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*On the generation
of Whitney forms*

*Just a chain map, from
singular to simplicial chains*

Posted as a complement to IMA May 13 2004 talk on Whitney forms

IMA talk suggested “Poincaré integration” as a way to generate the Whitney complex. Here an alternative view — which I now tend to prefer — is presented, where Whitney forms are conceived as a way to approximate manifolds (and more generally, singular chains) by simplicial chains. One requires this map χ to be a *chain map* ($\partial\chi = \chi\partial$). By duality, this gives a map from mesh cochains to differential forms with the desired commutative diagram property. Next five slides recall motivations for the latter in the context of the “generalized finite differences” discretization technique.

Captions such as this one are comments, summarizing oral part of original presentation at Caltech, October 4, 2003.

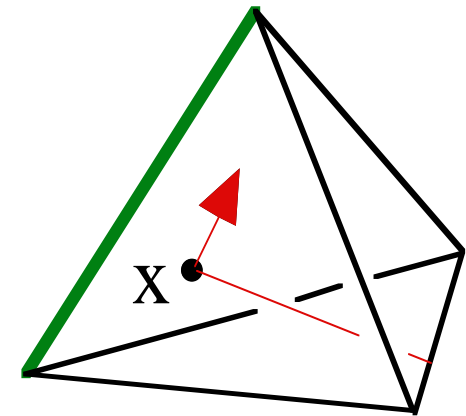
$$\sigma \partial_t \mathbf{A} + \text{rot}(\nu \text{rot } \mathbf{A}) = \mathbf{J}^S \quad (\mathbf{B} = \text{rot } \mathbf{A}, \mathbf{E} = -\partial_t \mathbf{A})$$

The eddy-current problem, "electric" formulation

$$\partial_t \int \sigma \mathbf{A} \cdot \mathbf{A}' + \int \nu \text{rot } \mathbf{A} \cdot \text{rot } \mathbf{A}' = \int \mathbf{J}^S \cdot \mathbf{A}'$$

$$\forall \mathbf{A}' \in \mathbf{A}$$

- \mathbf{A} : Finite-dimensional space of **curl-conformal** elements
- **Edge** circulations, physically meaningful, should be degrees of freedom



$$\mathbf{W}(\mathbf{x}) = \mathbf{a} \times \mathbf{x} + \mathbf{b}$$

Hence discrete form,

In the old days:

$$\sigma \partial_t \mathbf{a} + \mathbf{M} \mathbf{a} = \mathbf{j}^S$$

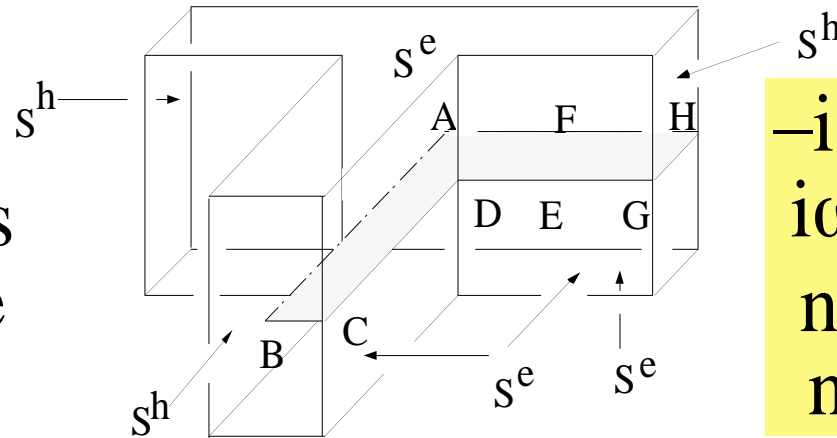
Now:

$$\sigma \partial_t \mathbf{a} + \mathbf{R}^t \nu \mathbf{R} \mathbf{a} = \mathbf{j}^S$$

Required: an interpolant from edge-circulations ...

The infamous "spurious modes", ca. 1989

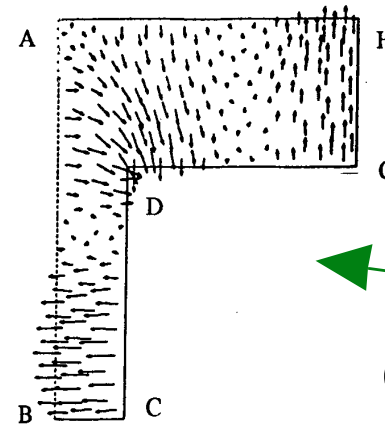
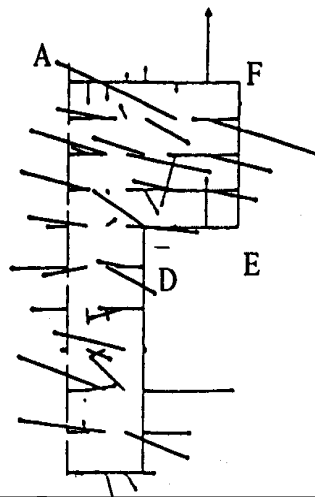
Compute resonant modes in a waveguide T-junction



$$\begin{aligned}
 -i\omega\epsilon\mathbf{E} + \text{rot } \mathbf{H} &= 0 \\
 i\omega\mu\mathbf{H} + \text{rot } \mathbf{E} &= 0 \\
 \mathbf{n} \times \mathbf{H} &= 0 \text{ on } S^h \\
 \mathbf{n} \times \mathbf{E} &= 0 \text{ on } S^e
 \end{aligned}$$

View of field \mathbf{E} in shaded plane section:

Using standard node-based vector-valued elements



Using edge elements

... with something more: "Spurious modes" issue motivates looking for whole complex of interpolants, with "exact sequence" property:

$$\text{rot}\left(\frac{1}{\mu}\text{rot } \mathbf{E}\right) = \omega^2 \varepsilon \mathbf{E} \quad \Rightarrow \quad \text{div}(\varepsilon \mathbf{E}) = 0$$

only **weakly** enforced

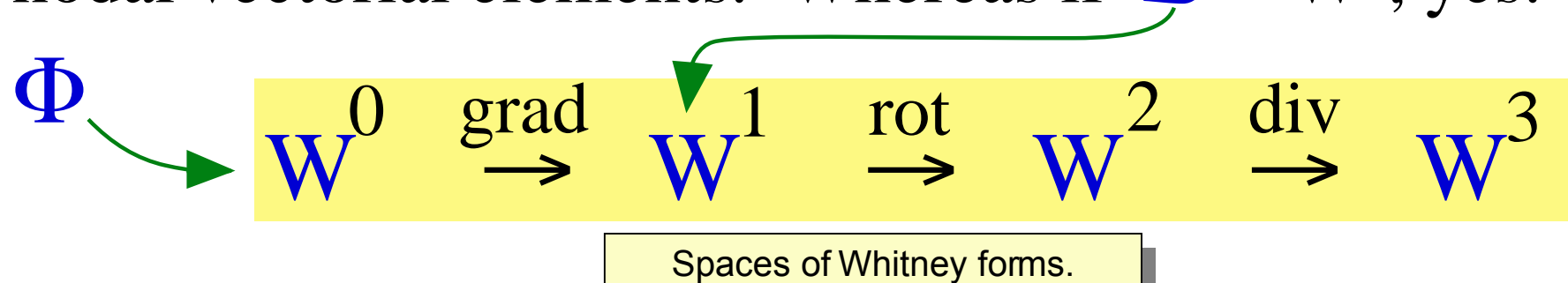
Find \mathbf{E} in \mathcal{E} (whose definition includes $\mathbf{n} \times \mathbf{E} = 0$ on S^e) *such that*

$$\int \text{rot } \mathbf{E} \cdot \text{rot } \mathbf{E}' = \omega^2 \int \varepsilon \mathbf{E} \cdot \mathbf{E}' \quad \forall \mathbf{E}' \in \mathcal{E}$$

Set $\mathbf{E}' = \text{grad } \varphi' \Rightarrow \int \varepsilon \mathbf{E} \cdot \text{grad } \varphi' = 0 \quad \forall \varphi' \in \Phi,$

the weak form of $\text{div}(\varepsilon \mathbf{E}) = 0$, so require $\text{grad } \Phi \subset \mathcal{E}$,

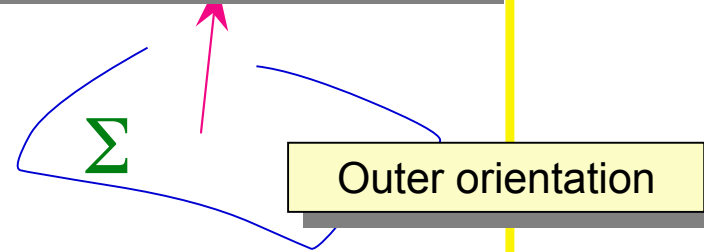
with Φ **large enough**. **Not** the case if \mathcal{E} spanned by nodal vectorial elements. Whereas if $\mathcal{E} = \mathbf{W}^1$, yes:



Maxwell, in terms of DF's:

Differential forms, seen as additive continuous maps from manifolds to reals

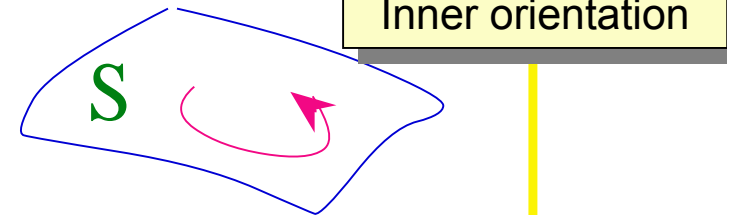
$$-\partial_t \int_{\Sigma} \mathbf{d} + \int_{\partial\Sigma} \mathbf{h} = \int_{\Sigma} \mathbf{j} \quad \forall$$



$$\mathbf{b} = \mu \mathbf{h}$$

$$\mathbf{d} = \epsilon \mathbf{e}$$

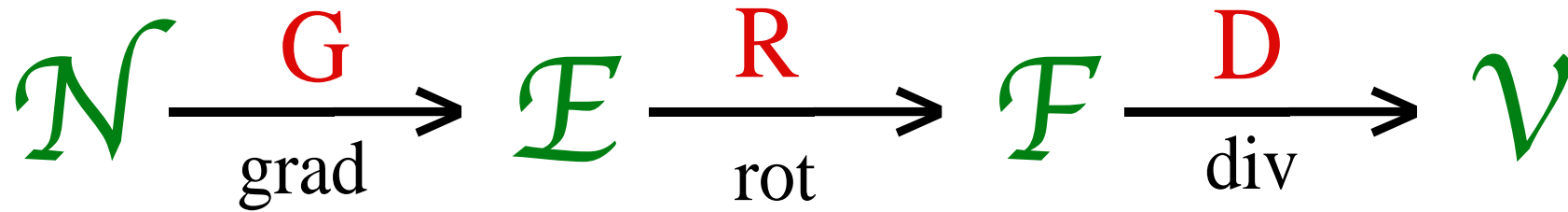
$$\partial_t \int_S \mathbf{b} + \int_{\partial S} \mathbf{e} = 0 \quad \forall$$



Note correspondences with standard vectorial formulation:

$$(-\partial_t \mathbf{D} + \text{rot } \mathbf{H} = \mathbf{J}, \quad \partial_t \mathbf{B} + \text{rot } \mathbf{E} = \mathbf{0})$$

A recall of the discretization technique. Start from eqns in integral form,

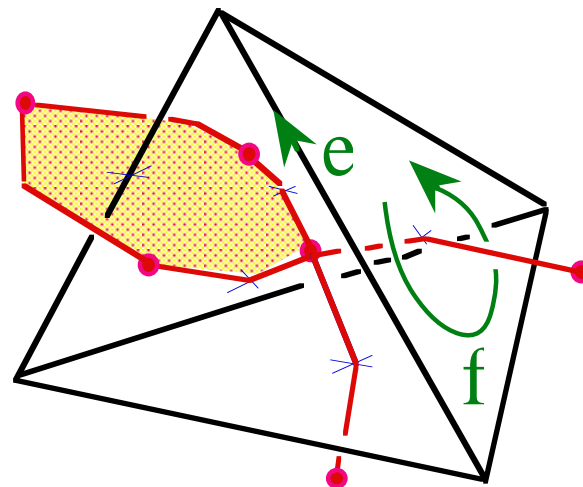


Approximate representation of the field by degrees of freedom assigned to both kinds of cells

b at faces

e, a

at edges



here, $R_{fe} = -1$

h at *dual* edges
(i.e., faces)

d, j

at *dual* faces

fluxes

$$\mathbf{b} = \{b_f : f \in \mathcal{F}\}$$

emf's

$$\mathbf{e} = \{e_e : e \in \mathcal{E}\}$$

$$\xrightarrow{\mathcal{V}}$$

$$\xrightarrow{\mathcal{E}}$$

$$\mathbf{h} = \{h_f : f \in \mathcal{F}\}$$

$$\mathbf{d} = \{d_e : e \in \mathcal{E}\}$$

mmf's

intensities

... assign DoF's to geometrical elements of both meshes,

... enforce previous laws for surfaces made of cells (primal or dual), define discrete Hodge matrices. Hence, “discretization toolkit”.

Next, be it for convergence studies or for exploitation of results, interpolants needed, to pass from calculated cochains to approximations of fields.

Whitney forms

Once obtained the cochains **b**, **e**, **h**, **d**,
what about the fields themselves?

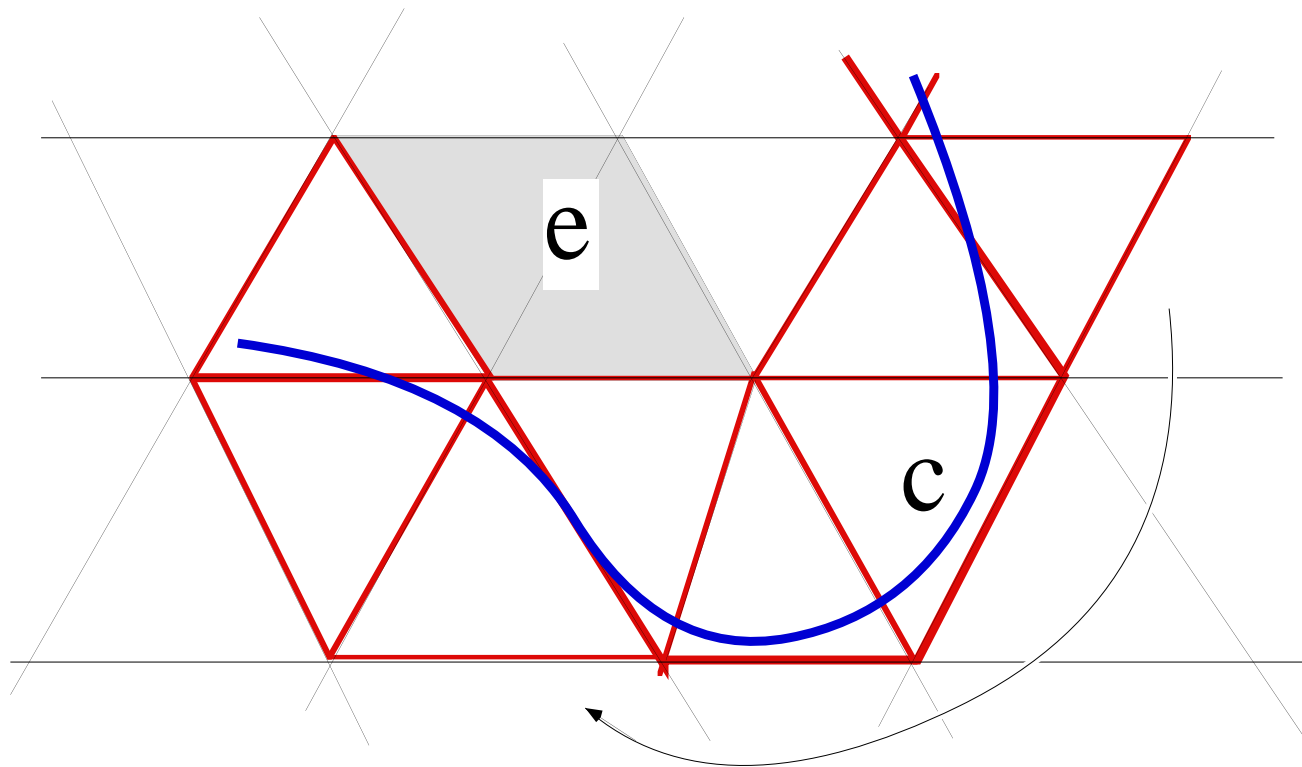
or else:

Are there objects that would be to
differential forms what finite elements are
to fonctions, i.e., to 0-forms?

$$\mathbf{e} \rightarrow \sum_{e \in \mathcal{E}} \mathbf{e}_e w^e$$

$$\mathbf{b} \rightarrow \sum_{f \in \mathcal{F}} \mathbf{b}_f w^f$$

Just as scalar interpolation amounts to representing a point as weighted sum of nodes...



Basic idea: Attribute "weight w.r.t. edge e" to line c

... from-edge-values-interpolation consists in approximating a line by a chain of primal edges.

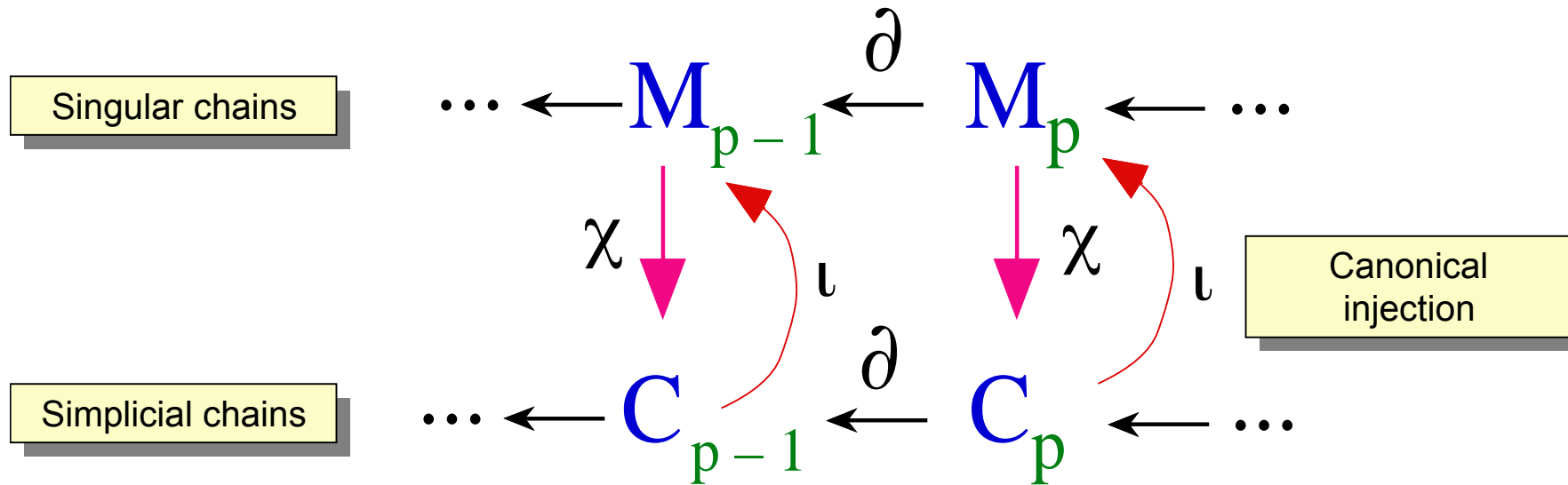
$$\text{If } c \sim \sum_{e \in \mathcal{F}} w^e(c) e, \text{ then } \int_c a \sim \sum_e w^e(c) \int_e a,$$

↑ chain denoted $\chi(c)$ below

$$\text{hence } \int_c a \sim \sum_e w^e(c) a_e, \text{ so } a \sim \sum_e a_e w^e$$

To-be-found chain coefficients denoted $w^e(c)$

A chain map:



Thm.: \exists unique χ such that

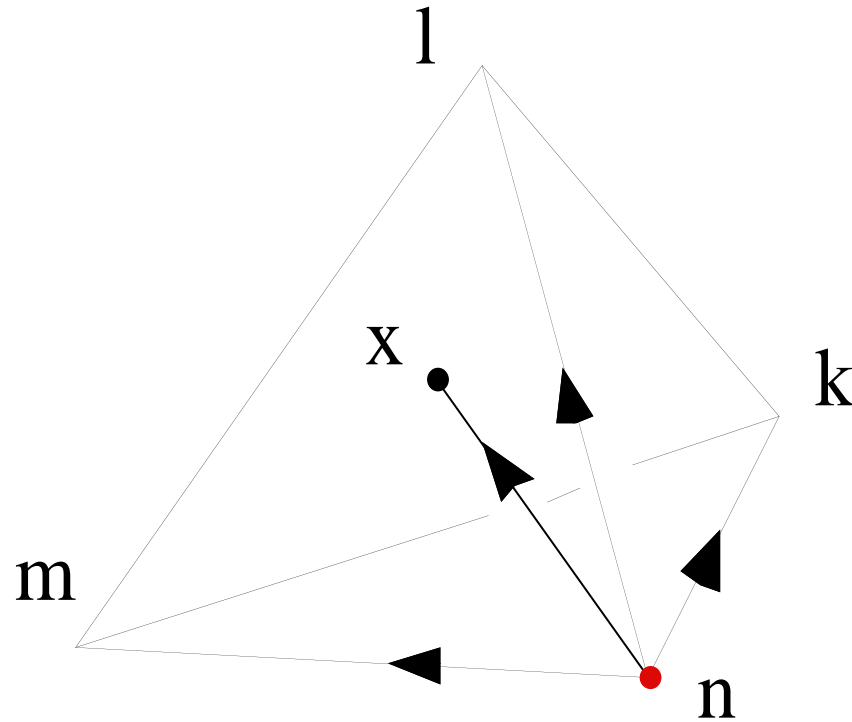
$$\chi\partial = \partial\chi, \quad \chi\iota = 1, \quad \chi(\mathbf{x}) = \sum_{\mathbf{n} \in \mathcal{N}} \lambda^{\mathbf{n}}(\mathbf{x}) \mathbf{n}$$

$$\varphi(\mathbf{x}) \approx \sum_{\mathbf{n}} \lambda^{\mathbf{n}}(\mathbf{x}) \varphi_{\mathbf{n}} \equiv \langle \sum_{\mathbf{n}} \lambda^{\mathbf{n}}(\mathbf{x}) \mathbf{n} ; \varphi \rangle = \langle \chi(\mathbf{x}) ; \varphi \rangle$$

How it works in standard scalar-interpolation case, hence leftmost χ . Now proceed recursively,

"Obviously", $\chi(nx) = \sum_i \lambda^i(x) ni$, $i \in \{k, l, m, n\}$

Only one reasonable way to express nx as a weighted sum of edges



More formally:

$$\chi \partial(nx) = \chi(x) - \chi(n) = \sum_i \lambda^i(x)(i - n) = \sum_i \lambda^i(x) \partial(ni)$$

$$\Rightarrow \chi(nx) = \sum_i \lambda^i(x) ni + \sum_{f \in \mathcal{F}} \alpha_f(x) \partial f \quad i \in \{k, l, m, n\}$$

with α_f linear, but $[\chi(nm) = nm] \Rightarrow \alpha_f = 0$.

To get $\chi(xy)$, write $\chi(xy) = \chi(ny) - \chi(nx) = \sum_i \lambda^i(y) n_i - \sum_i \lambda^i(x) n_i = \dots$, etc.. The edge element pops up:

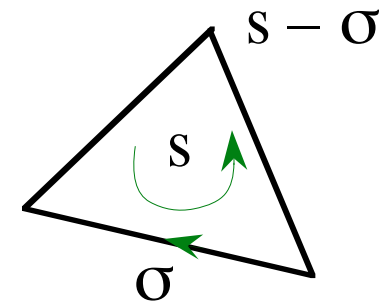
“If one knows how to represent a point (as a sum of nodes) one knows how to represent a segment (as a sum of edges)”, and the same, recursively, up in dimension. Details at

<http://www.icm.edu.pl/edukacja/mat.php>

in “Discretization of Electromagnetic Problems”, Chap. 4

All gory details omitted,

for a p -simplex s ,

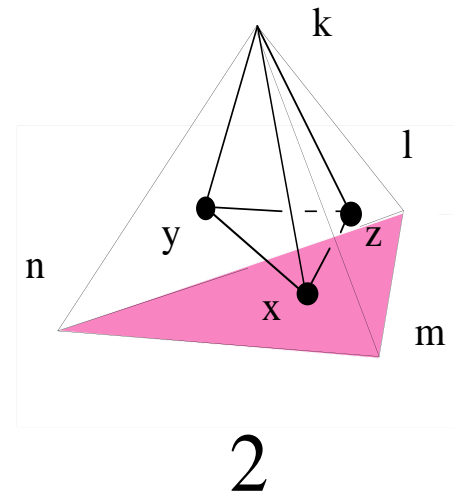
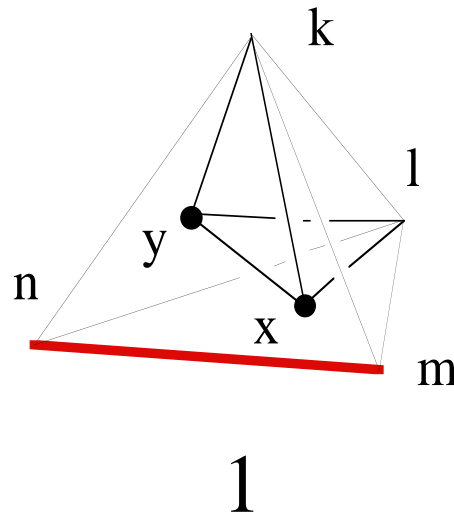
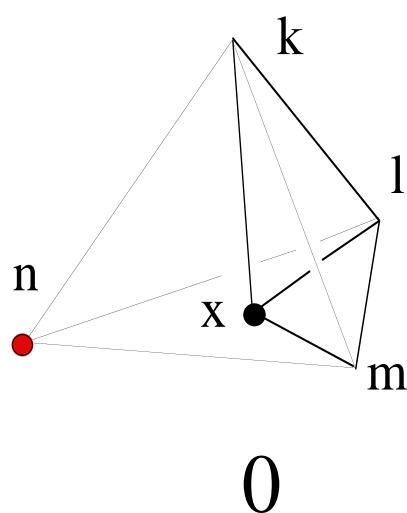


↓ Incidence matrix

$$w^s(\mathbf{x}) = \sum_{\sigma \in \{(p-1)\text{-simplices}\}} \partial_{\sigma}^s \lambda^{s-\sigma}(\mathbf{x}) dw^{\sigma}$$

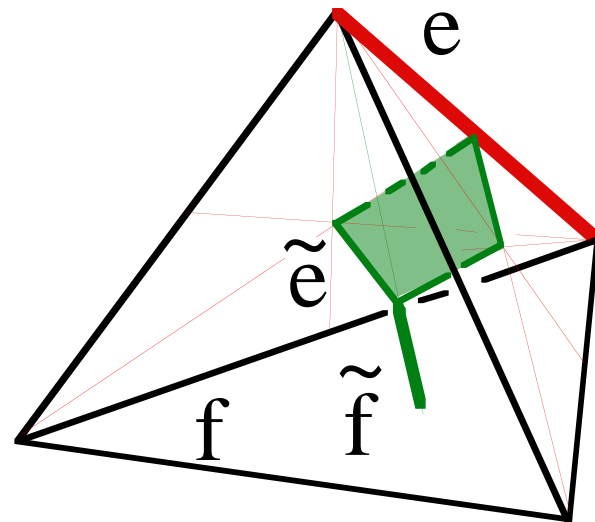
Recursive generating formula for Whitney forms

Weight of M w.r.t. simplex s is integral of Whitney form of s over M .



$$\int_{\tilde{f}} w^e = \int w^e \wedge w^f = \int_{\tilde{e}} w^f$$

A remarkable property of Whitney forms,
when dual mesh is the barycentric one

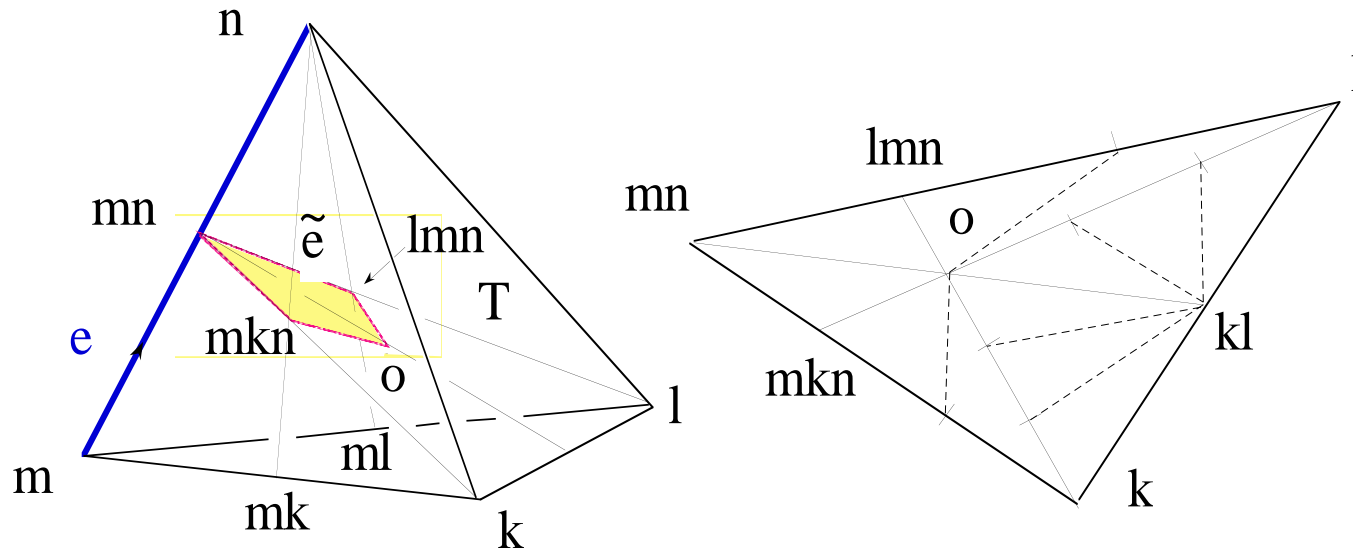


So if \mathbf{b} is constant, $\int w^e \wedge \mathbf{b} = \int_{\tilde{e}} \mathbf{b}$, i.e., " $\int w^e = \tilde{e}$ "

Integral of 1-form as (twisted) 2-vector

Pedestrian proof, on one example:

$$\int w^e = \tilde{e}, \quad \int w^f = \tilde{f}$$



$$\int_T \nabla w^n = \{k, l, m\}/3$$

$$\int_T w^m \nabla w^n - w^m \nabla w^n =$$

$$(\{k, l, m\}/3 + \{k, l, n\}/3)/4 = \tilde{e}$$

This observation, plus the one next slide, will make important point as regards Galerkin Hodge

Whitney forms as a *partition of unity*

○ $\sum_n w^n(x) = 1 \quad \forall x$

“Whitney map sends p-vector v to p-chain whose p-vector ‘mass’ is v itself”.
Results from its very construction.

0

○ $\sum_e w^e(x) \otimes e = 1 \quad \forall x$

1

i.e., $\sum_e (v \cdot w^e(x)) e = v \quad \forall x, v$

○ $\sum_f w^f(x) \otimes f = 1 \quad \forall x$

2

⊗ Ad-hoc notation

etc.

Consequence: The *mass matrix* M
of edge elements ...

Hybrid proof, using
vector proxies, and
assuming $\varepsilon = 1$.

$$\sum_{e'} \mathbf{w}^e(\mathbf{x}) \cdot \mathbf{w}^{e'}(\mathbf{x}) \mathbf{e}' = \mathbf{w}^e(\mathbf{x})$$

$$\sum_{e'} \left[\int_D \mathbf{w}^e(\mathbf{x}) \cdot \mathbf{w}^{e'}(\mathbf{x}) \, d\mathbf{x} \right] \mathbf{e}' = \int_D \mathbf{w}^e(\mathbf{x}) \, d\mathbf{x}$$

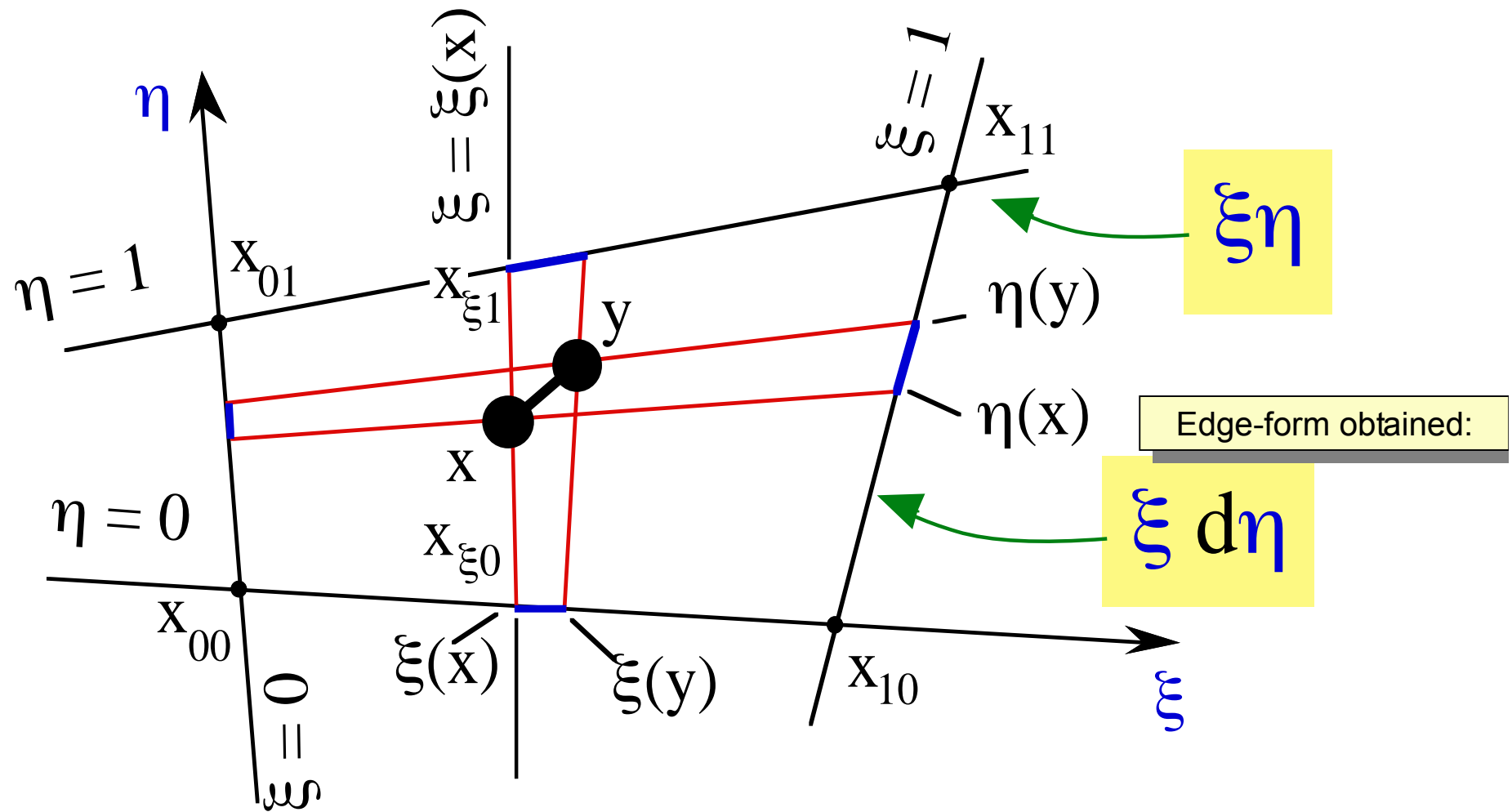
$$\sum_{e'} M^{ee'} \mathbf{e}' = \int \mathbf{w}^e(\mathbf{x}) = \tilde{\mathbf{e}}$$

Relation between mass-matrix coefficients, edges, and dual faces, that appeared as a consistency criterion in previous slide show.

... satisfies the *consistency* requirement

The basic idea, “express p -vector as a weighted sum of p -simplices”, and make sure this is a chain map, is quite productive. We next apply it to the construction of Whitney forms for other element-shapes than tetrahedra.

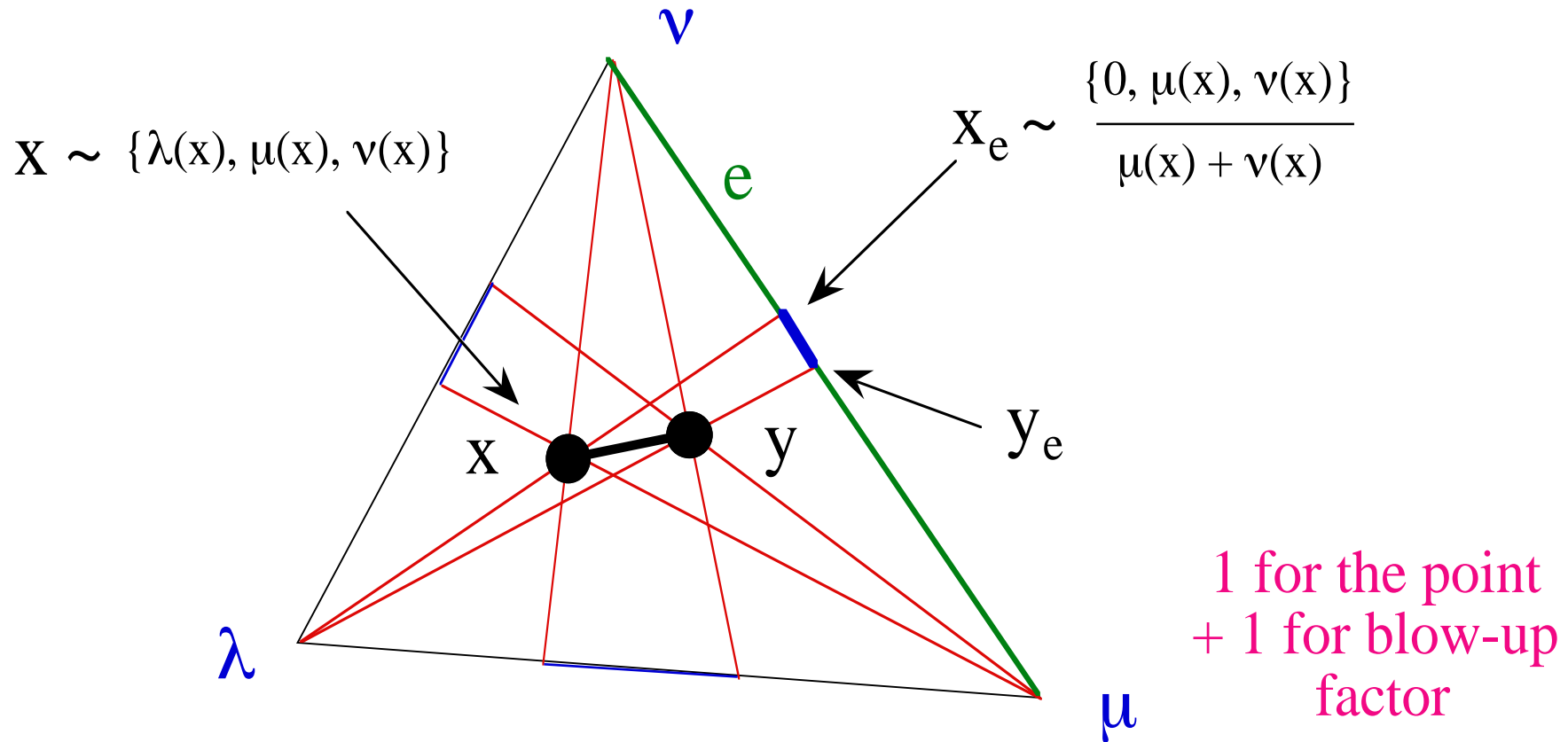
Given a quadrilateral, how to naturally represent a vector by a chain of edges? Start from representation of points (“isoparametric” stuff) and work upwards.



Weight of x w.r.t. x_{11} : $\xi(x) \eta(x)$

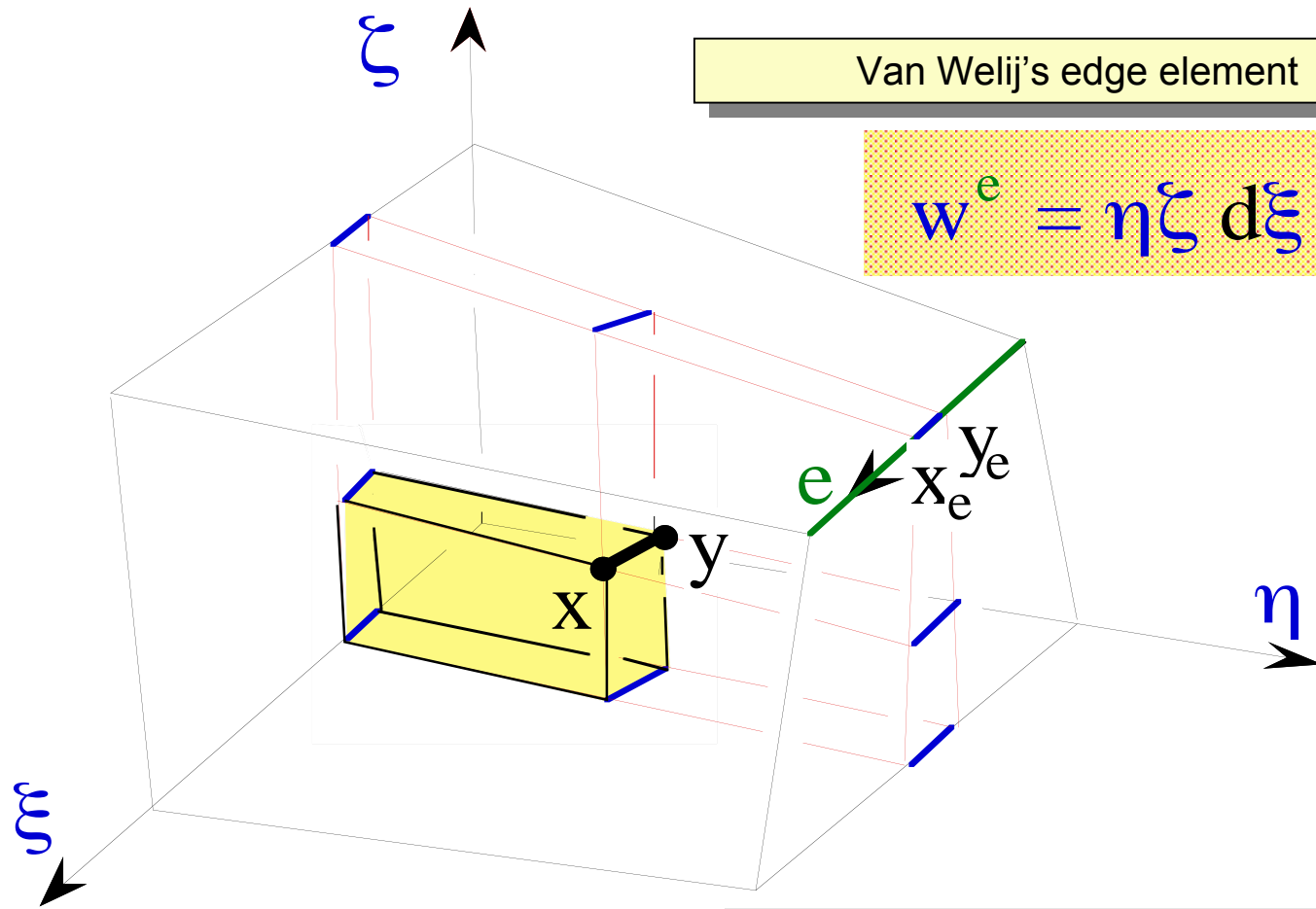
Weight of xy w.r.t. $x_{10} x_{11}$: $\frac{1}{2} [\xi(x) + \xi(y)](\eta(y) - \eta(x))$

Not so different from what was done on simplices:



Weight of xy w.r.t. e is $\frac{X_e Y_e}{X_\nu X_\mu} \times [(\mu + \nu) \left(\frac{x+y}{2}\right)]^2$

which is $(\nu d\mu - \mu d\nu)(xy)$ indeed



Van Welij's edge element

$$w^e = \eta \zeta d\xi$$

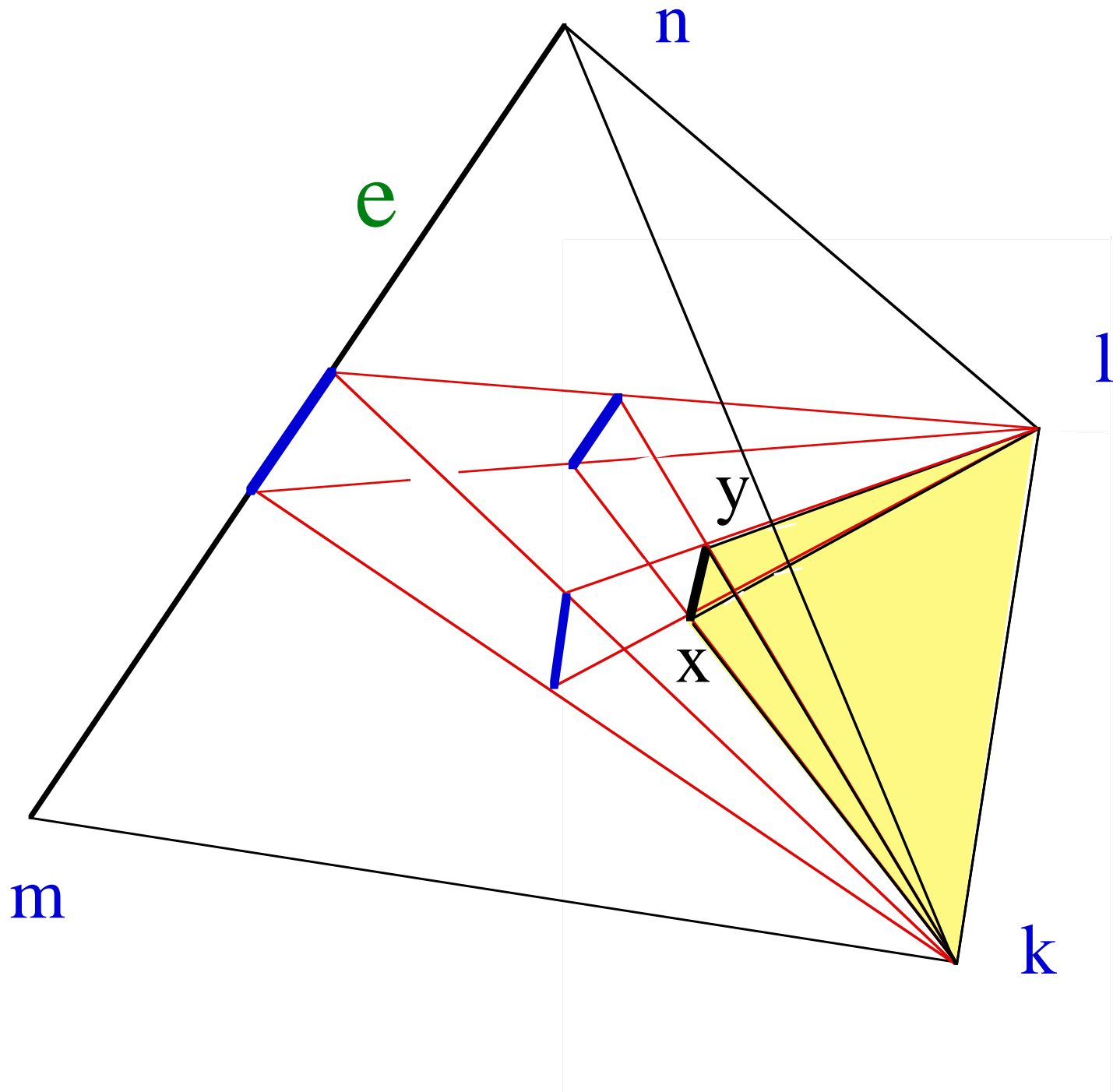
3D realization of same idea. "Share" total weight 1 for segment xy between iso-zeta surfaces $\zeta = 0$ and $\zeta = 1$. Then, recursively, share between iso-eta lines.

Weight of xy is product of these three

$$d\xi(xy) = \frac{x_e y_e}{e} \cdot \text{Weights } \zeta(x) \text{ and } \eta(x) \text{ gathered by projecting up then right.}$$

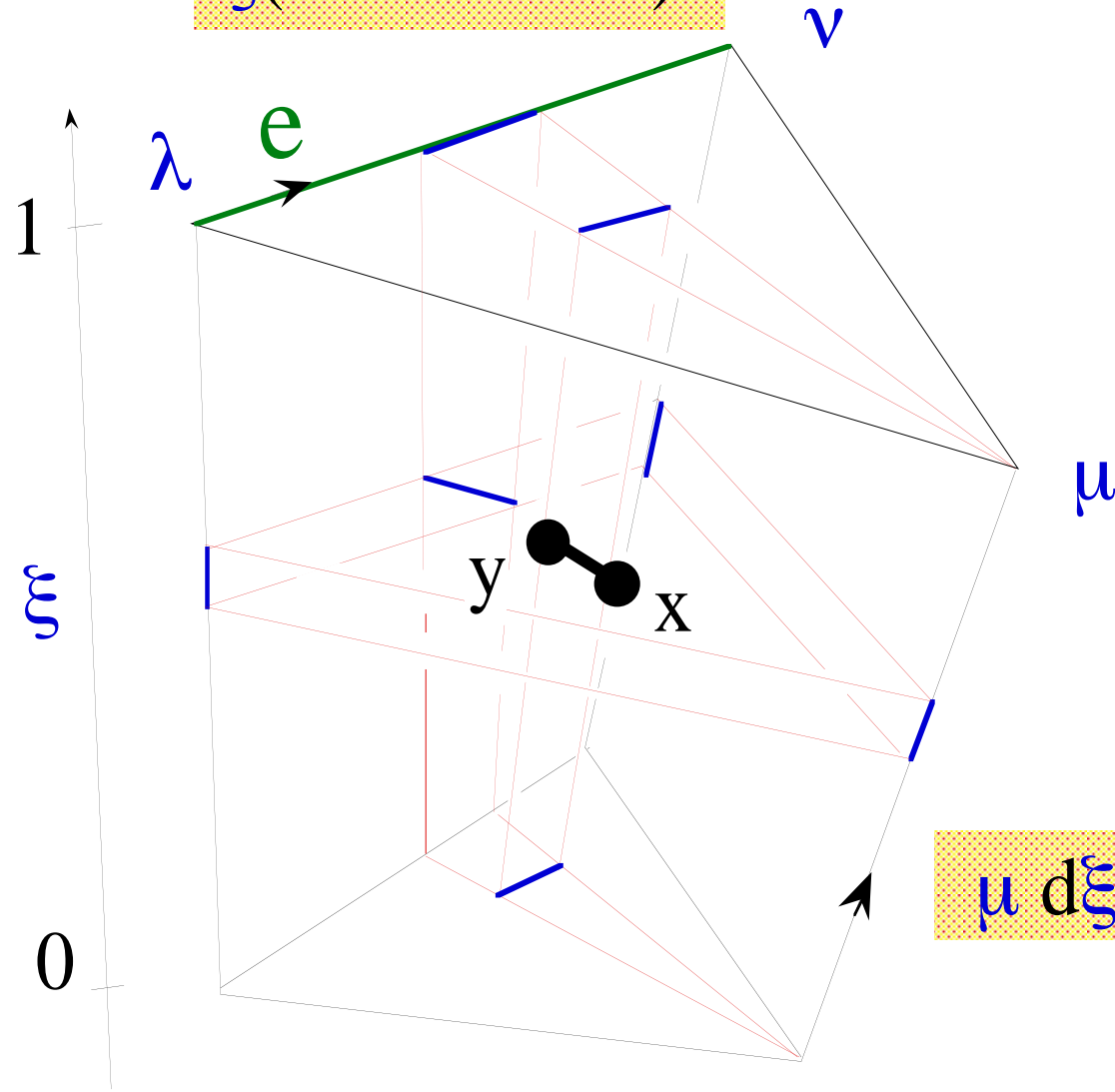
Note how weight assigned to edge e in representation of xy is relative volume (in $\xi\zeta\eta$ coordinates) of "hinder region" of xy w.r.t. e

... which is exactly as it went in simplicial case:



Obtained Whitney form:

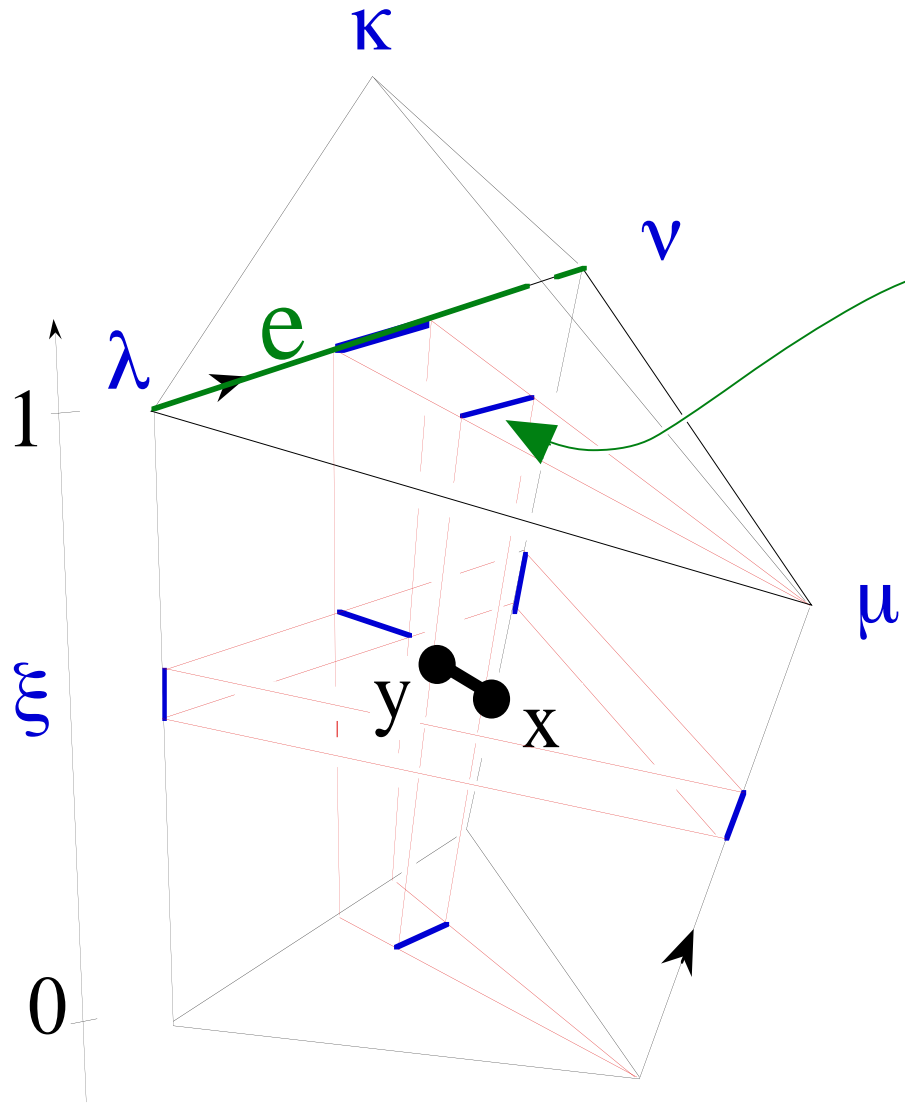
$$\xi(\lambda dv - v d\lambda)$$



Simplicial projection system in p dimensions, "isoparametric" one in $n - p$ other dimensions

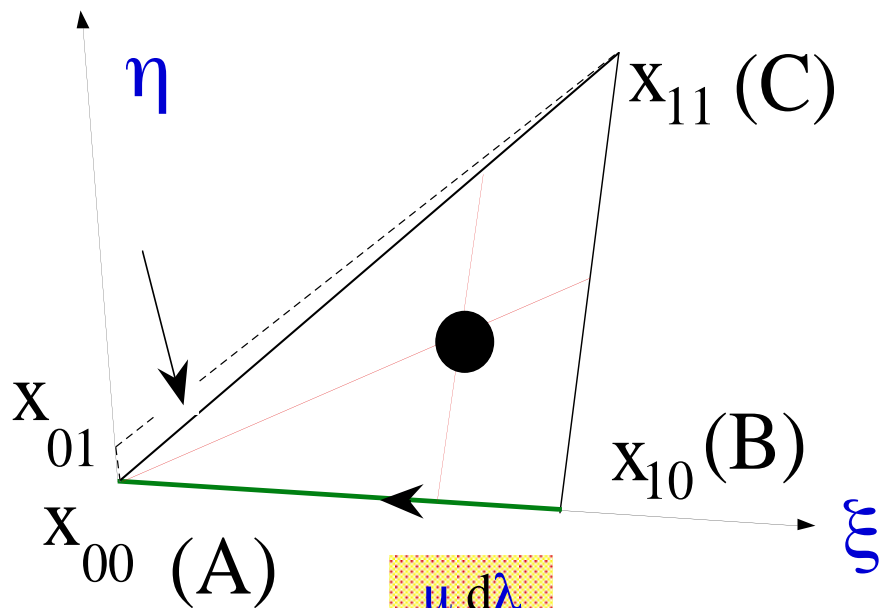
Now mix both systems (simplicial, and Cartesian) of weight attribution. In 3D, only one new possibility: prisms

Prism edge element and tetrahedral one get along well:



Same weight / e as evaluated from both sides, hence "conformity" of w^e

Weight attribution rule guarantees "conformity" at element interfaces



“Degeneracy”
as a way to
construct new
forms:
Example

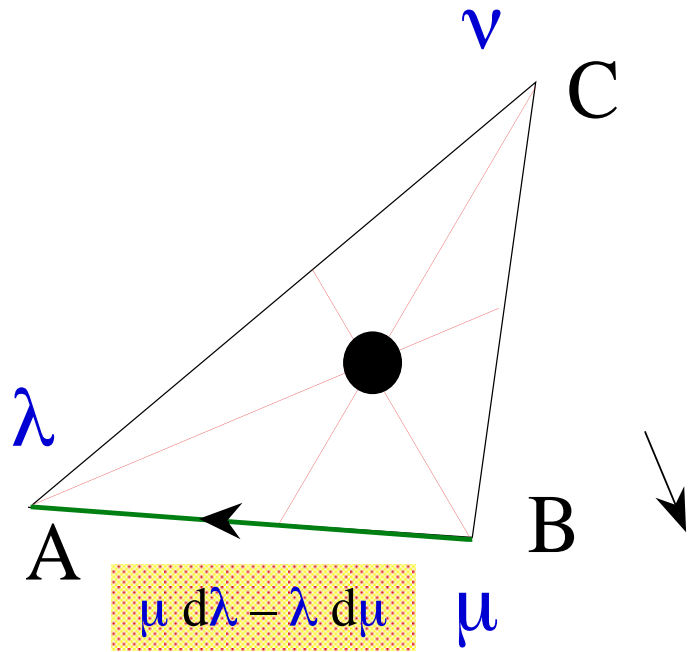
$$\eta = \frac{\nu}{\nu + \mu}$$

$$\xi = \nu + \mu$$

$$\frac{\mu d\lambda}{1 - \lambda}$$

$$(\eta - 1)d\xi$$

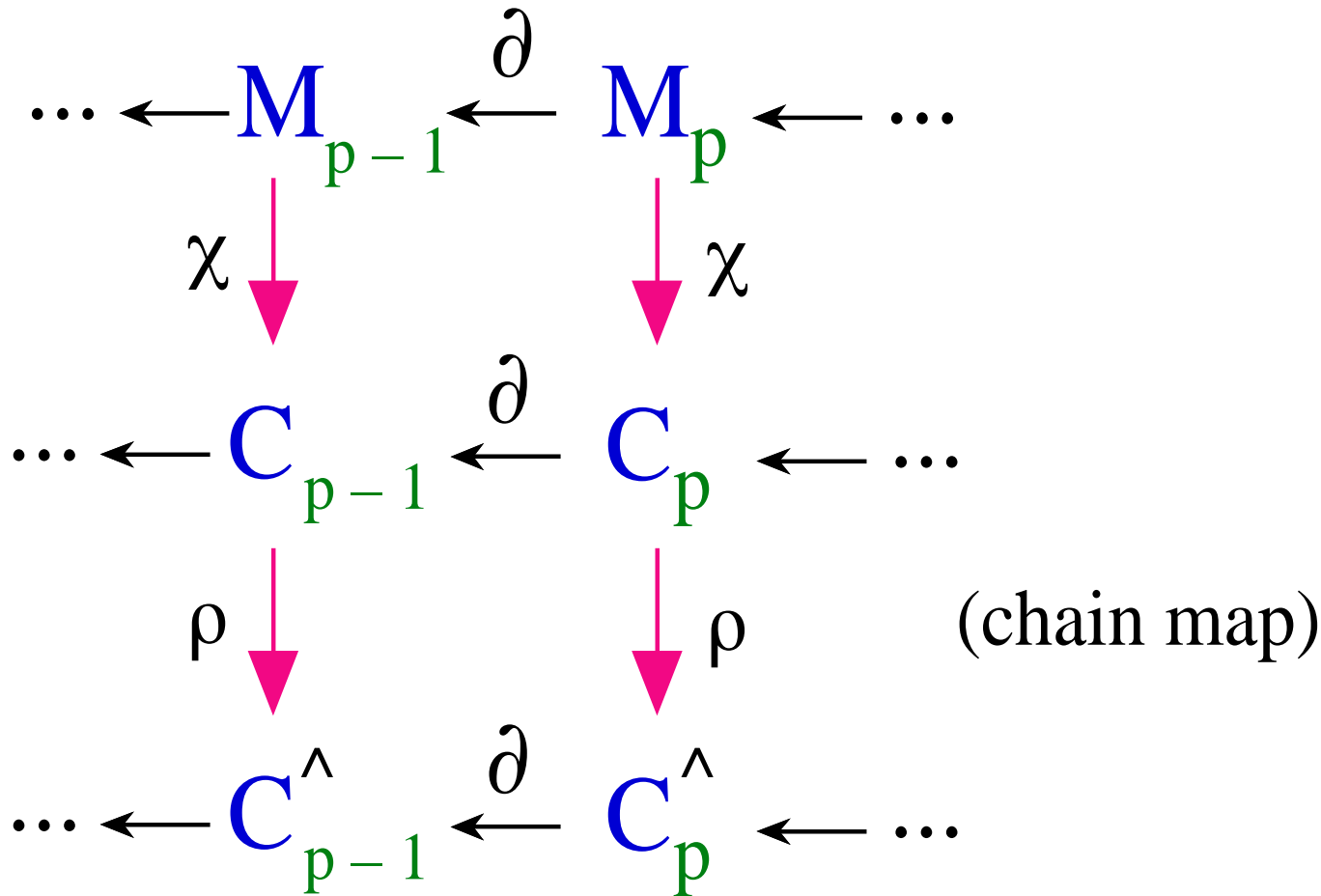
Not the same as



Using projection system of quadrilateral, let two nodes coalesce, take limit weights. Change old coordinates (ξ, η) to new ones (λ, μ, ν) to avoid singularities.

$$\mu d\lambda - \lambda d\mu$$

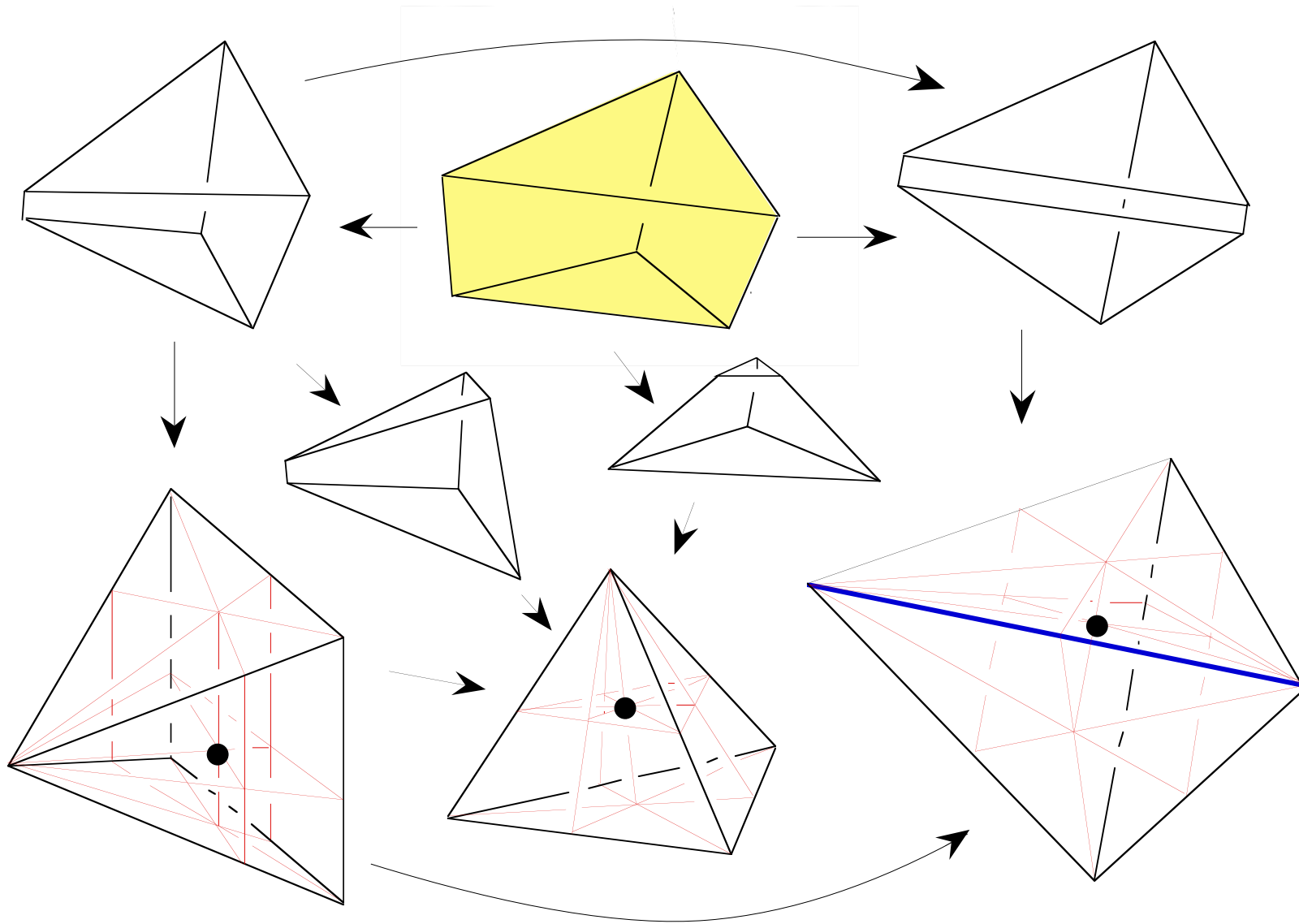
"Degeneracies": Theory



"Degeneracy" involves identification of nodes, edges, ... Hence chain-map ρ .

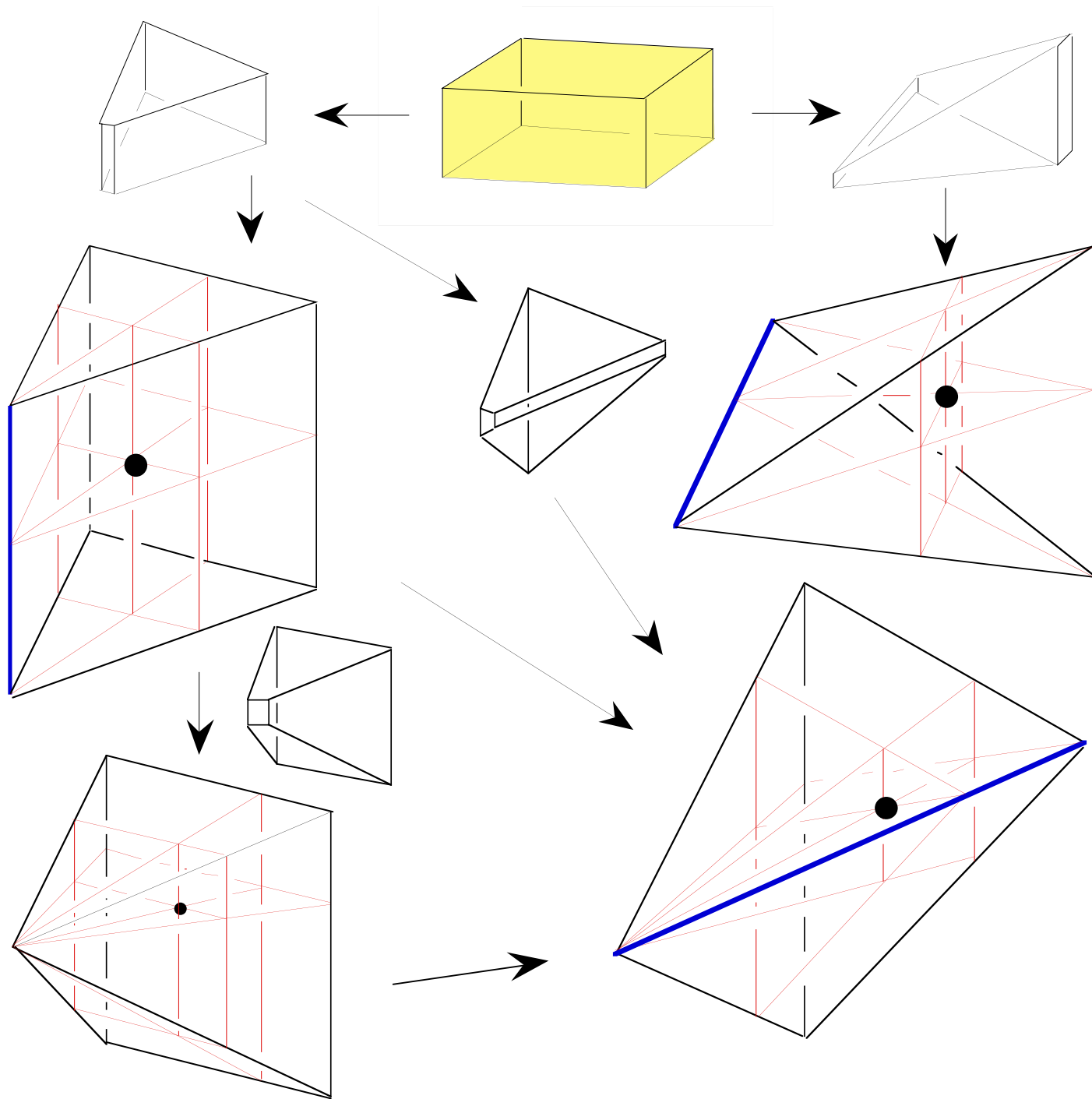
$$\text{Whitney}(\rho(s)) = \sum \text{Whitney}(\rho^{-1}(s))$$

Add up forms of preimage to get Whitney form on degenerate mesh

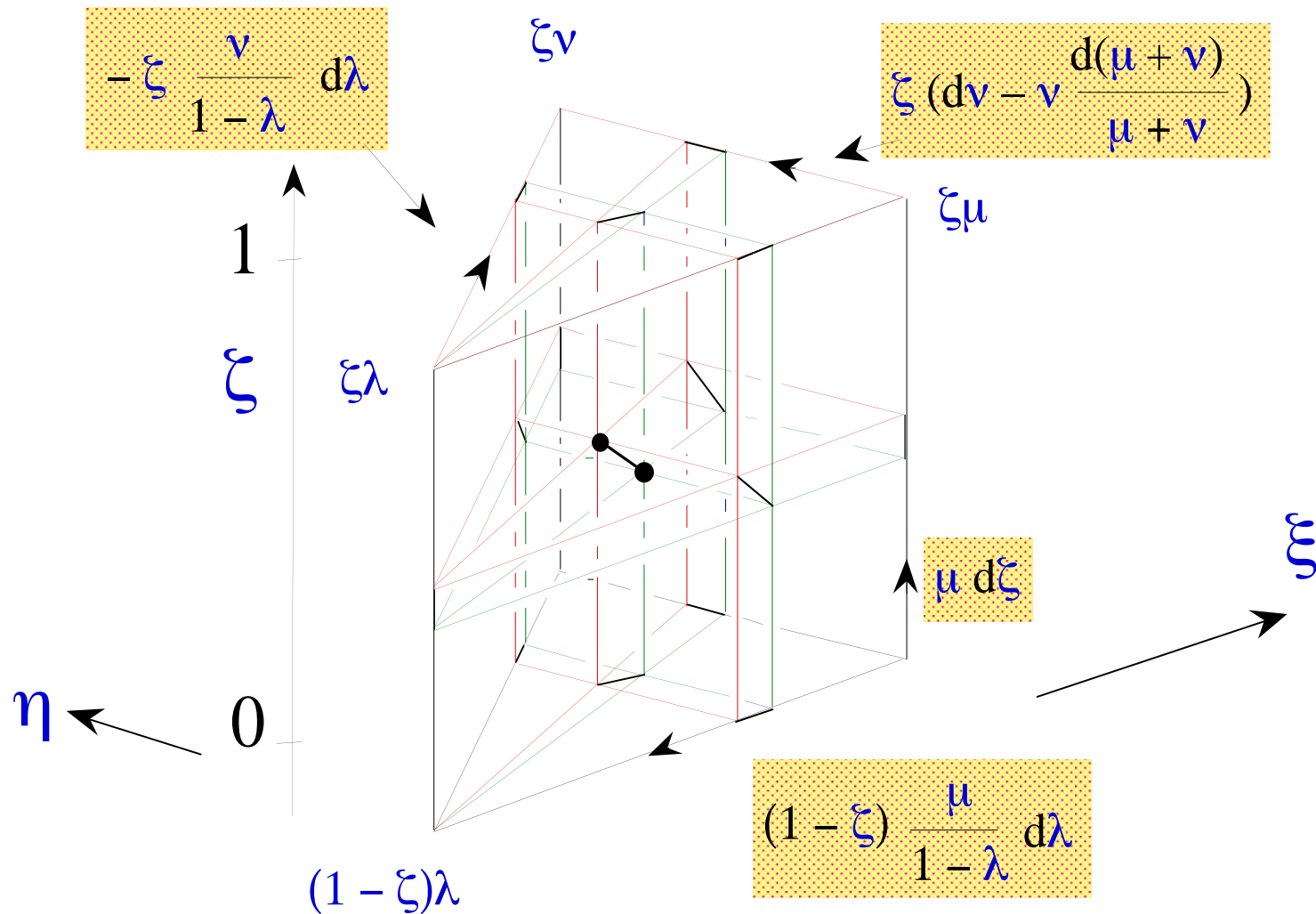


All ways a prism can degenerate (if faces have to stay planar), hence new projection systems, and new Whitney forms, on pyramid and tetrahedron

Problem: Interface conformity no more automatic



All ways a brick can degenerate (if faces have to stay planar), hence new projection systems, and new Whitney forms, on prism, pyramid and tetrahedron



Old coordinates ξ, η, ζ . New ones, ζ, λ, μ, v .

Example: new Whitney forms on prism