IMA “Hot Topics” Workshop: Compatible Spatial Discretizations for PDEs, Minneapolis, 11-15 May 2004

IMA, 13 05 04

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Computational electromagnetism and Whitney forms

Hindsight, hindsight...

50-mn talk, Minneapolis, May 13, 2004. Thanks to IMA and organizers.

Overlays, not part of presented slides, contain summary of oral comments

Content:

- An overview of the "Generalized finite differences" approach in electromagnetism
- Some history, from the edge element to Whitney forms
- Overfill material, posted

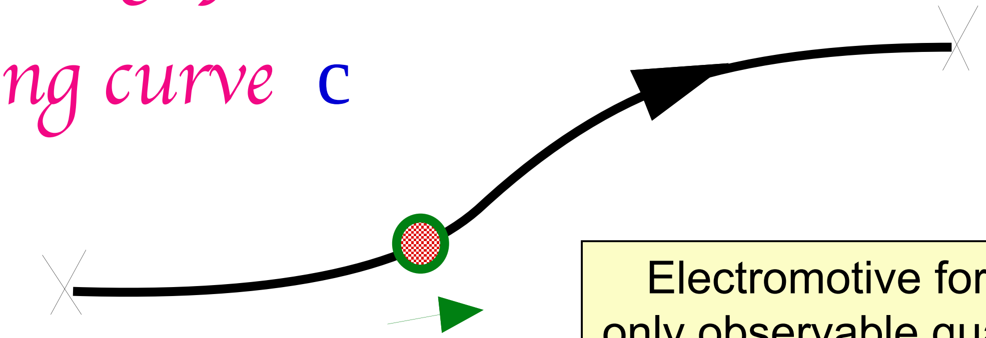
Method:

Occam's razor

I.e., model physical entities by "least assuming" mathematical ones

The electric field: a *mapping*

$\mathrm{e} : \textit{ORIENTED CURVE} \rightarrow \textit{REAL}$



Unit charge pushed along curve c

Work involved:

$$\underbrace{\int_c \mathrm{e}}_{\text{volts}}$$

Electromotive force, only observable quantity

with *additivity*: $\int_{c_1 + c_2} \mathrm{e} = \int_{c_1} \mathrm{e} + \int_{c_2} \mathrm{e}$,

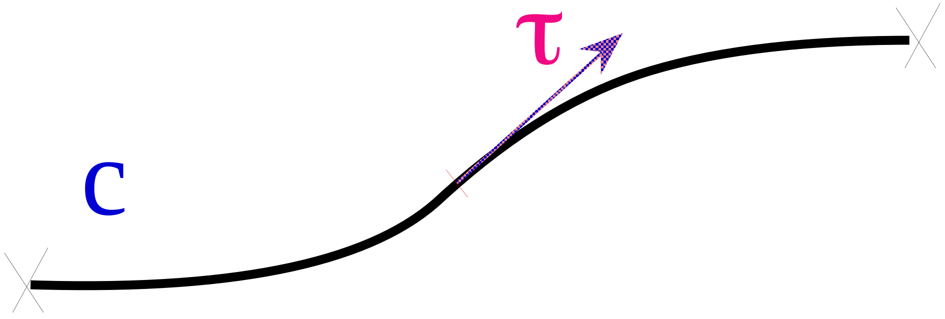
and *continuity* (w.r.t. variations of c)

Relevant properties of the mathematical object that will stand for (physical) electric field

E
(the vector field)

as a proxy for

e
(the 1-form)



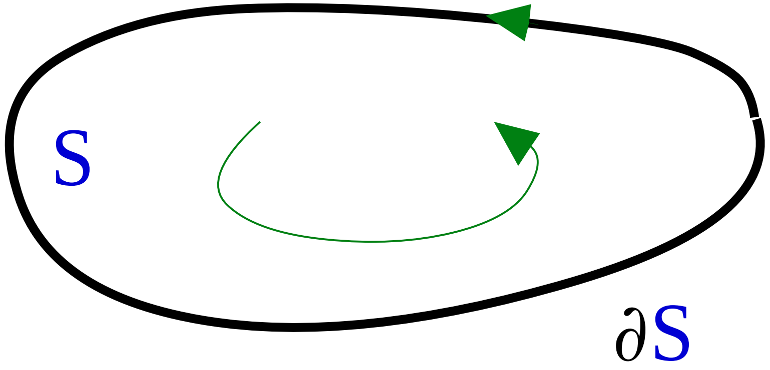
$$\int_c e = \int_c \tau \cdot E$$

Change "•", change E (and τ), for same e

The observable is not E but e, the differential form

Magnetic induction: a mapping

$b : \textit{ORIENTED SURFACE} \rightarrow \textit{REAL}$



$$\underbrace{\int_S b}_{\textit{webers}}$$

2-form

1-form

$$\boxed{\frac{d}{dt}\int_S b + \int_{\partial S} e = 0}$$

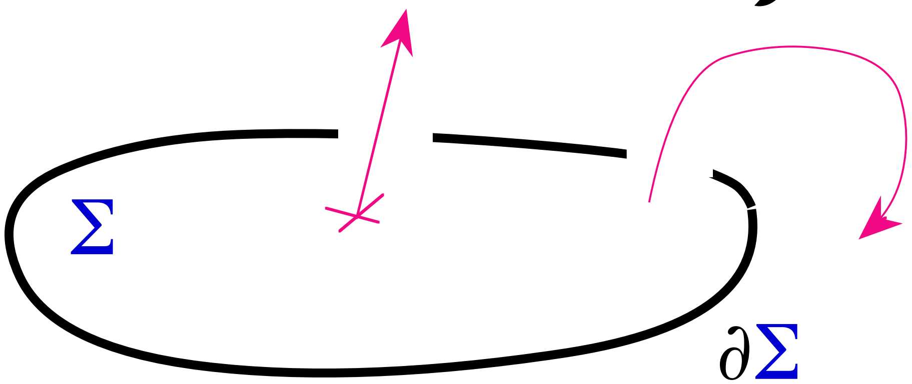
(Faraday.)

Metric-free expression.

or $\boxed{\partial_t b + de = 0}$, if d defined by $\int_S de = \int_{\partial S} e$

Again, observable b is a DF (of d° 2), and Faraday connects DF's

Current density j: Also a SURFACE → REAL map, but



with outer orientation (crossing direction)

j is a twisted (or odd, etc.) 2-form

(Ampère: $\boxed{\int_{\partial\Sigma} h = \int_{\Sigma} j}$)

“Displacement currents” ignored here

Again, metric-free expression

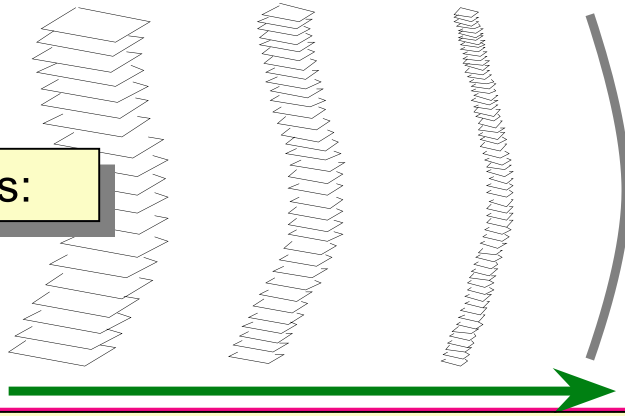
Mathematical formalization:

- Introduce the space of p-chains, i.e., formal sums of curves ($p = 1$), surfaces...

(**J. Phys. A32**, 28 (1999), pp. 5317-27)

- Topologize (cf. J. Harrison), obtain normed spaces*; complete, $\Rightarrow$ "chainlets"
- p-forms as dual objects
- d as dual of ∂

Cauchy sequence of 2-chains:



Support of limit 2-chainlet is a line.

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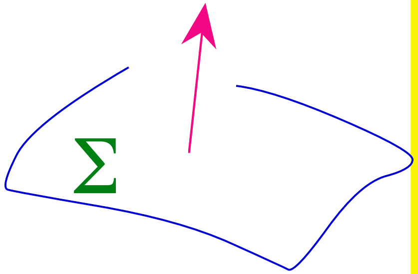
* Does not assume, and result does not depend on, a metric on underlying Euclidean space E_3

$$-\partial_t \int_\Sigma d + \int_{\partial\Sigma} h = \int_\Sigma j \quad \forall$$

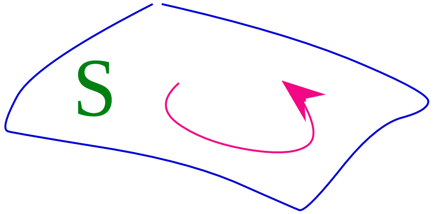


$$-\partial_t d + dh = j$$

source of field

$$b = \mu h, \qquad d = \varepsilon e$$

$$\partial_t \int_S b + \int_{\partial S} e = 0 \quad \forall$$



$$\partial_t b + de = 0$$

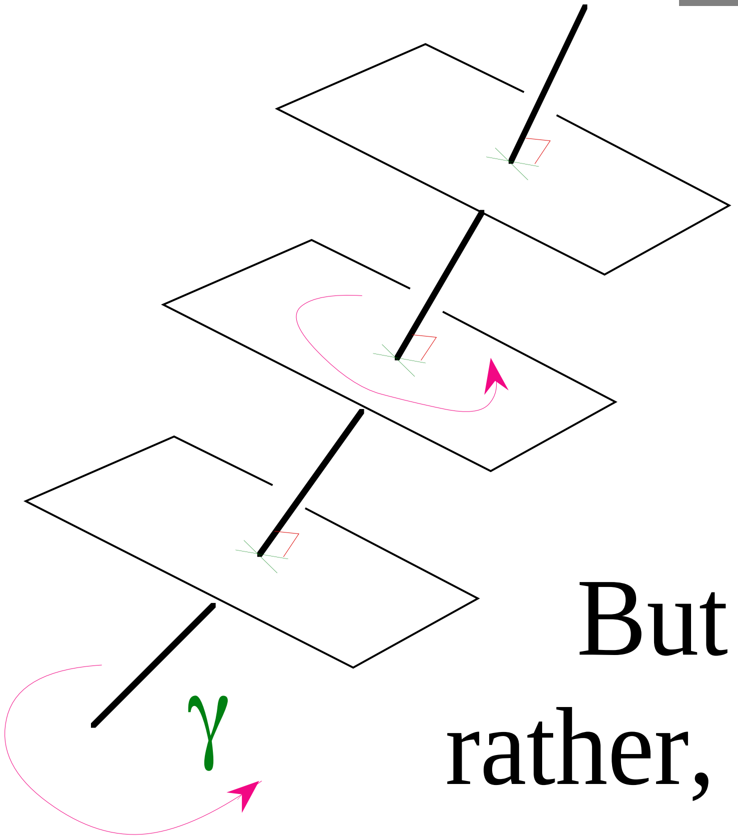
New status for ε and μ, now operators, of type 1-*FORM* → 2-*FORM* :

Knowing b, i.e. $\int_S b$ for all S, how do we know $h = \nu b$, i.e. $\int_\gamma h$ for all γ ?

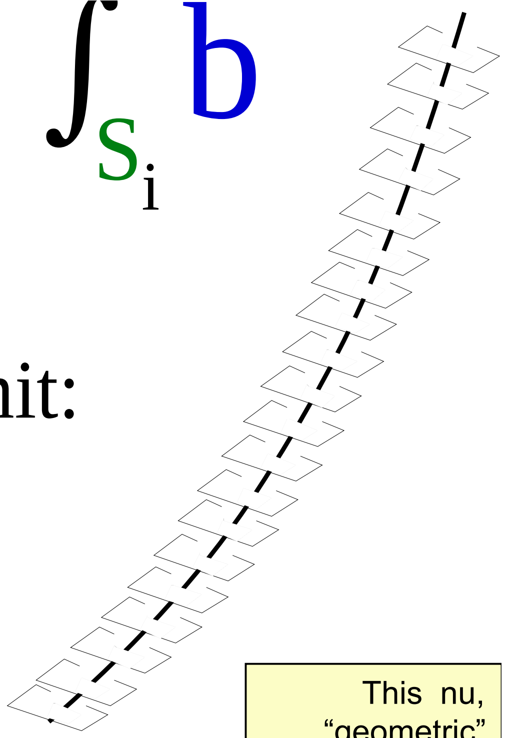
Chop γ into small segments γ_i, build orthogonal small surfaces S_i, with $\text{area}(S_i) = \nu \, \text{length}(\gamma_i)$, form

Apparent intervention of underlying metric of E_3

$$\int_\gamma h \approx \sum_i \int_{S_i} b$$



then go to the limit:



But why not,

rather, $\int_\gamma h = \int_S b$,

where S is chainlet $S = \nu\gamma$?

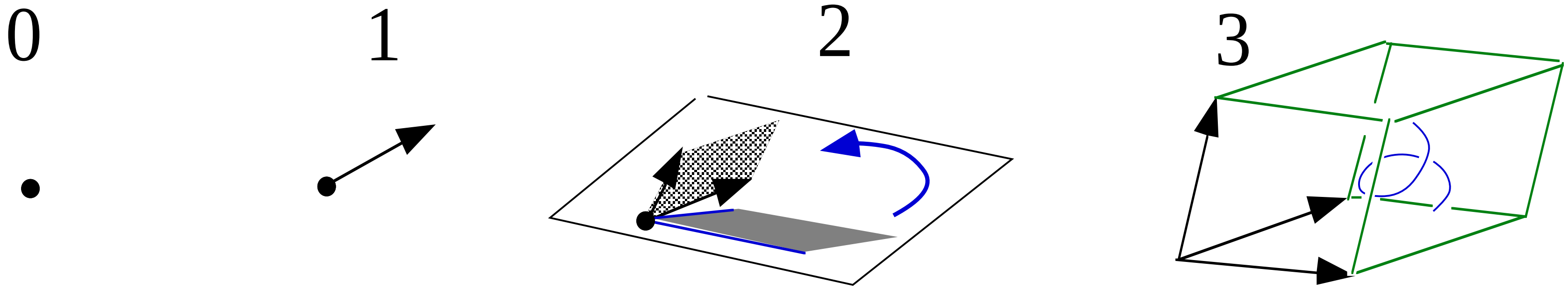
This nu, “geometric” Hodge, which merges const. law and underlying metric, is what is actually observable.

Define the ν in $h = \nu b$ as dual of geometrical one

$$\langle \gamma \,;\, h \rangle \equiv \langle \gamma \,;\, \nu b \rangle \hat{=} \langle \nu\gamma \,;\, b \rangle$$

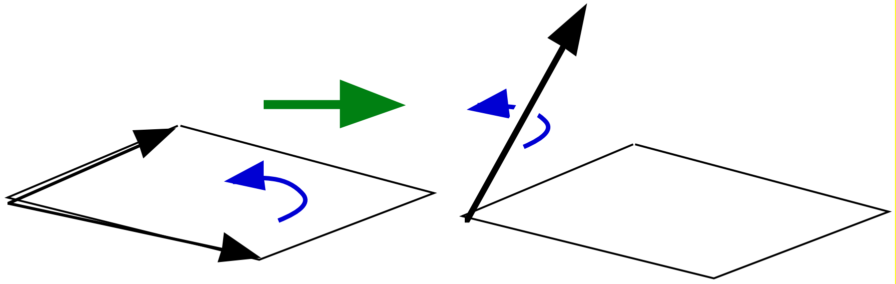
A truer view of the Hodge:

a p-*VECTOR* → (n – p)-*VECTOR* *map*

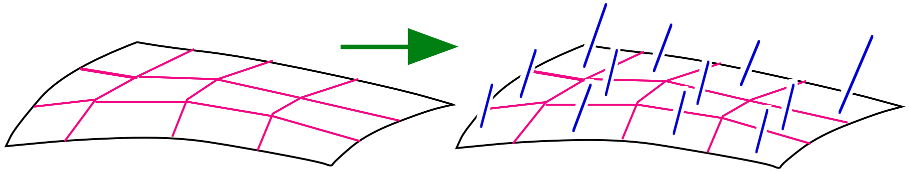


2-vector at x defined as equivalence class of oriented areas borne by 2D subspace containing x

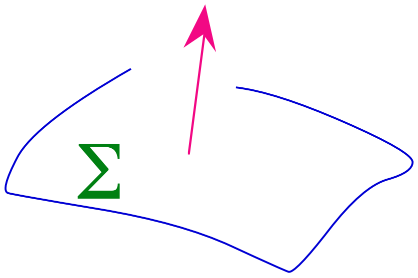
μ : (The geometrical one)



- Determines a metric ("μ-adapted")
- Induces a 2-*chain* → 1-*chain(let)* map



$$\left.\begin{aligned} -\partial_t \int_{\Sigma} d + \int_{\partial\Sigma} h &= \int_{\Sigma} j \\ \int_{\Sigma} d &= \int_{\varepsilon\Sigma} e \end{aligned}\right\} \forall$$

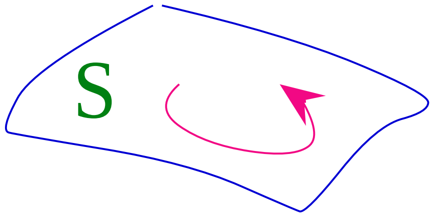


$$-\partial_t d + dh = j$$

$$d = \varepsilon e$$

Everything in global integral form, now

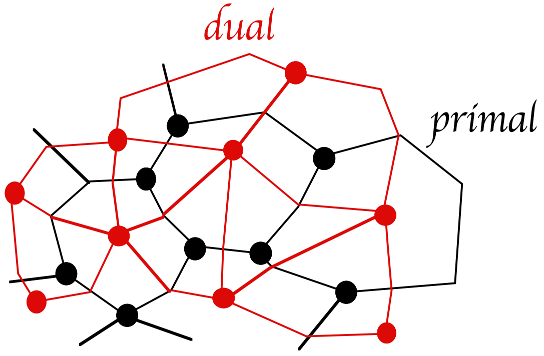
$$\left.\begin{aligned} \partial_t \int_{S} b + \int_{\partial S} e &= 0 \\ \int_{S} b &= \int_{\mu S} h \end{aligned}\right\} \forall$$



$$\partial_t b + de = 0$$

$$b = \mu h$$

Discretization strategy: Instead of all S, Σ, enforce this for surfaces spanned by faces of interlocked meshes:



Set of nodes

Incidence matrices

Set of faces

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\mathbf{G}"] & \mathcal{E} \arrow[r, "\mathbf{R}"] & \mathcal{F} \arrow[r, "\mathbf{D}"] & \mathcal{V} \end{tikzcd}$$

G, analog of grad

R, analog of rot

D, analog of div

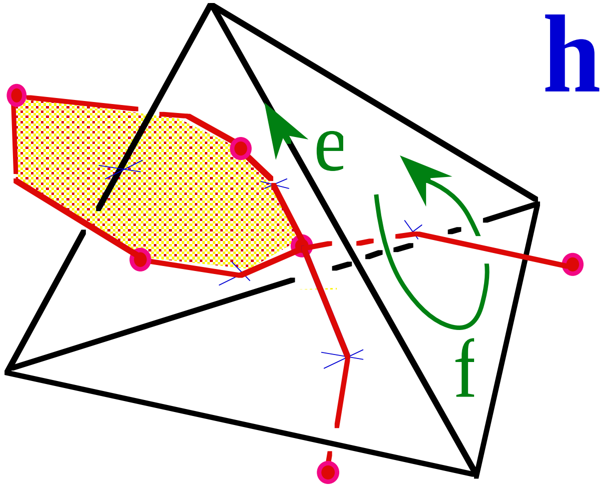
Dual mesh (n – p)-cells matched 1-to-1 with p-cells of primal mesh

b *at faces*

h *at dual edges (i.e., faces)*

e, a *at edges*

d, j *at dual faces*



here, $\mathbf{R}_{fe} = -1$

Everything lives on the discrete structure (mesh + some dual mesh) from now on

$$\mathbf{b} = \{b_f : f \in \mathcal{F}\} \xrightarrow{\quad \boldsymbol{\nu} \quad} \mathbf{h} = \{h_f : f \in \mathcal{F}\}$$

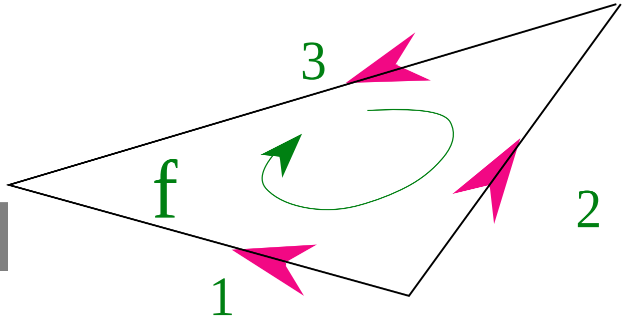
$$\mathbf{e} = \{e_e : e \in \mathcal{E}\} \xrightarrow{\quad \boldsymbol{\varepsilon} \quad} \mathbf{d} = \{d_e : e \in \mathcal{E}\}$$

To-be-specified square matrices

So, enforce Faraday's law, $\partial_t \int_S b + \int_{\partial S} e = 0$

for surfaces S made of primal faces, i.e., for each primal face f, which requires

$$\partial_t b_f + e_1 - e_2 - e_3 = 0$$



One such eqn per face. Together, in matrix form,

i.e.,

$$\partial_t \mathbf{b} + \mathbf{R}\mathbf{e} = 0$$

Same thing on dual network:

$$-\partial_t \mathbf{d} + \mathbf{R}^t \mathbf{h} = \mathbf{j}$$

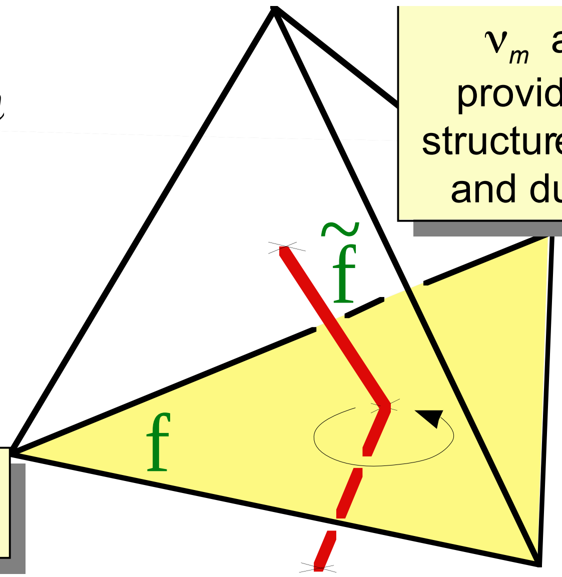
Which leaves
ε and **ν**
to be built

Boldface connotes (arrays of) DoF's. Red-boldface **ν** and **ε** are matrices, resp. face- and edge- indexed

Define approximation ν_m of Hodge ν by

$$\nu_m \tilde{f} = \sum_{f' \in \mathcal{F}} \boldsymbol{\nu}^{ff'} f'$$

↑ This is a 2-chain



ν_m and ε_m provide metric structure to primal and dual mesh

$$(\nu_m \tilde{f} \neq \nu \tilde{f})$$

← The Hodge of ~f (a chainlet)

Here's where error intrudes

$$\text{Then}\quad \mathbf{h}_f = \int_{\tilde{f}} h = \int_{\tilde{f}} \nu b = \boxed{\int_{\nu \tilde{f}} b \approx \int_{\nu_m \tilde{f}} b}$$

$$= \sum_{f' \in \mathcal{F}} \boldsymbol{\nu}^{ff'} \int_{f'} b = \sum_{f'} \boldsymbol{\nu}^{ff'} \mathbf{b}_{f'}$$

Hence $\mathbf{h} = \boldsymbol{\nu}\mathbf{b}$, and similarly, $\mathbf{d} = \boldsymbol{\varepsilon}\mathbf{e}$.

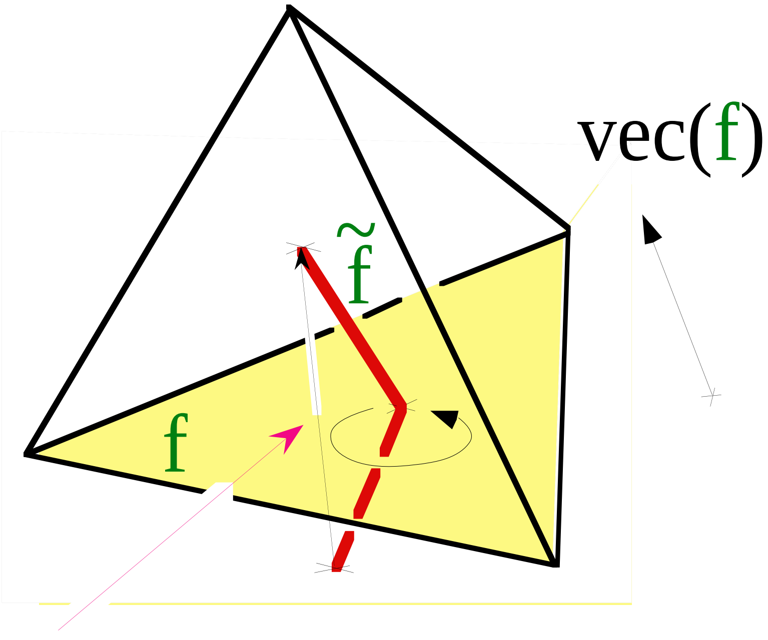
Weights of the chain representation of the Hodges of dual cells make entries of matrices $\boldsymbol{\nu}$ and $\boldsymbol{\varepsilon}$.

To minimize error due to

$$\int_{\nu \tilde{f}} b \approx \int_{\nu_m \tilde{f}} b$$

achieve equality for piecewise constant b, hence the

Consistency criterion



vec($\tilde{f}$): vector along dual edge $\tilde{f}$

vec(f): vectorial area of face f

A vector equality, now..

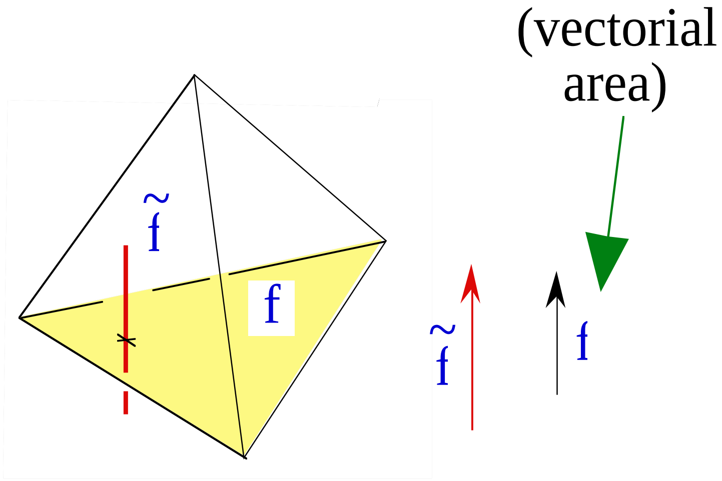
$$\sum_{f' \in \mathcal{F}} \nu^{ff'} \mathrm{vec}(f') = \nu \, \mathrm{vec}(\tilde{f})$$

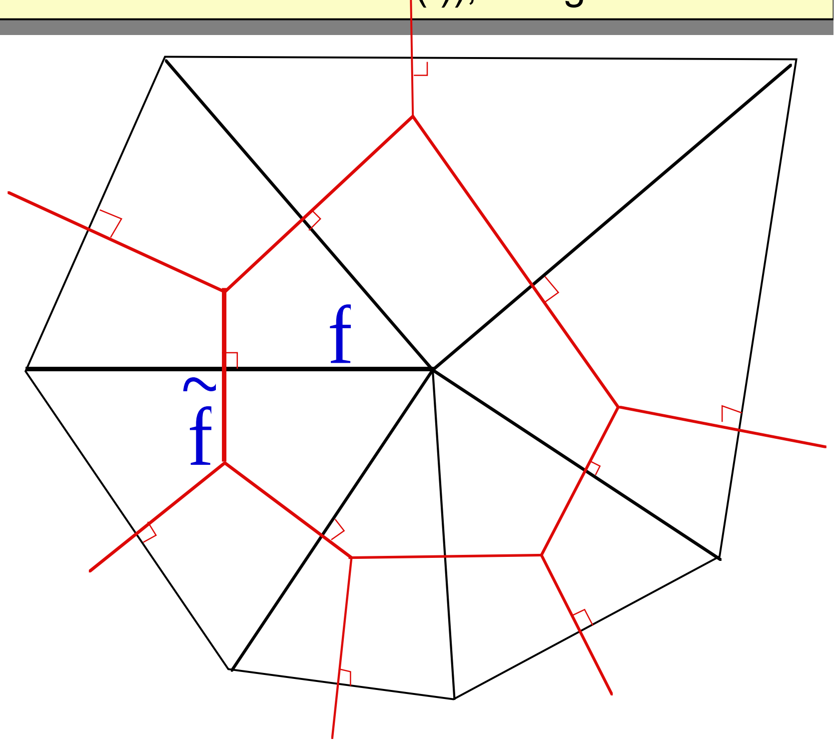
and similar about ε

(Actually, vec maps p-chains to p-vectors. But in 3D, 2-vecs isom to 1-vecs, by using metric.)

Criterion easily met if mutually orthogonal primal and dual meshes

Symbol f does double duty here (denotes both face f and its “vectorial area” vec(f)), for lighter notation





Here, $\tilde{f}$ // f, and criterion is

$$\boxed{\nu^{ff} f = \nu \tilde{f}} \Rightarrow \nu^{ff} = \nu \frac{\text{length}(\tilde{f})}{\text{area}(f)}$$

Weiland (“FIT” technique), Tonti (the “cell method”) do just that, with mutually orthogonal duals

The other standard way to manufacture discrete Hodges is good old Galerkin technique

If dual mesh barycentric, criterion met by the "Galerkin Hodge", defined as

$$\boldsymbol{\nu}^{ff'} = \int \nu \, \mathbf{w}^{f} \cdot \mathbf{w}^{f'}$$

$$\boldsymbol{\varepsilon}^{ee'} = \int \varepsilon \, \mathbf{w}^{e} \cdot \mathbf{w}^{e'}$$

where w^s is Whitney form of simplex s

We return to Whitney forms after this:

Discretization toolkit (a functor, in mathspeak) now available (replace b by **b**, rot by R, μ by **μ**, etc.).

Then, *automatic* spatial discretization of Maxwell's equations:

$$\partial_t \mathbf{b} + R\mathbf{e} = 0 \qquad -\partial_t \mathbf{d} + R^t \mathbf{h} = \mathbf{j}$$

$$\mathbf{h} = \boldsymbol{\nu}\, \mathbf{b} \qquad \mathbf{d} = \boldsymbol{\varepsilon}\, \mathbf{e}$$

hence a Yee-like scheme ("GFD"):

Generalizes [Yee 66]. Yee's primal and dual grids were staggered cubic lattices

$$\frac{\mathbf{b}^{k+1/2} - \mathbf{b}^{k-1/2}}{\delta t} + R\mathbf{e}^{k} = 0$$

Explicit if **ε** diagonal.

$$-\boldsymbol{\varepsilon}\,\frac{\mathbf{e}^{k+1} - \mathbf{e}^{k}}{\delta t} + R^t \boldsymbol{\nu}\, \mathbf{b}^{k+1/2} = \mathbf{j}^{k+1/2}$$

With orthogonal dual, this looks like finite-element free discretization. Are finite elements superfluous?

Of course not. Convergence study requires way to build approximate fields from DoF's.

$$\begin{tikzcd} p_m \mathbf{b} & b \arrow[d] \\ \mathbf{b} \arrow[u] & r_m b \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F}\text{orms} \arrow[d, bend right, "r_m"'] \\ \mathcal{D}.\text{o}.\mathcal{F}. \arrow[u, bend right, "p_m"'] \end{tikzcd}$$

$$\begin{tikzcd} p_m \mathbf{h} & h \arrow[d] \\ \mathbf{h} \arrow[u] & r_m h \end{tikzcd}$$

(degrees of freedom)

Computed fluxes

Computed mmf's

$$\mathbf{b} = \{\mathbf{b}_f : f \in \mathcal{F}\} \qquad \mathbf{h} = \{\mathbf{h}_f : f \in \mathcal{F}\}$$

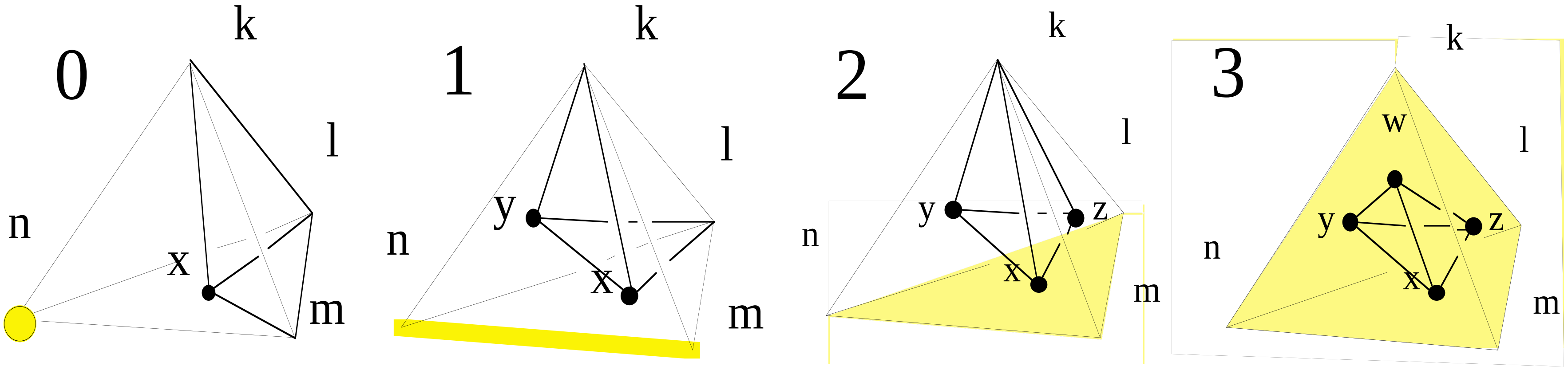
$$(r_m b)_f = \int_f b \qquad (r_m h)_f = \int_{\tilde{f}} h$$

True ones

m for mesh

The Whitney map: from DoF array **b**, build DF $p_m \mathbf{b}$ as $\sum_f \mathbf{b}_f \, w^f$

Whitney forms



w^n $\quad w^{\{m,\, n\}}$ $\quad w^{\{l,\, m,\, n\}}$ $\quad w^{\{k,\, l,\, m,\, n\}}$

λ^n

Integral of w^{m, n} along xy is vol(lxyk)/vol(lmnk). Same for other degrees.

$$\lambda^n \, \nabla\lambda^m - \lambda^m \, \nabla\lambda^n$$

← ⇓ Vector-proxies notation here

$$2[\lambda^l \; \nabla\lambda^m \times \nabla\lambda^n + \ldots + \ldots]$$

Circular permutation

$1/\mathrm{vol}$

At levels 1 and 2, the edge- and face-elements.

Commutative diagram:

$$\begin{tikzcd}
\text{Proxies} & \overset{0}{\circ} \arrow[r, "\mathrm{grad}"] \arrow[d, leftrightarrow, "\#"', "\flat"] & \overset{1}{\circ} \arrow[r, "\mathrm{rot}"] \arrow[d, leftrightarrow, "\#"', "\flat"] & \overset{2}{\circ} \arrow[r, "\mathrm{div}"] \arrow[d, leftrightarrow, "\#"', "\flat"] & \overset{3}{\circ} \arrow[d, leftrightarrow, "\#"', "\flat"] \\
\text{Forms} & \circ \arrow[r, "\mathrm{d}"] & \circ \arrow[r, "\mathrm{d}"] & \circ \arrow[r, "\mathrm{d}"] & \circ \\
\text{Primal DoF arrays} & \circ \arrow[r, "\mathrm{G}"] \arrow[u, "p_m"] & \circ \arrow[r, "\mathrm{R}"] \arrow[u, "p_m"] & \circ \arrow[r, "\mathrm{D}"] \arrow[u, "p_m"] & \circ \arrow[u, "p_m"]
\end{tikzcd}$$

First step, compare computed DoF's with face- and edge-integrals of actual fields

$$\mathbf{D}\mathbf{b} = 0, \; \mathbf{b} = \boldsymbol{\mu}\mathbf{h}, \quad \mathbf{R}^{t}\mathbf{h} = \mathbf{j}$$

Magnetostatics, discretized form

$$\mathrm{D}\, r_m b = 0 \qquad\qquad \mathrm{R}^{t}\, r_m h = r_m j$$

$$(\mathrm{D}\, r_m = r_m \operatorname{div}) \qquad\qquad (\mathrm{R}^{t}\, r_m = r_m \operatorname{rot})$$

$$\underbrace{(\mathbf{b} - r_m b)}_{\in \ker(\mathrm{D})} - \boldsymbol{\mu}\underbrace{(\mathbf{h} - r_m h)}_{\in \ker(\mathrm{R}^{t})} = (\boldsymbol{\mu} r_m - r_m \mu) h$$

$$\equiv \boldsymbol{\mu}\,(r_m \nu - \boldsymbol{\nu} r_m) b$$

a term which is small ($o(\gamma_m)$) if consistency criterion is met

grain of the mesh

Notation: $\|\mathbf{b}\|^2_{\nu} = \sum_{f,\,f'} \boldsymbol{\nu}^{ff'} \mathbf{b}_f \mathbf{b}_{f'}$ ("ν–norm"), $(\mathbf{b}, \mathbf{h}) = \sum_f \mathbf{b}_f \mathbf{h}_f$

"ν-square" the equality

$$\mathbf{b} - r_m b - \boldsymbol{\mu}(\mathbf{h} - r_m h) = (\boldsymbol{\mu} r_m - r_m \mu) h$$

Observe that

$$(\mathbf{b} - r_m b,\ \mathbf{h} - r_m h) = (R\mathbf{a},\ \mathbf{h} - r_m h) = (\mathbf{a},\ R^t(\mathbf{h} - r_m h)) = 0,$$

for some **a**, *hence*

$$\|\mathbf{b} - r_m b\|^2_{\nu} + \|\mathbf{h} - r_m h\|^2_{\mu} = \|(\boldsymbol{\nu} r_m - r_m \nu) b\|^2_{\mu}$$

$$\equiv \|(\boldsymbol{\mu} r_m - r_m \mu) h\|^2_{\nu}$$

Evaluate this last term. Easy for diagonal **μ**. Show it small if above criterion upheld.

The proof follows the lines of Lax's classic argument. Stability comes from mesh uniformity.

$$\text{Consistency} \left\{ \begin{array}{l} p_m r_m b \to b \qquad \text{when "}m \to 0\text{"} \\ \|(\boldsymbol{\nu} r_m - r_m \nu) b\|_\mu \to 0 \end{array} \right.$$

$$+$$

$$\text{Stability :} \qquad \alpha \, \|p_m \mathbf{b}\|_\nu \le \|\mathbf{b}\|_\nu$$

$$=$$

$$\text{Convergence :} \quad \|p_m(\mathbf{b} - r_m b)\|_\nu \le \frac{1}{\alpha} \|\mathbf{b} - r_m b\|_\nu$$

$$\le \frac{1}{\alpha} \|(\boldsymbol{\nu} r_m - r_m \nu) b\|_\mu \to 0 \qquad \Rightarrow \boxed{p_m \mathbf{b} \to b}$$

No infsup argument there.
See why in A.B.: "A new viewpoint on mixed elements", **Meccanica, 27** (1992), pp. 3-11.

$$\partial_t \mathbf{b} + R\mathbf{e} = 0 \qquad -\partial_t \mathbf{d} + R^t \mathbf{h} = \mathbf{j}$$

$$\mathbf{h} = \boldsymbol{\nu}\, \mathbf{b} \qquad \mathbf{d} = \boldsymbol{\varepsilon}\, \mathbf{e}$$

Use $DR = 0$, $G^t R^t = 0$ to get

$$-G^t \mathbf{d} = \enclose{circle}{\mathbf{q}}$$

$$\mathbf{d} = \varepsilon\, \mathbf{e}$$

$$R\mathbf{e} = -\partial_t \mathbf{b}$$

← K's node law →

$$(\partial_t \mathbf{q} - G^t \mathbf{j} = 0)$$

← K's loop law →

$$D\mathbf{b} = 0$$

$$\mathbf{h} = \nu\, \mathbf{b}$$

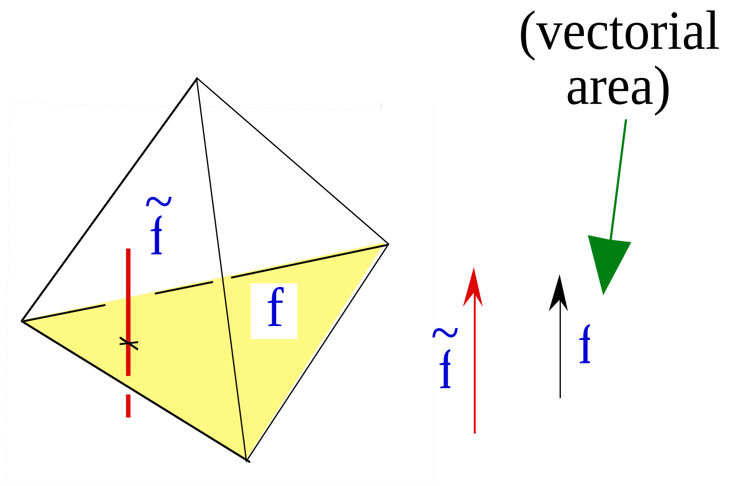
$$R^t \mathbf{h} = \enclose{circle}{\mathbf{j}} + \partial_t \mathbf{d}$$

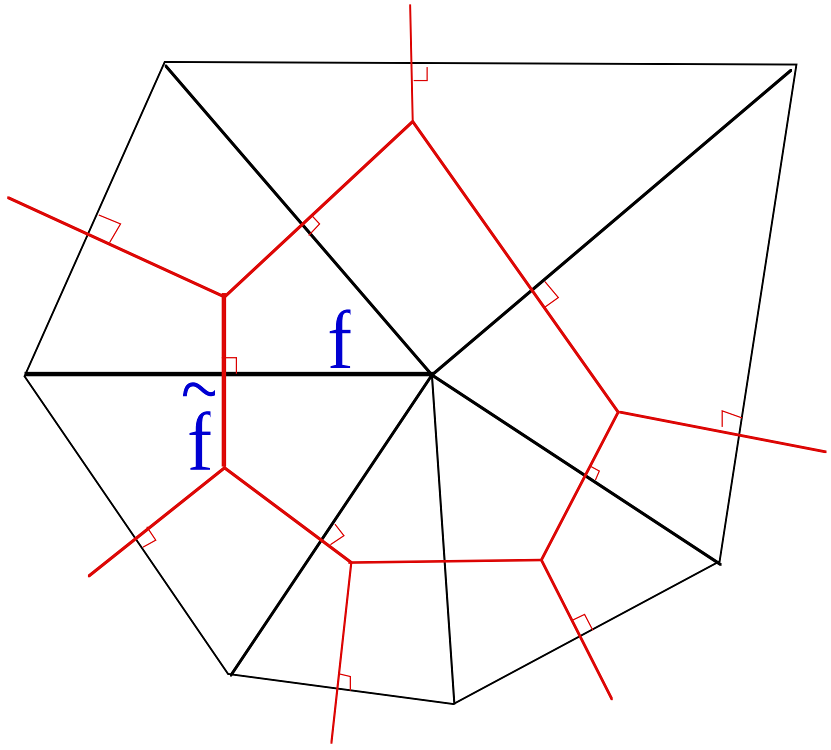
Two interlocked networks

If ε and ν diagonal, ε^{ee} and ν^{ff} can be seen as branch impedances

… for this reason, and also for explicitness of evolution scheme,

Highly desirable mutual orthogonality of primal and dual meshes

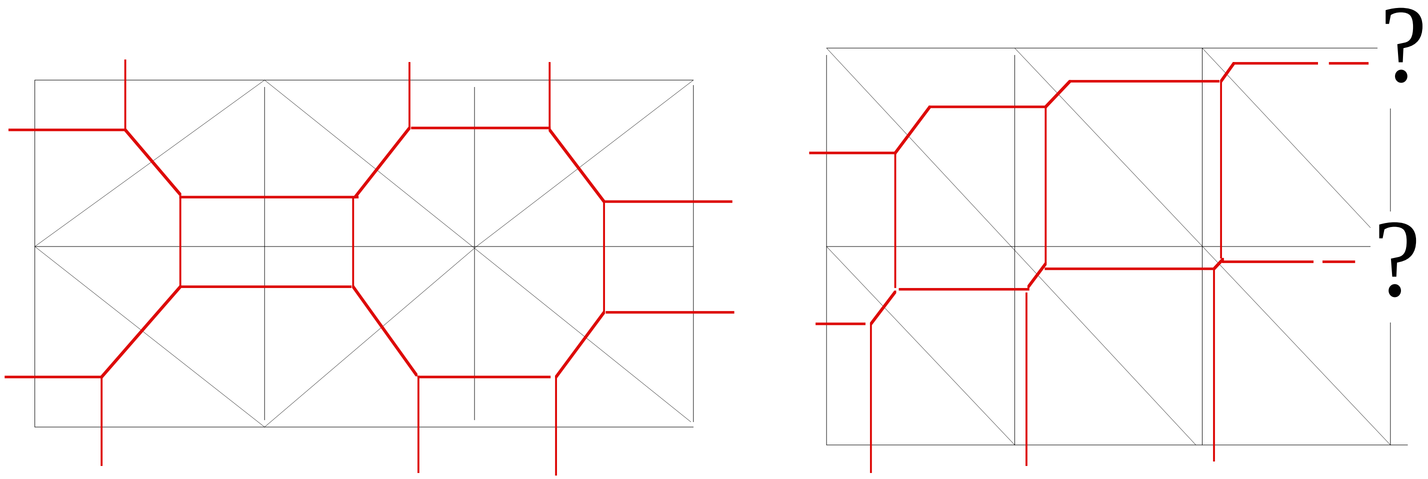




Here, $\tilde{f}$ // f, and criterion

is $\boxed{\nu^{ff} f = \nu \tilde{f}}$ $\Rightarrow \nu^{ff} = \nu \dfrac{\text{length}(\tilde{f})}{\text{area}(f)}$

Alas ...

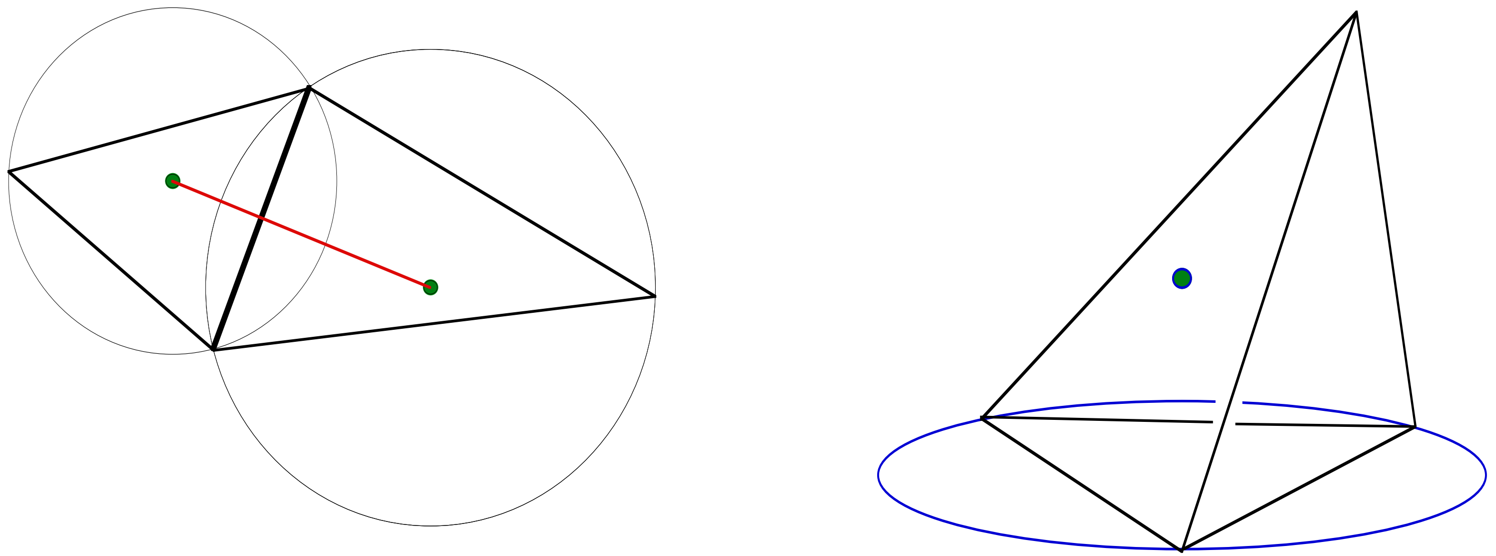


Only specially designed primal meshes will admit an orthogonal dual

Voronoi cells may flunk this test.

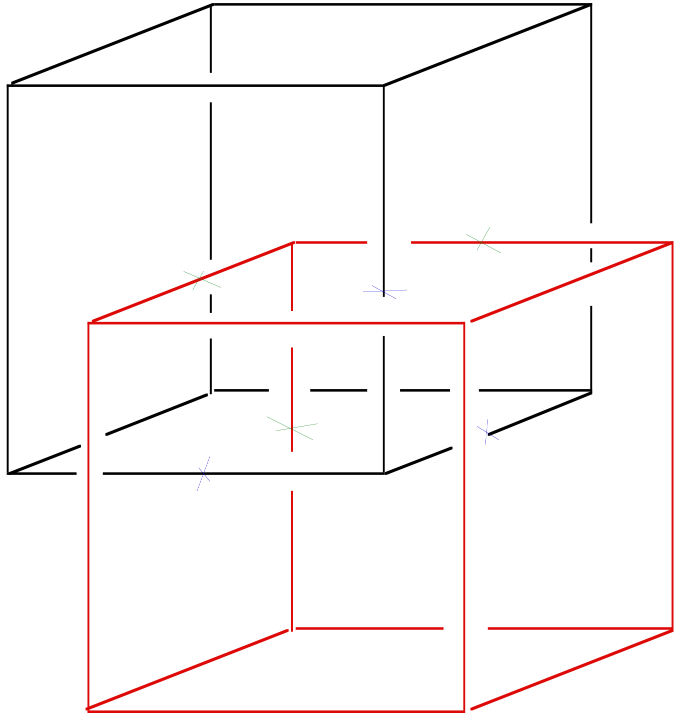
A *sufficient* condition:

The "circumcenter inside" property



Not equivalent to “acute dihedral angles”.

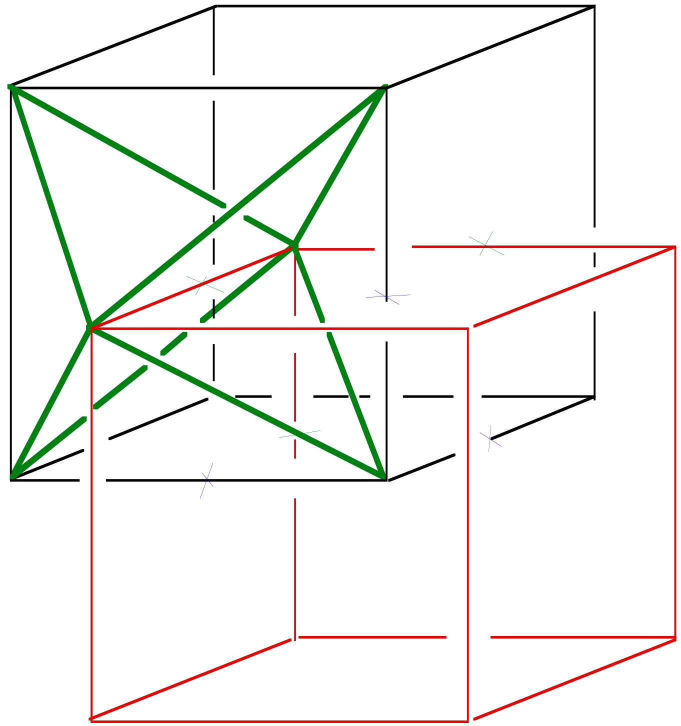
The Sommerville mesh



Start from two staggered cubic lattices ...

... and join the nodes:

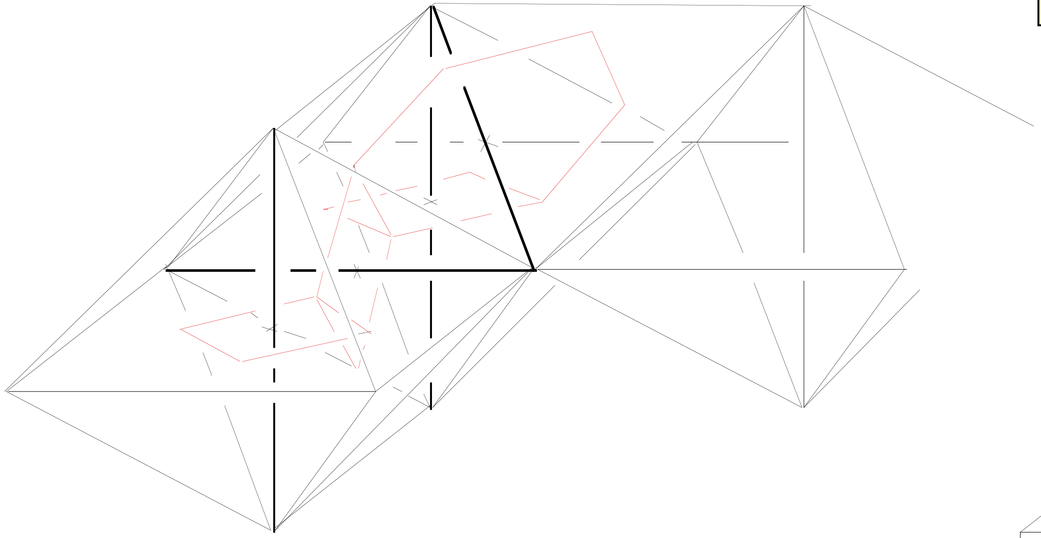
Lo the Sommerville (1923) tetrahedron, a space-filler.

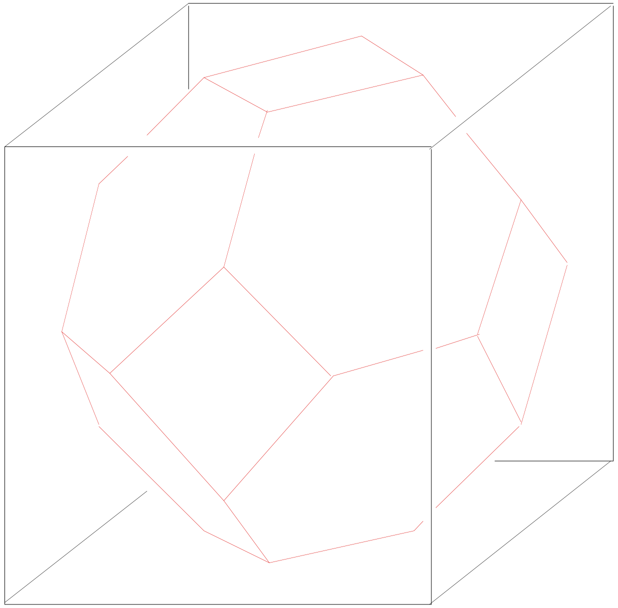


Circumcenter well inside. Property robust w.r.t. mesh deformations.

The dual mesh:

Roundness of dual volume suggests good dispersion properties. Indeed, cf. Monk–Parrott, SIAM, 1994.





(truncated octahedron, aka tetrakaidecahedron)

Two interlocked networks:

$$-G^t\mathbf{d} = \enclose{circle}{\mathbf{q}} \quad \longleftarrow \text{K's node law} \longrightarrow \quad D\mathbf{b} = 0$$

$$\mathbf{d} = \varepsilon\,\mathbf{e} \qquad (\partial_t\mathbf{q} - G^t\mathbf{j} = 0) \qquad \mathbf{h} = \nu\,\mathbf{b}$$

$$R\mathbf{e} = -\partial_t\mathbf{b} \quad \longleftarrow \text{K's loop law} \longrightarrow \quad R^t\mathbf{h} = \enclose{circle}{\mathbf{j}} + \partial_t\mathbf{d}$$

e, b at the edges and faces of one mesh (primal),

h, d at the edges and faces of the other (dual)

An old idea: Kron, Yee... Was "in the air" circa 1978

… that is, when I began to work on that. What follows is not science history, just an account of some developments I happened to witness or be involved in

From papers such as

J. Carpenter: "A network approach to the numerical solution of eddy-current problems", **IEEE Trans., MAG-11,** 5 (1975), pp. 1517-22.
["The essence of the present method is to form separate, but linked, network models of the various flux and current-flow paths, and then examine the linkages, and thus the interactions, between these two networks.”]

and preprint versions of

J.A.M. Davidson, M.J. Balchin: "Experimental verification of network method for calculating flux and eddy-current distributions in three dimensions", **IEE Proc., 128, Pt. A,** 7 (1981), pp. 492-96.

L.R. Turner, R.J. Lari: "Developments in Eddy Current Computation with EDDYNET", **IEEE Trans., MAG-19**, 6 (1983), pp. 2577-80.

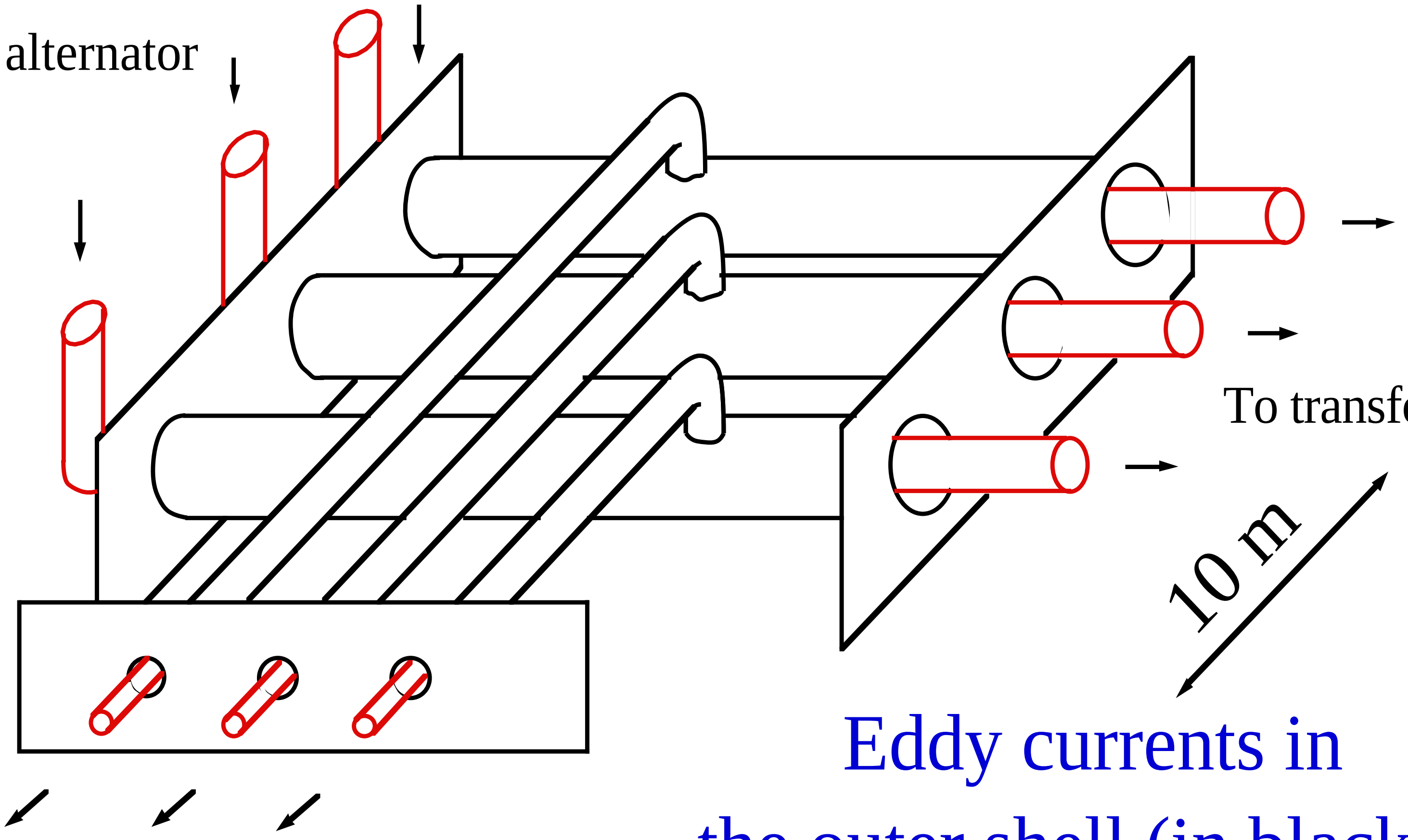
(to say nothing of those I should have read but didn’t, such as

R.H. MacNeal: "An asymmetrical finite difference network", **Quart. Appl. Math.,** 11 (1953), pp. 295-310.
[Delaunay mesh. "Non negative resistors." There is a suggestion here that networks in duality (in 2D) could be characterized by 1) nodes of one inside cells of the other, 2) edges of ones orthogonal to the edges of the other.]
)

… it was quite obvious that DoF’s relative to E or H had to be assigned to edges. The question was, how to interpolate from such edge values, in order to derive, via the variational method, coupling coefficients between the two networks.

Such research was supported as part of the nuclear energy program. For instance, we [J. Planchard, J.C. Vérité and I, at EdF, and our consultant J.C. Nédélec] had to analyze the following eddy-current problem on the aluminum sheath of an alternator output:

Problems like that one (a 1350 MW alternator output):



Eddy currents in
the outer shell (in black) ?
Vibrations ?

... spurred an interest for topology *in electromagnetism*

The problem was solved (cf.

J.C. Vérité: "Computation of Eddy Currents on the Alternator Output Conductors by Finite Element Methods", **Electrical Power and Energy Systems, 1,** 3 (1979), pp. 193-98.
)

by the "j method" described next, but (since conductors were thin) with surface current density, expressed as the rotated gradient of a scalar stream function (since div j = 0). Investment on algebraic topology, necessary to understand how to "cut" the surface, in order to make this function single-valued, made it easier, later, to understand the Whitney connection.

Had the conductors be thicker, a full 3D analysis would have been required, for which the j-formulation (next slide) could be adopted, but the "stream function" would then be vector-valued, and edge based, without any obvious finite-element interpolant.

Meanwhile, we published results obtained in 3D via the j-formulation, cf.

J.C. Vérité, in **COMPUMAG Conference on the Computation of Magnetic fields,** Communications Proceedings, Grenoble, 4, 5, 6 Sept. 1978, ENSEGP, paper 7.3.

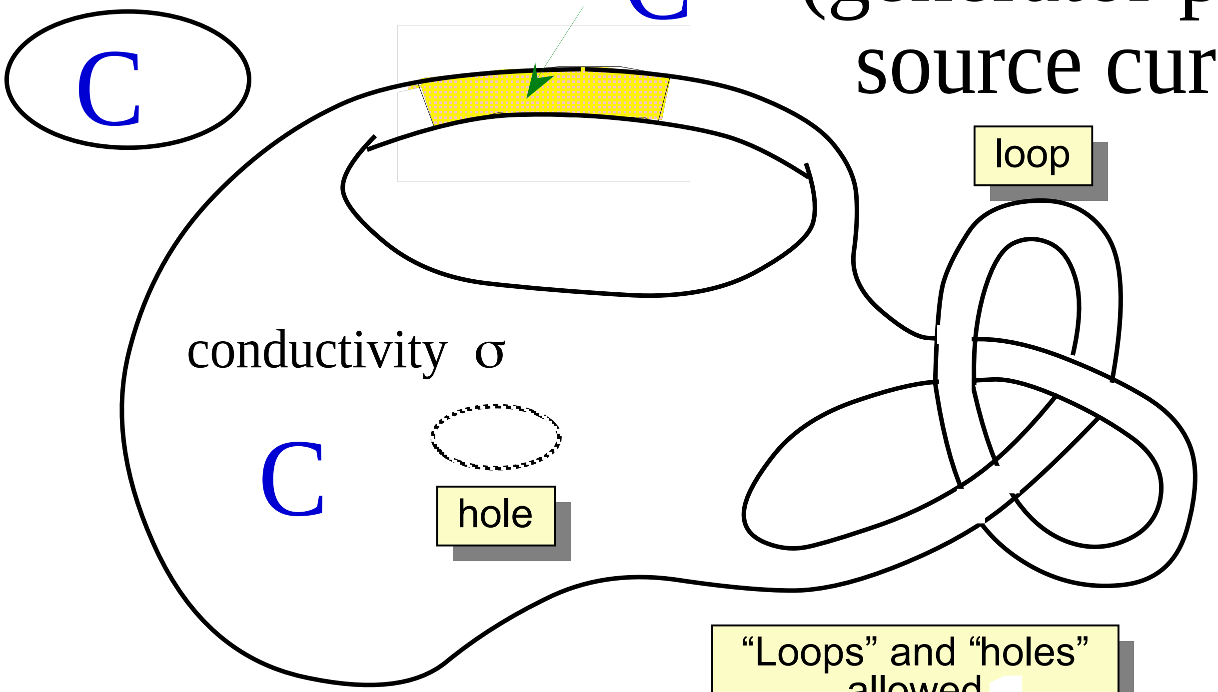
A. Bossavit: "On the Numerical Analysis of Eddy-Current Problems", **Comp. Meth. Appl. Mech. Engng., 27** (1981), pp. 303-18.

(where I goofed by confusing homology and homotopy). The equivalence between j-formulation and h-formulation (next slides) was clear, but we lacked an interpolant from edge values. In the summer of 79, Nédélec provided just that (**Numer. Math., 35** (1980), pp. 315-41): the field $x \longrightarrow a \times x + b$, whose 6 parameters matched the six edge DoFs of a tetrahedron. The road was clear to a 3D eddy current code:

A. Bossavit, J.C. Vérité: "A Mixed FEM-BIEM Method to Solve Eddy-Current Problems", **IEEE Trans., MAG-18,** 2 (1982), pp. 431-35; "The TRIFOU Code: Solving the 3-D Eddy-Currents Problem by Using H as State Variable", **IEEE Trans., MAG-19,** 6 (1983), pp. 2465-70.

The j-formulation

C^s (generator part, given source current j^s)



"Loops" and "holes" allowed

$$a(x) = \frac{\mu_0}{4\pi}\int \frac{j(y)}{|x-y|}\,dy$$

$$a = Gj$$

Green's operator (not the previous "red G")

$$J^s = \{ j \in L^2(C) : \ \operatorname{div} j = 0, \ j = j^s \ \text{in} \ C^s \}$$

Find $j \in J^s$ *such that*

$$\int_C \partial_t G j \cdot j' + \int_C \frac{1}{\sigma} j \cdot j' = 0 \ \ \forall\, j \in J^0$$

This is the eddy currents model: $\varepsilon\, \partial_t E$ neglected (note "infinite celerity of signal" implied in definition of G)

$$J^s = \{ j \in L^2(C) : \ \operatorname{div} j = 0, \ \ j = j^s \ \text{in} \ C^s \}$$

Find $j \in J^s_m$ *such that*

$$\int_C \partial_t G j \cdot j' + \int_C \frac{1}{\sigma} j \cdot j' = 0 \ \ \forall j \in J^0_m$$

Straigthforward Galerkin discretization

J_m : piecewise constant fields with face intensities $\mathbf{j}_f$ as DoF's, and $\mathbf{j} = \{\mathbf{j}_f : f \in \mathcal{F}\}$

constrained by $\mathbf{D}\mathbf{j} = 0$

Mixed system

$$\partial_t \mathbf{G}\mathbf{j} + \boldsymbol{\sigma}^{-1}\mathbf{j} + \mathbf{D}^t\boldsymbol{\psi} = 0$$

$$\mathbf{D}\mathbf{j} \ = 0$$

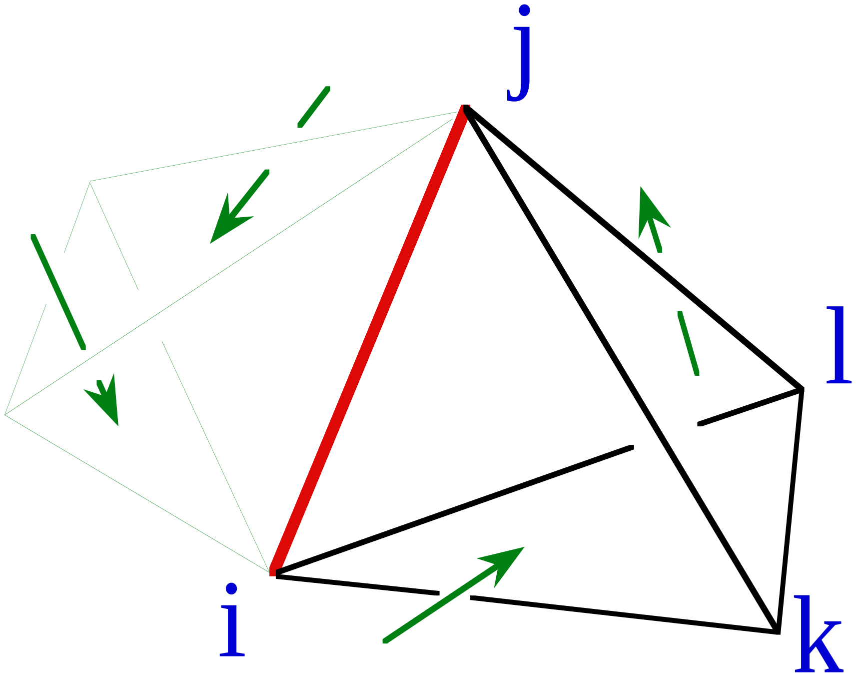
Electric potential

(block-centered)

A basis for J_m?

All face intensities = 1 around edge ij

i.e., mmf = 1 along ij



These “edge currents”, div-free, span J_m

Suggests the curl of some "edge element", but of which form?

With such an element, one could discretize, Galerkin style, the equivalent "H formulation":

$$\mathrm{H}^{s} = \{\mathrm{H} \in \mathrm{L}^{2}(\mathrm{E}_{3}) : \mathrm{rot}\,\mathrm{H} = \mathrm{j}^{s} \ \text{in}\ \mathrm{C},\ 0\ \text{out of}\ \mathrm{C}\}$$

Find $\mathrm{H} \in \mathrm{H}^{s}$ *such that*

$$\int_{\mathrm{E}_3} \mu\, \partial_t \mathrm{H} \cdot \mathrm{H}' + \int_{\mathrm{C}} \frac{1}{\sigma}\, \mathrm{rot}\,\mathrm{H} \cdot \mathrm{rot}\,\mathrm{H}' = 0 \quad \forall\, \mathrm{H}' \in \mathrm{H}^{0}$$

over all space, but sparse matrices!

Field outside C can be handled by boundary-integral method, hence a discretization over C, in fine

What follows is a reconstruction, with the benefit of hindsight, of the logical process by which the edge element *could* have been derived. (The actual process may have been less direct.)

For electrical engineers, the "Poincaré gauge" (next slide), $A(x) \cdot x = 0$, where x denotes both point x and vector from origin to x, is a sometimes useful alternative to the "Coulomb gauge" (div $A = 0$). Starting from div-free B, the very integration process by which mathematicians prove the Poincaré lemma (call that "Poincaré mapping", denoted $d+$ below) provides a Poincaré-gauged A such that $B = \text{rot}\ A$.

The Poincaré mapping itself derives from an integration of the Lie derivative of a diff. form along the flow of a vector field that achieves a "retraction" of the region of interest (next slide). By a particular choice of the retraction (slide further down) one can obtain the edge element as the Poincaré-mapping image of the above edge current.

This applies to all degrees, hence a generation of the whole Whitney complex by starting from piecewise constant functions (DF's of degree 3) on tetrahedra, and integrating upwards (slides 3 to 7 down the road). But it's only in 1995 that I saw that clearly (IEE conference on CEM, Bath, U.K., April 96).

From E, build

$$\boxed{\varphi(x) = \int_0^1 E(\lambda x)\cdot x \, d\lambda}$$

If rot $E = 0$ then $E = \operatorname{grad} \varphi$

From B, build

$$\boxed{A(x) = \int_0^1 \lambda \, B(\lambda x) \times x \, d\lambda}$$

If div $B = 0$ then $B = \operatorname{rot} A$

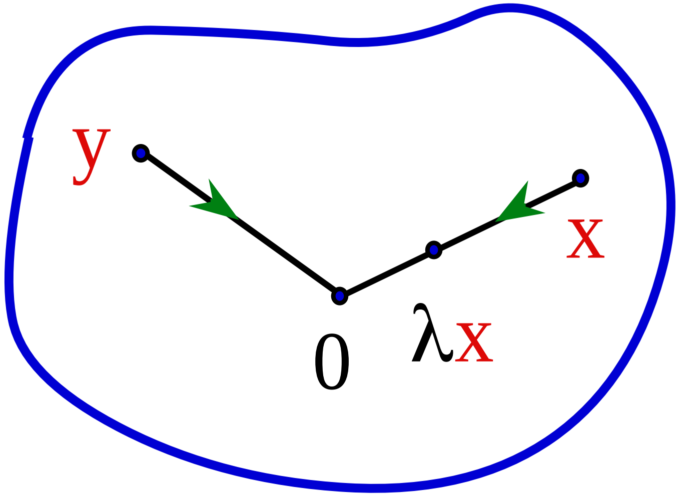
From Q, build

$$\boxed{J(x) = \int_0^1 \lambda^2 \, Q(\lambda x) \, x \, d\lambda}$$

then $Q = \operatorname{div} J$

"Poincaré gauge"

$(A(x) \cdot x = 0)$

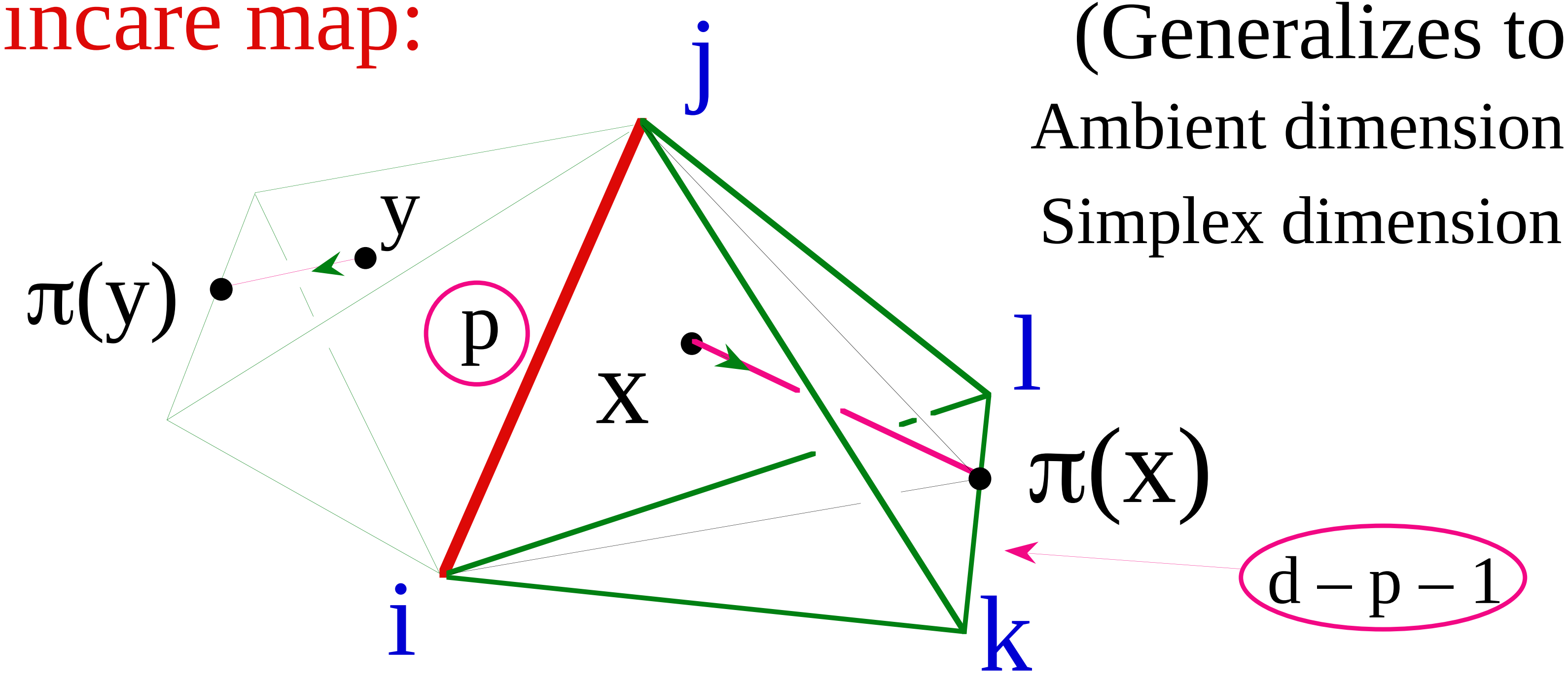


retraction

Homotopy between maps x --> x and x --> 0.

Retraction there is not to a point, but from “cluster” around ij, minus ij itself, to equatorial “belt” of cluster.

From B to A by variant of Poincaré map:



(Generalizes to
Ambient dimension d
Simplex dimension p)

$$A(x) = \int_0^1 \lambda\, B(\pi(x) + \lambda(x - \pi(x))) \times (x - \pi(x))\, d\lambda$$

If $B = 2\,\nabla\lambda^i \times \nabla\lambda^j \equiv \frac{kl}{3\,\mathrm{vol}}$, then $A(x) = \frac{1}{6\,\mathrm{vol}}\, kl \times (x - \pi(x))$

the edge element !

Note that kl x (x – π(x)) = kl x (x – k))

Taking all d-conformal forms of pol. degree 1 does **not** make an exact sequence (as simple counting shows

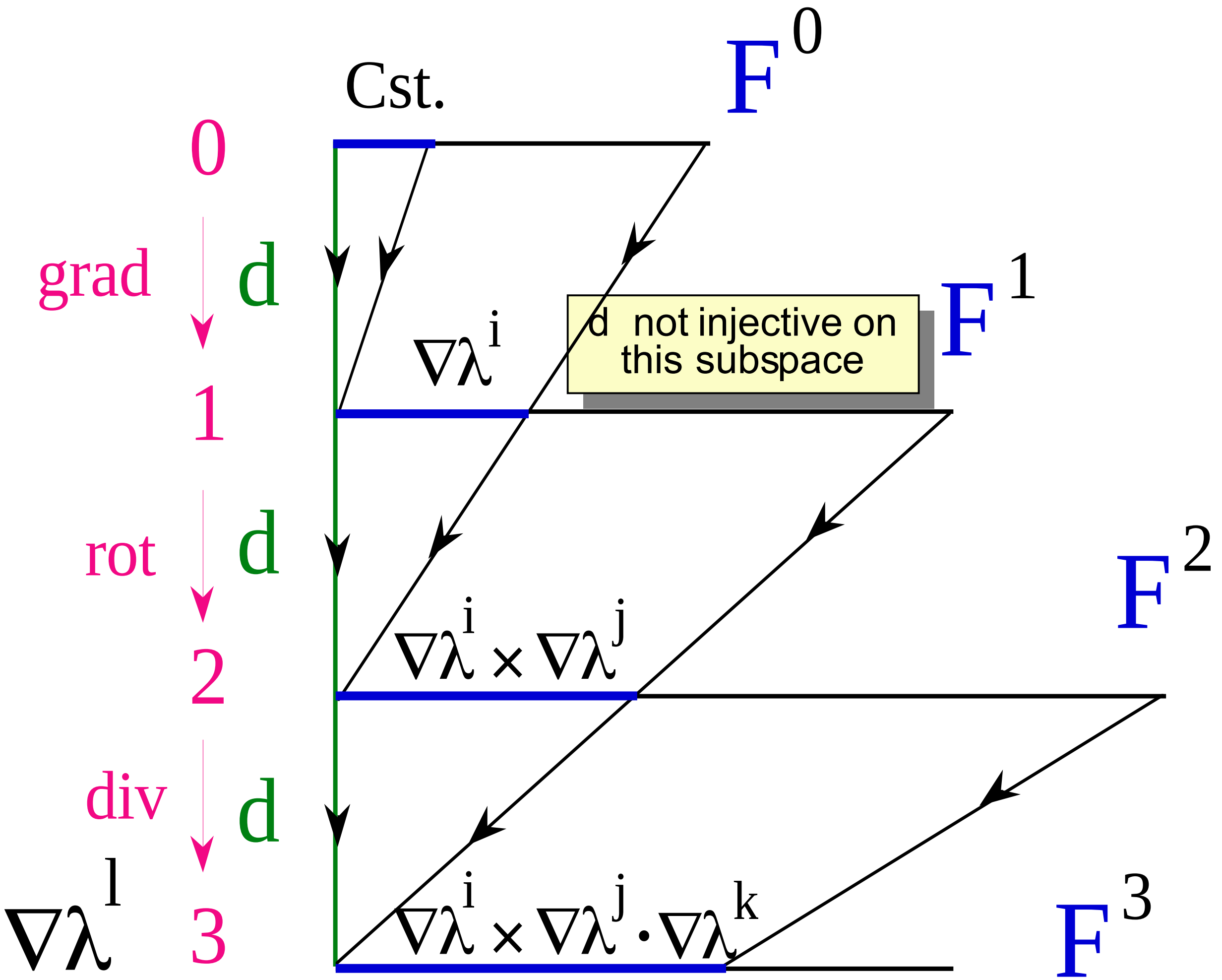
N λ^i

2E $\lambda^i \nabla\lambda^j$

“Tangentially continuous” — but this is not enough

3F $\lambda^i \nabla\lambda^j \times \nabla\lambda^k$

4V $\lambda^i \nabla\lambda^j \times \nabla\lambda^k \cdot \nabla\lambda^l$



d not injective on this subspace

Dimension of space, for
N nodes,
E edges,
etc.

$$N - E + F - V = \chi$$

1 7 12 6

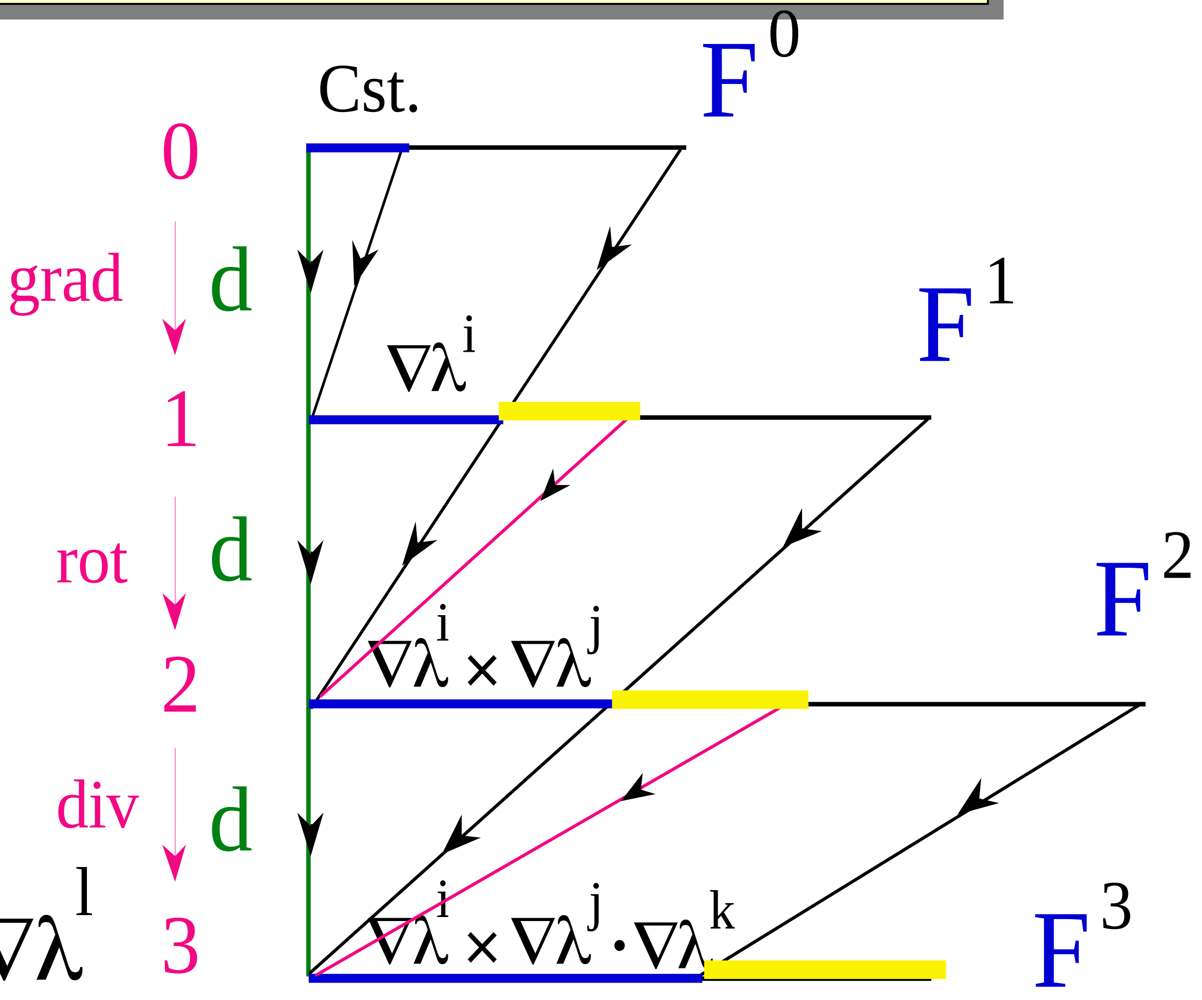
"Culling": Yellow subspaces should be excised to get exact sequence.
For instance, at d° 1, gradients of quadratics not wanted.)

N λ^i

2E $\lambda^i \; \nabla\lambda^j$

3F $\lambda^i \; \nabla\lambda^j \times \nabla\lambda^k$

4V $\lambda^i \; \nabla\lambda^j \times \nabla\lambda^k \cdot \nabla\lambda^l$



$$\underset{1}{N} - \underset{7}{E} + \underset{12}{F} - \underset{6}{V} = \chi$$

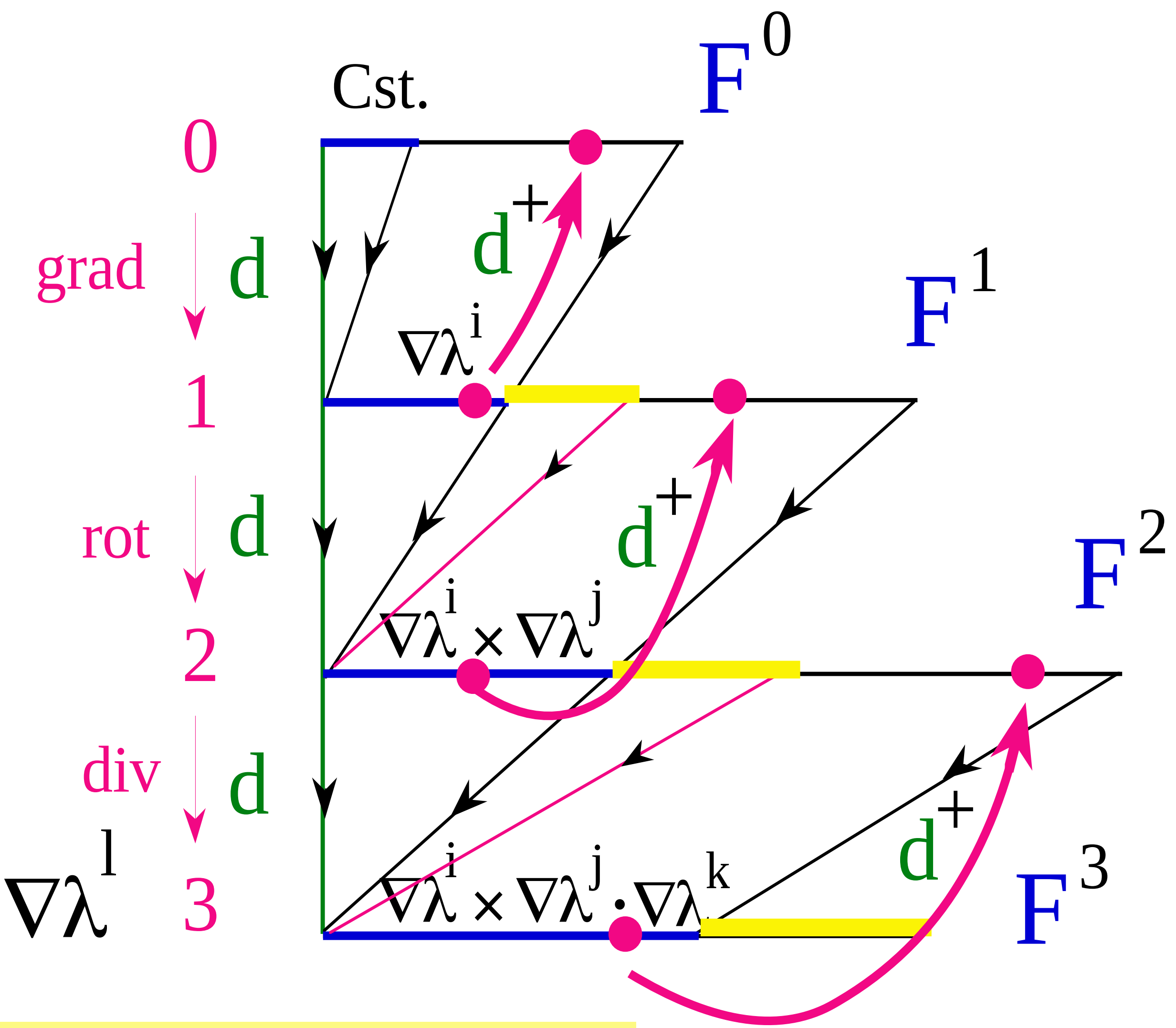
Poincaré map d^+ gives subspaces to be kept

$N \quad \lambda^i$

$2E \quad \lambda^i \, \nabla\lambda^j$

$3F \quad \lambda^i \, \nabla\lambda^j \times \nabla\lambda^k$

$4V \quad \lambda^i \, \nabla\lambda^j \times \nabla\lambda^k \cdot \nabla\lambda^l$



$$N - E + F - V = \chi$$

1 7 12 6

The structure of the Whitney complex:

Since

$$d\,d^{+}d = d$$

one has

$$\boxed{W^{p} = dF^{p-1} \oplus d^{+}d\,F^{p}}$$

{Wh. forms of degree p}

d-conformal forms of {degree p – 1} {degree p}

Same thing for hexahedra. Same thing for higher polynomial degrees (cf. Hiptmair):

Uniform polynomial degree

+

"Exact sequence" requirement

+

(adapted) Poincaré gauge

=

Whitney forms

But back in 1981, none of us knew about Whitney forms, nor were aware that Dodziuk[#] and Müller[*] had used them for numerical work (not related with electrical engineering, however). I wanted a rationale for this outlandish $a \times x + b$ expression. After several false starts, I found the right clue in Kotiuga's thesis (next slide) after our Dec. 84 encounter. In July 85, reading [†], and setting $n = 3$ and $p = 1$ in the formula for Whitney forms given there, I found myself staring at the edge element $(\lambda^i \nabla\lambda^i - \lambda^j \nabla\lambda^i$, i.e. $a \times x + b$, with the advantage of a *basis* function over a *shape* function).

[#] J. Dodziuk: "Finite-Difference Approach to the Hodge Theory of Harmonic Forms", **Amer. J. Math., 98**, 1 (1976), pp. 79-104.

[*] W. Müller: "Analytic Torsion and R-Torsion of Riemannian Manifolds", **Advances in Mathematics, 28** (1978), pp. 233-305.

[†] J. Komorowski: "On Finite-dimensional Approximations of the Exterior Differential, Codifferential and Laplacian on a Riemannian Manifold", **Bull. Acad. Polonaise Sc**. (Math., Astr., Phys.), **23,** 9 (1975), pp. 999-1005.

> *"A fundamental connection between homology groups and finite element interpolation will be recognized in the next few years"*

KOTIUGA, P.R., *Hodge Decompositions and Computational Electromagnetics* (Thesis), Dpt. of Electrical Engng., Mc Gill University, Montréal, 1984.

Moreover, the “commutative diagram property” illuminated many numerical facts. For one, that our out-of-conductor discretization (by $H = \text{grad}\,\phi$, with ϕ a weighted sum of scalar nodal elements), “got along” so well with the edge element inside the conductor. Perhaps more importantly (as we slowly realized between 85 and 88, along with coding developments), it allowed one to interpret the discrete H-formulation as a realization of the two-network idea:

Discrete form of H-formulation:

$$H^{s}_{m} = \{\mathrm{H} = \sum_{e \in \mathcal{E}} \mathbf{H}_{e}\, w^{e} \;:\; \operatorname{rot} \mathrm{H} = j^{s} \;\text{ in }\; C^{s}, \quad 0 \;\text{ out of }\; C\}$$

Find $\mathrm{H} \in H^{s}_{m}$ *such that*

$$\int_{E_3} \mu \partial_t \mathrm{H} \cdot \mathrm{H}' + \int_{C} \frac{1}{\sigma} \operatorname{rot} \mathrm{H} \cdot \operatorname{rot} \mathrm{H}' = 0 \quad \forall\, \mathrm{H}' \in H^{0}_{m}$$

By the exact sequence property, $\operatorname{rot} w^{e} = \sum_{f \in \mathcal{F}} R_{fe}\, w^{f}$,
hence (with $\boldsymbol{\mu}$ and $\boldsymbol{\rho}$ Galerkin hodges of μ and $1/\sigma$)

$$\boldsymbol{\mu}\, \partial_t \mathbf{H} + R^{t} \underbrace{\boldsymbol{\rho} R(\mathbf{H} - \mathbf{H}^{s})}_{\mathbf{E}} = 0,$$

indeed the 2-network system, if one sets $\mathbf{j} = R\mathbf{H}$.

Others, meanwhile, were developing this equivalent-networks idea, notably T. Weiland [W], E. Tonti [T], but without using interpolants. Similar ideas appeared in other communities ("mimetic" methods, cf. Hyman–Shashkov, finite volumes, cf. Nicolaides, Yotov, lattice discretizations, cf. Teixeira–Chew, etc.). In joint work with L. Kettunen, we progressively shifted attention to the discretization of the Hodge operator, till the point where all methods (networks, finite differences, finite elements, finite volumes) appeared as different realizations of the same paradigm -- the difference being in their treatment of the Hodge map [BK].

But above all, the commutative diagram property helped solve the "spurious modes" conundrum (next two slides, cf. also D. Boffi's talk), which was probably the most potent factor in the adoption of edge elements from 1989 on. The property retains heuristic value nowadays (L. Demkowicz's presentation), and not only in electromagnetics [A].

[BK] A. Bossavit, L. Kettunen: "Yee-Like Schemes on Staggered Cellular Grids: A Synthesis Between FIT and FEM Approaches", **IEEE Trans., MAG-36,** 4 (2000), pp. 861-7 (errata, **MAG-36,** 6 (2000), p. 4050).

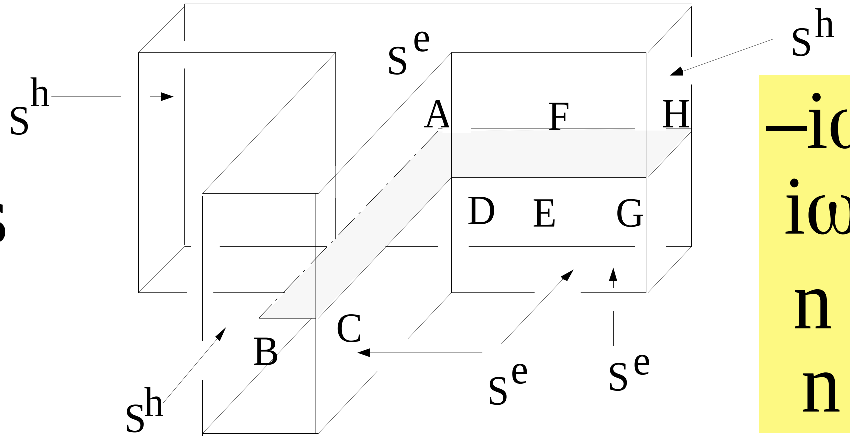
[T] E. Tonti: "A Direct Formulation of Field Laws: The Cell Method", **CMES, 2,** 2 (2001), pp. 237-58.

[W] T. Weiland: "Three dimensional resonator mode computation by finite difference methods", **IEEE Trans., MAG-21,** 6 (1985), pp. 2340-3.

[A] D.N. Arnold: "Differential complexes and numerical stability", **ICM** address (Beijing, Aug. 24, 2002).

The infamous "spurious modes", ca. 1989

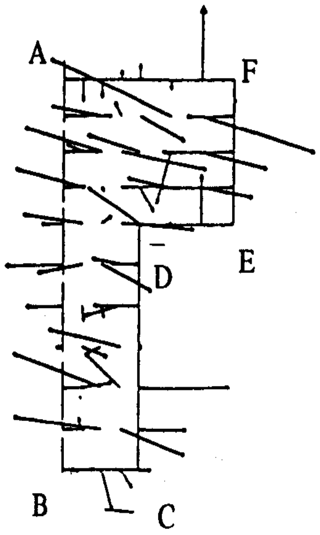
Compute
resonant modes
in a waveguide
T-junction

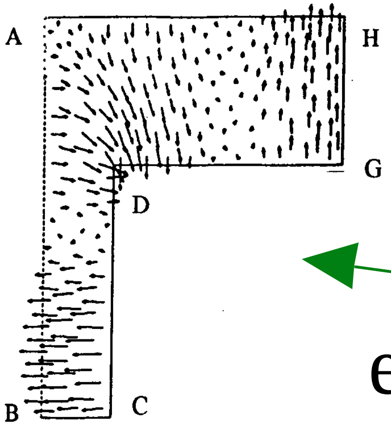


$$-i\omega\varepsilon E + \mathrm{rot}\ H = 0$$
$$i\omega\mu H + \mathrm{rot}\ E = 0$$
$$n \times H = 0 \ \text{ on } \ S^h$$
$$n \times E = 0 \ \text{ on } \ S^e$$

View of field E in shaded plane section:

Using standard
node-based
vector-valued
elements





Using
edge elements

After M.-F. Wong et al., **J. Phys. III** (1992), pp. 2083-99.

$$\mathrm{rot}\left(\frac{1}{\mu}\mathrm{rot}\ E\right) = \omega^2 \varepsilon E \quad \Rightarrow \quad \mathrm{div}(\varepsilon E) = 0$$

only weakly enforced

Find E *in* $\mathcal{E}$ (whose definition includes $n \times E = 0$ on S^e) *such that*

$$\int \mathrm{rot}\ E \cdot \mathrm{rot}\ E' = \omega^2 \int \varepsilon\ E \cdot E' \quad \forall\ E' \in \mathcal{E}$$

Set $E' = \mathrm{grad}\ \varphi'$ $\Rightarrow$ $\int \varepsilon E \cdot \mathrm{grad}\ \varphi' = 0 \quad \forall\ \varphi' \in \Phi$,

the weak form of $\mathrm{div}(\varepsilon E) = 0$, so require $\mathrm{grad}\ \Phi \subset \mathcal{E}$,

with Φ large enough. Not the case if $\mathcal{E}$ spanned by nodal vectorial elements. Whereas if $\mathcal{E} = W^1$, yes:

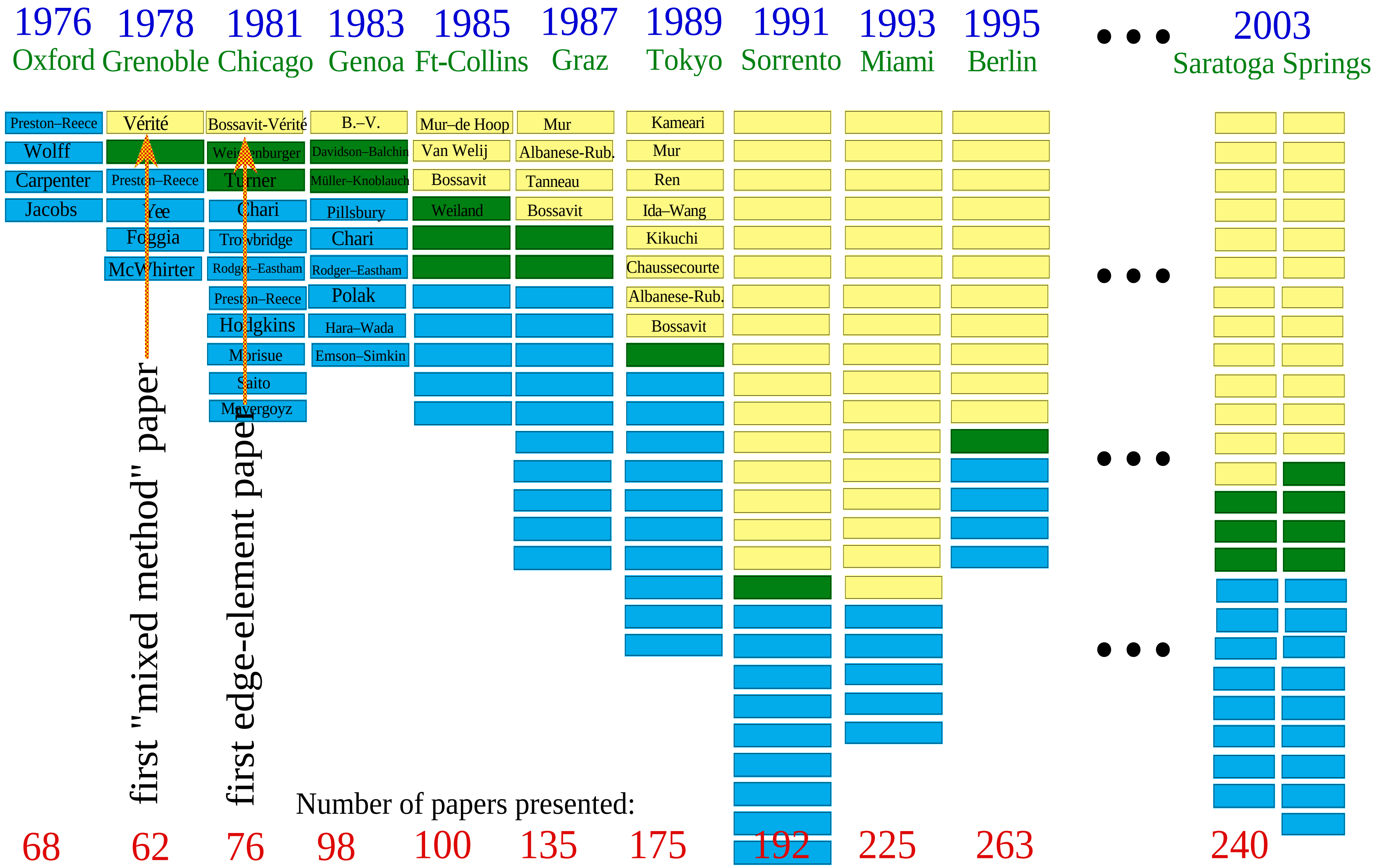
$$\begin{tikzcd}
\Phi \arrow[dr] & & & & \\
& W^0 \arrow[r, "\mathrm{grad}"] & W^1 \arrow[r, "\mathrm{rot}"] & W^2 \arrow[r, "\mathrm{div}"] & W^3
\end{tikzcd}$$

The chart next slide shows how the edge element idea was received by the “low frequency” segment of the CEM community, as evidenced by the written record of the well-established “Compumag” conference. (Since 1981, proceedings are published in IEEE Transactions on Magnetism.) Papers mentioned are all those concerned with numerical methods for the computation of eddy currents in three dimensions. In yellow, the edge element ones. In green, the “network methods” ones.

After a period of ten years, edge elements were widely known, if not adopted by everyone (some commercial code makers were a bit lukewarm) and the record shows a shift to other interests. Past 1993, the methodological issues were essentially solved, and the corpus loses homogeneity. The sample of 2003, given for reference, contains mainly application papers. (Selected here, those describing studies which “could have used edge elements and did or didn’t”.)

This same year, about ten papers explicitly used differential geometric concepts, an indication that awareness of their importance may grow in the near future.

"Edge element" papers in the Compumag record



1976 Oxford: Preston–Reece, Wolff, Carpenter, Jacobs

1978 Grenoble: Vérité, Preston–Reece, Yee, Foggia, McWhirter

1981 Chicago: Bossavit-Vérité, Weissenburger, Turner, Chari, Trowbridge, Rodger–Eastham, Preston–Reece, Hodgkins, Morisue, Saito, Mayergoyz

1983 Genoa: B.–V., Davidson–Balchin, Müller–Knoblauch, Pillsbury, Chari, Rodger–Eastham, Polak, Hara–Wada, Emson–Simkin

1985 Ft-Collins: Mur–de Hoop, Van Welij, Bossavit, Weiland

1987 Graz: Mur, Albanese-Rub., Tanneau, Bossavit

1989 Tokyo: Kameari, Mur, Ren, Ida–Wang, Kikuchi, Chaussecourte, Albanese-Rub., Bossavit

1991 Sorrento

1993 Miami

1995 Berlin

• • •

2003 Saratoga Springs

first "mixed method" paper

first edge-element paper

Number of papers presented:

1976	1978	1981	1983	1985	1987	1989	1991	1993	1995	2003
68	62	76	98	100	135	175	192	225	263	240