Div-curl lemma for edge elements

Snorre H. Christiansen
CMA, c/o Dept. Math., University of Oslo

Abstract

A variant of Murat and Tartar's div-curl lemma is stated and proved for Nedelec's edge elements. In the Galerkin finite element setting one can expect control of L^2 norms of vectorfields and also sufficient control of their curls in H^{-1}. But the divergence is usually just controlled trough the integral of the vectorfields against a finite dimensional space of gradients. The proposed lemma is designed to handle this case. The proof uses a uniform norm equivalence related to discrete compactness properties of vector FE spaces and a super-approximation property of scalar FE spaces.

Compensated compactness

•Tool for homogenisation (oscillatory material coefficients) and some non-linear PDE (e.g. conservation laws).

•Weakly converging sequence of functions $\mathbb{R}^n \to \mathbb{R}^p$. Compose with $Q : \mathbb{R}^p \to \mathbb{R}$ quadratic form.

$$u_h \to u \text{ weakly in } \mathrm{L}^2(\mathbb{R}^n; \mathbb{R}^p) \tag{1}$$

$$\text{convergence of } Q \circ u_h \to Q \circ u \text{ ?} \tag{2}$$

•Main example: div-curl lemma:

$$u_h \to u, \;\; u'_h \to u' \text{ weakly in } \mathrm{L}^2(\mathbb{R}^3; \mathbb{R}^3) \tag{3}$$

$$\text{convergence of } u_h \cdot u'_h \to u \cdot u' \text{ ?} \tag{4}$$

It is sufficient that:

$$(\operatorname{div} u_h) \text{ is precompact in } \mathrm{H}^{-1}(\mathbb{R}^3; \mathbb{R}) \tag{5}$$

$$(\operatorname{curl} u'_h) \text{ is precompact in } \mathrm{H}^{-1}(\mathbb{R}^3; \mathbb{R}^3) \tag{6}$$

to ensure that for all $\phi \in \mathrm{C}^\infty_c$:

$$\int (u_h \cdot u'_h)\phi \to \int (u \cdot u')\phi. \tag{7}$$

Finite elements

•U bounded convex and smooth domain in $\mathbb{R}^3$.
Uniform curved tetrahedral meshes $\mathcal{T}_h$ with max diameter h.
Study the limit $h \to 0$.

•X_h lowest order Nédélec edge elements on $\mathcal{T}_h$.
A finite dimensional space of vector fields $u \in \mathrm{L}^2(U; \mathbb{R}^3)$ s.t. $\mathrm{curl}\, u \in \mathrm{L}^2(U; \mathbb{R}^3)$ (defined a priori as a distribution).

•Vectorfields $u_h \in X_h$ have continuous tangential traces across faces of mesh, but normal component may jump.

•Most popular finite element space for electromagnetics:
– Good approximation properties on non-smooth domains,
– No spurious eigenvalues for $\mathrm{curl}\,\mathrm{curl}$ operator.
(contrary to other nodal finite element spaces which have continuous vectorfields) (Costabel, Dauge, Boffi, Brezzi et al., Monk, Demkowicz et al.).

•Y_h space of continuous piecewise linear functions on $\mathcal{T}_h$.
Then the following sequence (exists and) is exact.

$$Y_h \xrightarrow{\mathrm{grad}} X_h \xrightarrow{\mathrm{curl}} \mathrm{L}^2, \tag{8}$$

Variational formulations and motivation

•Variational formulation of a problem from electromagnetics.
Find $u_h \in X_h$ s.t. $\forall u'_h \in X_h$:

$$\int u_h \cdot u'_h + \int \operatorname{curl} u_h \cdot \operatorname{curl} u'_h = \int f(u_h) u'_h. \quad (9)$$

•A priori estimates:
– Energy: typically $u'_h = u_h$ gives bound on $\|u_h\|_{\mathrm{L}^2}$ and $\|\operatorname{curl} u_h\|_{\mathrm{L}^2}$. In particular $(\operatorname{curl} u_h)$ is precompact in H^{-1}.
– Divergence: test with $u'_h = \operatorname{grad} p'_h$ where $p'_h \in Y_h \cap \mathrm{H}^1_0$, gives estimate on $\int u_h \cdot \operatorname{grad} p'_h$.

•Best case: $\int u_h \cdot \operatorname{grad} p'_h = 0$ for all $p'_h \in Y_h \cap \mathrm{H}^1_0$, i.e. u_h is discrete divergence free.

•Question:
If (u_h) is bounded in L^2 and discrete divergence free,
is $(\operatorname{div} u_h)$ precompact in H^{-1}?

Div-curl lemma for Edge elements

•It's an open problem (much stronger than discrete compactness property of Kikuchi).

•Prove: If $u_h, u'_h \in X_h$, weakly converging in L^2 and u_h is discrete div free and $(\mathrm{curl}\, u'_h)$ is precompact in H^{-1}, then for all $\phi \in \mathrm{C}_c^\infty(U)$:

$$\int (u_h \cdot u'_h)\phi \to \int (u \cdot u')\phi. \tag{10}$$

•Corollary: if $u_h = v_h + \mathrm{grad}\, p_h$ with v_h discrete divergence free and $p_h \in Y_h \cap \mathrm{H}_0^1$ precompact in H^1, then ok. (because each part also converges weakly).

•Difficulty: p_h contains less information than $\mathrm{div}\, u_h$.

Continuous and discrete Hodge decompositions

• L^2 vectorfields on U can be uniquely decomposed as:

$$u = v + \operatorname{grad} p \text{ with } \operatorname{div} v = 0 \text{ and } p \in \mathrm{H}^1_0(U). \qquad (11)$$

Dirichlet problem gives p. Neumann also useful.

• Discrete analogues: $u_h \in X_h$ can be uniquely decomposed as:

$$u_h = v_h + \operatorname{grad} p_h \text{ with } p \in Y_h \text{ and } v_h \perp \operatorname{grad} Y_h. \qquad (12)$$

(Neumann type)

• Weak convergence: if u_h converges weakly in L^2
then v_h converges weakly in L^2 and p_h converges weakly in H^1.

Estimates on Hodge decompositions

•Mixed regularity (div or curl control) can be easily read off Hodge decomposition.

•In $\mathrm{L}^2(U;\mathbb{R}^3)$ let V be the orthogonal of $\mathrm{grad}\,\mathrm{H}^1(U)$. Divergence free and 0 normal component on ∂U.

•Continuous estimate:
On V, $\|\mathrm{curl}(\cdot)\|_{-1}$ is a norm equivalent to the L^2 norm.
Hard part: $\mathrm{curl}\,\mathrm{L}^2 \to \mathrm{H}^{-1}$ has closed range (regularity of ∂U).

•Discrete analogue. $V_h \subset X_h$ the L^2 orthogonal of $\mathrm{grad}\, Y_h$. We have $V_h \not\subset V$. Nevertheless:

•There is $C > 0$ such that $\forall h$:

$$\forall v_h \in V_h \quad \|v_h\|_0 \leq C\|\mathrm{curl}\, v_h\|_{-1}. \tag{13}$$

Technicalities similar to proof of discrete compactness (Girault).

Application

●Proposition:

Let (v_h) be a sequence with $v_h \in V_h$.
If $(\mathrm{curl}\, v_h)$ converges to $\mathrm{curl}\, v$ in H^{-1} for some $v \in V$,
then (v_h) converges to v in L^2.

●Proof:

– Denote by P_V the L^2 orthogonal projection onto V.
$(P_V v_h)$ converges to v in L^2.
– Denote by P_h the L^2 orthogonal projection onto X_h.
$(P_h v)$ converges to v in L^2 and $P_h v \in V_h$.
– Write:

$$\begin{aligned} \|P_h v - v_h\|_0 &\leq C\|\,\mathrm{curl}\, P_h v - \mathrm{curl}\, v_h\|_{-1} && (14) \\ &\leq C\|\,\mathrm{curl}\, P_h v - \mathrm{curl}\, P_V v_h\|_{-1} && (15) \\ &\leq C\|P_h v - P_V v_h\|_0. && (16) \end{aligned}$$

Hence (v_h) converges to v.

Super-approximation

•Proposition: For any function $\phi \in \mathrm{C}_c^\infty$ (smooth and compactly supported) we have:

$$\lim_{h \to 0} \sup_{p_h \in Y_h} \inf_{\tilde{p}_h \in \tilde{Y}_h} \|\phi p_h - \tilde{p}_h\|_1 / \|p_h\|_1 = 0. \tag{17}$$

•That is: Y_h is almost stable under multiplication by ϕ, even though it is not a compact endomorphism of $\mathrm{H}^1(U)$.

•Proof:

Let Π_h be the nodal interpolator. For all $p_h \in Y_h$ we have:

$$\|\Pi_h(\phi p_h) - \phi p_h\|_1 \leq Ch \sum_T |\phi p_h|_{T,2} \leq Ch\|p_h\|_1, \tag{18}$$

(Leibniz' rule and second order derivatives of p_h vanish)

Application of super-approximation

•Proposition: Suppose vectorfields $v_h \to v$ weakly in L^2, with $v_h \in X_h$ discrete divergence free.
Suppose scalar fields $p_h \to p$ weakly in H^1 with $p_h \in Y_h$.
Then for all $\phi \in \mathrm{C}_c^\infty$:

$$\int (v_h \cdot \mathrm{grad}\, p_h)\phi \to \int (v \cdot \mathrm{grad}\, p)\phi. \tag{19}$$

•Proof: Pick $\phi \in \mathrm{C}_c^\infty$. We have:

$$\int (v_h \cdot \mathrm{grad}\, p_h)\phi = \int v_h \cdot \mathrm{grad}(\phi p_h) - \int (v_h \cdot \mathrm{grad}\, \phi) p_h. \tag{20}$$

–The first term can be replaced by: $\int v_h \cdot \mathrm{grad}(\phi p_h - \widetilde{p}_h)$, for any $\widetilde{p}_h \in \widetilde{Y}_h$. It converges to 0 by superapproximation.
–In the second term $(v_h \cdot \mathrm{grad}\, \phi) \to v \cdot \mathrm{grad}\, \phi$ weakly in L^2, whereas $p_h \to p$ strongly in L^2 (compact injection $\mathrm{H}^1 \to \mathrm{L}^2$).
– Therefore:

$$\int (v_h \cdot \mathrm{grad}\, p_h)\phi \;\to\; -\int (v \cdot \mathrm{grad}\, \phi)p = \int (v \cdot \mathrm{grad}\, p)\phi. \tag{21}$$

Div-Curl lemma for Edge Elements (proof)

•Lemma: If $u_h, u'_h \in X_h$, weakly converging in L^2 and $u_h = v_h + \operatorname{grad} p_h$ with v_h discrete div free, $p_h \in Y_h \cap \mathrm{H}^1_0$ precompact in H^1, and $(\operatorname{curl} u'_h)$ is precompact in H^{-1}, then for all $\phi \in \mathrm{C}^\infty_c(U)$:

$$\int (u_h \cdot u'_h)\phi \to \int (u \cdot u')\phi. \tag{22}$$

•Proof:

– $\operatorname{grad} p_h \to \operatorname{grad} p$ strongly in L^2 hence $\int (\operatorname{grad} p_h \cdot u'_h)\phi \to \int (\operatorname{grad} p \cdot u')\phi$.

– Write $u' = v' + \operatorname{grad} p'$ with $v' \in V$ and $p' \in \mathrm{H}^1$, and $u'_h = v'_h + \operatorname{grad} p'_h$ with $v'_h \in V_h$ and $p'_h \in Y_h$.

– We have $\operatorname{curl} v'_h \to \operatorname{curl} v$ in H^{-1} hence $v'_h \to v$ in L^2 by uniform norm estimate on dicrete Hodge decompositions. Hence:

$$\int (v_h \cdot v'_h)\phi \to \int (v \cdot v')\phi. \tag{23}$$

– By superapproximation we get:

$$\int (v_h \cdot \operatorname{grad} p'_h)\phi \to \int (v \cdot \operatorname{grad} p')\phi. \tag{24}$$

Variants and related properties

• Div-curl lemma can be stated for differential forms in arbitrary dimensions. Discrete variants can be obtained for discrete differential forms (Whitney forms), following a procedure similar to the one eposed here.

• Discrete compactness: the injection $\mathrm{H}^1 \to \mathrm{L}^2$ is compact. Similarly if $v_h \in X_h$ is bounded in L^2, has bounded curl in L^2, is discrete divergence free and satisfies some boundary condition, then has subsequence converging in L^2. Direct consequence of uniform norm equivalence.

• All families of spaces for which the discrete div-curl lemma is true satisfy a local discrete compactness property.

• Reference and Bibliography in preprint:
`http://www.math.uio.no/eprint/pure_math/2003/30-03.html`
Revised September 16, 2003.