational compatibility conditions are not constructive!

These conditions are not very helpful in finding the stable spaces and may be difficult to verify. Creative application of non-trivial tricks required, e.g.,

— Fortin’s operator
— Verfurth’s method
— Boland & Nicolaides’s method

Inf-sup fear and loathing still common!
“Pure” Direct Discretizations

Algebraic model

Kinematic relation

\[
\begin{align*}
  u_1 &= p_2 - p_1 \\
  u_2 &= p_3 - p_2 \\
  u_3 &= p_4 - p_1 \\
  u_4 &= p_5 - p_2 \\
  u_5 &= p_6 - p_3 \\
  u_6 &= p_5 - p_4 \\
  u_7 &= p_6 - p_5 \\
  u_8 &= p_7 - p_4 \\
  u_9 &= p_8 - p_5 \\
  u_{10} &= p_9 - p_6 \\
  u_{11} &= p_8 - p_7 \\
  u_{12} &= p_9 - p_8
\end{align*}
\]

Constitutive equation

\[ v_i = \rho_i u_i \]

Continuity relation

\[
\begin{align*}
  -v_1 - v_3 &= 0 \\
  +v_1 - v_2 - v_4 &= 0 \\
  +v_2 - v_5 &= 0 \\
  +v_3 - v_6 - v_8 &= 0 \\
  +v_4 + v_6 - v_7 - v_9 &= 0 \\
  +v_7 + v_5 - v_{10} &= 0 \\
  +v_8 - v_{11} &= 0 \\
  +v_9 + v_{11} - v_{12} &= 0 \\
  +v_{10} + v_{12} &= 0
\end{align*}
\]

Reduced system

\[ p \rightarrow "\text{pressure}" \]

\[ u \rightarrow "\text{velocity}" \]

\[ \rho \rightarrow "\text{density}" \]

\[ v \rightarrow "\text{flow}" \]
A possible “physical” interpretation of Hodge: (Franco’s question)

Conversion of velocity (measured along a line) into a flow (measured across a surface)
# Problems with identical reduced systems

<table>
<thead>
<tr>
<th></th>
<th>Potential flow</th>
<th>Thermal diffusion</th>
<th>Electrostatics</th>
<th>Linear elasticity</th>
<th>Electrical network</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>Pressure</td>
<td>Temperature</td>
<td>Potential</td>
<td>Displacement</td>
<td>Potential</td>
</tr>
<tr>
<td>$u$</td>
<td>Velocity</td>
<td>Heat flux</td>
<td>Electric field</td>
<td>Strain</td>
<td>Voltage</td>
</tr>
<tr>
<td>$A^{-1}$</td>
<td>Permeability</td>
<td>Thermal conductivity</td>
<td>Conductivity Ohm’s law</td>
<td>Compliance Hook’s law</td>
<td>Conductivity Ohm’s law</td>
</tr>
<tr>
<td>$v$</td>
<td>Flow rate</td>
<td>Heat flow</td>
<td>Current</td>
<td>Stress</td>
<td>Current</td>
</tr>
<tr>
<td>$f$</td>
<td>Fluid Source</td>
<td>Heat Source</td>
<td>Source Current</td>
<td>Applied load</td>
<td>Applied current</td>
</tr>
<tr>
<td>$g$</td>
<td>N/A</td>
<td>Heat battery</td>
<td>Battery</td>
<td>N/A</td>
<td>Battery</td>
</tr>
</tbody>
</table>
## Matrix Form

### Kinematic

\[ u + B^T p = g \]

### Constitutive

\[ u = Av \]

### Continuity

\[ Bu = f \]

Note that if we were to **build** the reduced system, its behavior will be described **exactly** by this algebraic equation!

\[
\begin{pmatrix}
A & B^T \\
B & 0
\end{pmatrix}
\begin{pmatrix}
v \\
p
\end{pmatrix} =
\begin{pmatrix}
g \\
f
\end{pmatrix}
\]

\[ -BA^{-1} B^T p = f - BA^{-1} g \]
Geometric compatibility

Geometrically compatible discretization:

**algebraic equations that describe “actual” physical systems.**

Requires to discover structure and invariants of physical systems and then copy them to a discrete system

- Fields are observed *indirectly* by measuring **global** quantities (flux, circulation, etc)
- Physical laws are **relationships** between **global** quantities (conservation, equilibrium)

**Differential forms provide the tools to encode** such relationships

- **Integration:** an abstraction of the *measurement* process
- **Differentiation:** gives rise to **local invariants**
- **Poincare Lemma:** expresses **local geometric** relations
- **Stokes Theorem:** expresses **global relations** (differentiation + integration)
How to achieve geometric compatibility?

Algebraic topology provides the tools to copy the structure

1. System states are **differential forms** reduced to **co-chains**
2. Exterior **differentiation** approximated by the **co-boundary** operator
3. **Dual** operators defined using **Hodge** * operator


Mimetic and co-volume methods fit this reduction model

- **Vector fields** represented by their integrals (fluxes or circulations)
- **Differential operators** defined via Stokes Theorem (coordinate-invariant)
- **Primal and dual** equations/operators (B and B^T) and an inner product (A)
Algebraic Topology Approach

1. System reduction

3 exact sequences: \((W_0, W_1, W_2, W_3), (C_0, C_1, C_2, C_3), (C^0, C^1, C^2, C^3)\)

\[
\begin{align*}
\text{forms} & \quad \cdots W_k \xrightarrow{d} W_{k+1} \cdots \\
\downarrow & \quad \downarrow \\
\text{co-chains} & \quad \cdots C^k \xrightarrow{\delta} C^{k+1} \cdots \\
\uparrow & \quad \uparrow \\
\text{chains} & \quad \cdots C_k \xrightarrow{\partial} C_{k+1} \cdots \\
\end{align*}
\]

DeRham map

Fundamental property: \(R d = \delta R\)

\[
\langle \delta R \omega, c \rangle = \langle R \omega, \partial c \rangle = \int_c \omega = \int_c d\omega = \langle R d\omega, c \rangle
\]

\{G, D, C\} \leftarrow \delta \text{ approximates } d \rightarrow \{\text{grad, curl, div}\}
Example

chains

\[ \int \omega \]

\[ N \xrightarrow{\delta} E \xrightarrow{\delta} F \xrightarrow{\delta} K \]

coop-gravity

\[ K \xrightarrow{\partial} \partial K \xrightarrow{\partial} \partial \partial K = 0 \]

\[ \langle \delta c^k, c_{k+1} \rangle = \langle c^k, \partial c_{k+1} \rangle \]

co-chains

\[ \delta \delta = 0 \]
Algebraic Topology Approach
2. Inner products and dual operators

Inner product $W_k \times W_k$

$*: W_k \to W_{n-k} \quad (\omega, \varphi)_W = \int_\Omega \omega \wedge *\varphi$

Inner product $C^k \times C^k$

$I : C^k \to W_k \quad (a, b)_c = (Ia, Ib) = \int_\Omega Ia \wedge *Ib = a^T Mb$

Dual operators

$(\delta a, b)_c = (a, \delta^* b) \quad \rightarrow \quad G^*, C^*, D^*$

$C^*G^* = D^*C^* = 0$ requires $dI = I\delta$

Commuting Diagram II

$C^k \xrightarrow{\delta} C^{k+1}$

$I \downarrow \quad \downarrow I$

$W_k \xrightarrow{d} W_{k+1}$
Examples

Co-volume

Mimetic

Whitney

\[ \mathbf{M} = \begin{pmatrix} h_1 h_1 \perp \\ h_2 h_2 \perp \\ h_3 h_3 \perp \end{pmatrix} \begin{pmatrix} \frac{V_2}{\sin^2 \phi_2} + \frac{V_3}{\sin^2 \phi_3} \\ \frac{V_3 \cos \phi_3}{\sin^2 \phi_3} \\ \frac{V_1 \cos \phi_1}{\sin^2 \phi_1} \end{pmatrix} \begin{pmatrix} \frac{V_2}{\sin^2 \phi_2} + \frac{V_3}{\sin^2 \phi_3} \\ \frac{V_3 \cos \phi_3}{\sin^2 \phi_3} \\ \frac{V_1 \cos \phi_1}{\sin^2 \phi_1} \end{pmatrix} \begin{pmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} (w_{ij}, w_{kl}) \end{pmatrix} \]

Nicolaiades, Trapp (1992-04)

Hyman, Shashkov, Steinberg (1985-04)

Dodzuik (1976)

Hyman, Scovel (1988)
**Properties**

**Co-volume** inner product is the unique inner product that is

- **diagonal**
- **exact** for constant vector fields

⇒ Important computational property:
  - **dual** co-volume operators have **local** stencils

**Action** of co-volume and mimetic products coincides if

\[ V_i = |t| \frac{\tan \phi_i}{\sum_k \tan \phi_k} \]  

(Trapp, 2004)

**Approximation**

\[ I_{\text{Mim/co}} (R \omega) - \omega = O(h^2)/O(h) \]  

(Shashkov, Wheeler, Yotov 2004/ Trapp, 2004)

\[ I_{\text{Whitney}} (R \omega) - \omega = O(h) \]  

(Dodzuik, 1976)
Algebraic Topology Framework: Summary

1. Structures:
   \[(W_0, W_1, W_2, W_3)\] Forms
   \[(C_0, C_1, C_2, C_3)\] Chains
   \[(C^0, C^1, C^2, C^3)\] Co-chains

2. De Rham map
   \[R : W_k \rightarrow C^k\] \[R \ d = \delta R\]

3. Interpolation operator
   \[I : C^k \rightarrow W_k\] \[dI = I \delta\]

4. Inner product
   \[(a, b)_c = (Ia, Ib)\] \[M\]

5. Primal and dual operators
   \[\{G, C, D\} \& \{G^*, C^*, D^*\}\]
Direct discretization of a div-curl system

\[ \mathbf{n} \times \mathbf{u} = h \quad \text{on } \Gamma \]
\[ \nabla \times \mathbf{u} = \mathbf{f} \quad \text{in } \Omega \]
\[ \nabla \cdot \mathbf{u} = g \quad \text{in } \Omega \]
\[ \mathbf{h} = \mathbf{n} \cdot \mathbf{u} \quad \text{on } \Gamma \]

\[ \mathbf{u} \in C^1 \rightarrow \begin{cases} 
C : C^1 \rightarrow C^2 \\
D^* : C^1 \rightarrow C^0 
\end{cases} \]
\[ \nabla \times \rightarrow d_1 \]
\[ \nabla \cdot \rightarrow d_2 \]
\[ \mathbf{C}^* : C^2 \rightarrow C^1 \\
D : C^2 \rightarrow C^3 \]

\[ \mathbf{u} = h \quad \text{on } C^1 / C^1 \Gamma \]
\[ \mathbf{C} \mathbf{u} = \mathbf{f} \quad \text{in } C^2 \]
\[ \mathbf{C}^* \mathbf{u} = \mathbf{f} \quad \text{in } C^1 \]
\[ \mathbf{D}^* \mathbf{u} = \mathbf{g} \quad \text{in } C^0 \]
\[ \mathbf{D} \mathbf{u} = \mathbf{g} \quad \text{in } C^3 \]
\[ \mathbf{u} = h \quad \text{on } C^2 / C^2 \Gamma \]

Examples:

Co-volume: Nicolaides et. al. 1992-2004
Finite difference: Yee, 1966
Finite volume: Weiland, 1977
### Direct discretization of a div-grad system

<table>
<thead>
<tr>
<th>Condition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \cdot u = h$ on $\Gamma$</td>
<td>$\nabla \cdot u = f$ in $\Omega$&lt;br&gt;$\nabla \varphi + u = 0$ in $\Omega$</td>
</tr>
</tbody>
</table>

| $u = h$ on $C^2 / C^2_\Gamma$ | $D u = f$ in $C^3$<br>$G^* \varphi + u = 0$ in $C^2$ | $D^* u = f$ in $C^2$<br>$G \varphi + u = 0$ in $C^1$ | $\varphi = h$ on $C^0 / C^0_\Gamma$ |

**Eliminations**

$-D G^* \varphi = f$<br>$-D^* G \varphi = f$<br>$-BA^{-1} B^T$

**Examples**

- **Mimetic:** Shashkov et. al. 1995-2004
- **Finite volume:** The box integration method: Mock, 1983
What does geometric compatibility buy you?

Co-cycles of \((W_0, W_1, W_2, W_3)\) \(\xrightarrow{R} \) co-cycles of \((C^0, C^1, C^2, C^3)\)

\[
d\omega = 0 \quad \Rightarrow \quad \delta R \omega = 0
\]

**Discrete Poincare lemma** (existence of potentials in contractible domains)

\[
d\omega_k = 0 \quad \Rightarrow \quad \omega_k = d\omega_{k+1} \quad \quad \delta c^k = 0 \quad \Rightarrow \quad c^k = \delta c^{k+1}
\]

**Discrete Stokes Theorem**

\[
\langle d\omega_{k-1}, c_k \rangle = \langle \omega_{k-1}, \partial c_k \rangle \quad \quad \langle \delta c^{k-1}, c_k \rangle = \langle c^{k-1}, \partial c_k \rangle
\]

**Discrete “Vector Calculus”**

\[
dd = 0 \quad \quad \delta\delta = 0 \quad \Rightarrow \quad CG = DC = 0; \quad C^*G^* = D^*C^* = 0
\]

Any feature of the continuum system that is implied by differential forms calculus is inherited by the discrete model

Called **mimetic** property by Hyman and Scovel (1988)
Solvability: free of charge

**Div-curl system:** Discrete Helmholtz orthogonality

\[
\begin{align*}
Cu = 0 \\
D^*v = 0
\end{align*} \implies (u,v)_{C^1} = 0
\]

\[
\begin{align*}
Cu = 0 \\
D^*u = 0
\end{align*} \implies (u,u)_{C^1} = 0 \implies u \equiv 0
\]

**Div-grad system:** Commuting diagram property

Unique solvability: \( G^* \varphi = 0 \implies \varphi = 0 \)

Assume: \( \varphi \in C^3; G^* \varphi = 0 \) but \( \varphi \neq 0 \)

\[
\begin{array}{ccc}
W_2 & \xrightarrow{d} & W_3 \quad \text{surjection} \\
R & \downarrow & R \\
C^2 & \xrightarrow{\delta} & C^3 \quad \text{surjection}
\end{array}
\]

\( 0 = (u_\varphi, G^* \varphi) = (Du_\varphi, \varphi) = (\varphi, \varphi) \neq 0 \), a contradiction!
Variational vs. geometric

Variational

- **Operator**-centric point of view
  - Problem = *operator equation* on function spaces
  - Discretization = *operator equation* + *functional approximation*

- Stability conditions
- Error estimates

Geometric

- **Topology**-centric point of view
  - Problem = *equilibrium relation* on manifolds
  - Discretization = *equilibrium relation* + *manifold approximation*

- Forces **physically compatible** discretization patterns
- Preserves problem structure
Variational and geometric

We can benefit from combining both approaches

D. Arnold  
stable mixed spaces designed by association of the problem with a differential complex

M. Shashkov  
error analysis of mimetic schemes enabled by identification with a mixed Galerkin method and a proper quadrature selection.

I will now examine **connections** between **geometrical** and **variational** compatibility that validate such collaborations using **Kelvin’s principle** as a prototype problem

\[
\begin{align*}
\int_{\Omega} \psi \nabla \cdot v d\Omega &= \int_{\Omega} \psi f d\Omega \quad \forall \psi \in S \\
\int_{\Omega} v \cdot w - \phi \nabla \cdot w d\Omega &= 0 \quad \forall w \in V
\end{align*}
\]

\[
\min_{v \in V} \max_{q \in S} \frac{1}{2} \int_{\Omega} |v|^2 d\Omega - \int_{\Omega} \phi (\nabla \cdot v - f) d\Omega
\]
Early examples

Grid Decomposition Property

\[ \begin{align*}
\mathbf{v} & \in L^2 \quad \mathbf{v} = \mathbf{w} + \mathbf{z} \\
\nabla \cdot \mathbf{z} &= 0
\end{align*} \]

geometry

\[ \begin{align*}
\mathbf{v}^h &= \mathbf{w}^h + \mathbf{z}^h \\
\nabla \cdot \mathbf{z}^h &= 0
\end{align*} \]

Helmholtz

\[ (\mathbf{w}, \mathbf{z}) = 0 \]

metric

\[ \begin{align*}
(\mathbf{w}^h, \mathbf{z}^h) &= 0 \\
\|\mathbf{w}^h\|_0 &\leq C (\|\nabla \cdot \mathbf{v}^h\|_1 + \|\nabla \cdot \mathbf{v}^h\|_0)
\end{align*} \]

GDP

Theorem

GDP is \textit{necessary} and \textit{sufficient} for stable, optimally accurate mixed discretization of the Kelvin principle.


Similar GDP exists for the Dirichlet principle but is trivial to satisfy!
Early examples

Fortin Lemma

$$(V^h, S^h)$$ verify inf-sup condition for the Kelvin principle iff:

$$\Pi^h : V \rightarrow V^h \begin{cases} \int \nabla \cdot (\Pi^h \psi) \psi^h d\Omega = \int \nabla \cdot \psi \psi^h d\Omega \quad \text{geometry} \\
\|\Pi^h \psi\|_V \leq C\|\psi\|_V \quad \text{metric} \end{cases}$$

Geometric assumption: equivalent to a \textit{commuting diagram}!

$$\int \psi^h \nabla \cdot (\Pi^h \psi) d\Omega = \int \psi^h \nabla \cdot \psi d\Omega$$

$$\nabla \cdot (I_2) = I_3 (\nabla \cdot)$$

Can this be an accident?

We see:

- conditions that combine geometric and metric properties
- the ubiquitous commuting diagram...

The French Connection

* Bossavit, Nedelec, Verite, 1982-88 and Kotiuga, 1984, were first from the finite element community to notice and document an uncanny connection between unusual, i.e., *not nodal*, finite element spaces and *Whitney forms*. 
Elsewhere...

FINITE-DIFFERENCE APPROACH TO THE HODGE THEORY OF HARMONIC FORMS,*

By JOZEF DODZIUK.

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0 Introduction
1 Whitney Forms
2 Standard Subdivision of a Complex
3 Approximation Theorem
4 Inner Product in Cochain Spaces. Combinatorial and Continuous Hodge Theories
5 Eigenvalues of the Laplacian Acting on Functions

G. Strang informed us that the techniques used in this paper are very closely related to finite element method of solving partial differential equations numerically.
CDP 1 + CDP 2 = VC

Geometric compatibility

\[ W_k \xrightarrow{d} W_{k+1} \]

CDP 1 \[ R \downarrow \quad \downarrow R \]

\[ C^k \xrightarrow{\delta} C^{k+1} \]

\[ C^k \xrightarrow{\delta} C^{k+1} \]

CDP 2 \[ I \downarrow \quad \downarrow I \]

\[ W_k \xrightarrow{\delta} W_{k+1} \]

Variational compatibility

\[ W_k \xrightarrow{d} W_{k+1} \]

Forms

\[ R \downarrow \quad \downarrow R \]

DOFs

\[ C^k \xrightarrow{\delta} C^{k+1} \]

\[ I \downarrow \quad \downarrow I \]

FEMs

\[ W_k^h \xrightarrow{d} W_{k+1}^h \]

\[(I \circ R) \circ d = d \circ (I \circ R)\] CDP

CDP is equivalent to stability of mixed FEM

CDP and GDP are also equivalent!
There’s only one low-order compatible method

Well, up to a choice of an inner product…

And a quadrature rule…

And a cell shape…

FEM shapes restricted to those that have a “reference element”!

Whitney, 1957 simplex
Nedelec, 1980-85 cube, prism
Van Welij, 1985 hexahedron
BD(F)M, 80°-90° many shapes
There are more high-order methods

But they are mostly FEM….Why?

Direct methods:
  reliance on the De Rham map limits DOFs to co-chains: stencils expand!

Variational methods:
  order = degree of complete polynomials contained in the space (Bramble-Hilbert)

Allows to automate formulation of high-order spaces:
  - Define reference space containing desired polynomials
  - Glue together into piecewise polynomial space
  - Coordinate interpolation and DOFs to provide CDP

\[ \Pi d = d\Pi \]

Conclusions

Stronger in metric-dependent aspects:
- assessment of the asymptotic behavior (error, stability)
- formulation of higher-order methods

Variational:

Weaker in structure-dependent aspects:
- compatibility conditions not constructive, difficult to verify
- FEM restricted to special cell shapes

Weaker in metric-dependent aspects:
- uniform stability of systems, errors, harder to prove
- higher-order methods not easy to define directly

Geometric:

Stronger in structure-dependent aspects:
- structure of the problem copied automatically
- local/global relationships and invariants preserved
- admit a wider set of cell shapes
Conclusions

Variational + Geometric is better

Enjoy the workshop!
Another viewpoint

Recall the discrete network of pipes...

\[
\begin{align*}
  u_1 &= p_1 - p_i \\
  u_2 &= p_2 - p_i \\
  u_3 &= p_4 - p_i \\
  u_4 &= p_5 - p_i \\
  u_5 &= p_6 - p_i \\
  u_6 &= p_3 - p_i \\
  u_7 &= p_6 - p_i \\
  u_8 &= p_7 - p_i \\
  u_9 &= p_8 - p_i \\
  u_{10} &= p_9 - p_i \\
  u_{11} &= p_4 - p_7 \\
  u_{12} &= p_5 - p_8 \\
  v_1 &= \rho \mu
\end{align*}
\]

Kinematic and continuity relations depend only on “network topology” (incidence matrices!)

- **Metric** is introduced by the constitutive equation.

This distinct pattern appears over and over in physical models (Tonti, 1974).

It can be used to provide an additional insight into compatible discretizations.
Factorization (Tonti) diagrams

De Rham complex

**Primal**

\[
\begin{align*}
W^0 & \xrightarrow{\nabla} W^1 \\
& \xrightarrow{\nabla \times} W^2 \\
& \xrightarrow{\nabla} W^3 \\
& \xrightarrow{\nabla \cdot} W^4
\end{align*}
\]

\[\uparrow \quad \uparrow \quad \text{Hodge} \ast \uparrow \quad \uparrow\]

**Dual**

\[
\begin{align*}
\tilde{W}^3 & \xleftarrow{\nabla \ast} \tilde{W}^2 \\
& \xleftarrow{\nabla \times} \tilde{W}^1 \\
& \xleftarrow{\nabla} \tilde{W}^0
\end{align*}
\]

Discrete De Rham complex

**Primal**

\[
\begin{align*}
W_h^0 & \xrightarrow{\nabla \ast_{d_h}} W_h^1 \\
& \xrightarrow{\nabla \times_{d_h}} W_h^2 \\
& \xrightarrow{\nabla} W_h^3 \\
& \xrightarrow{\nabla \cdot} W_h^4 \\
& \xrightarrow{\nabla \ast_{d_h}} W_h^0
\end{align*}
\]

\[\uparrow \quad \uparrow \quad \text{Hodge} \ast_{h} \uparrow \quad \uparrow\]

**Dual**

\[
\begin{align*}
\tilde{W}_h^3 & \xleftarrow{\nabla \ast_{d_h}} \tilde{W}_h^2 \\
& \xleftarrow{\nabla \times_{d_h}} \tilde{W}_h^1 \\
& \xleftarrow{\nabla} \tilde{W}_h^0
\end{align*}
\]

“All” 2nd order PDE’s

\[
\begin{align*}
\nabla a &= -b \\
\nabla \cdot \beta &= -\alpha \\
\alpha &= \ast_{\mu} a \\
\beta &= \ast_{\epsilon} b \\
-\nabla \cdot \epsilon \nabla a + \mu a &= f
\end{align*}
\]

Elimination “All” Methods **Primal-dual**

- One DDF set used
- One set eliminated
- One \( d \) is exact
- One \( d \) is weak
- One grid only
- Typical: Mixed FEM
  - Two DDF sets used
  - Two \( d \)'s are exact
  - Two grids (P&D)
  - Typical: Co-Volume
  - Typical: Staggered grid
  - Typical: Mimetic FD