**Div-curl lemma for edge elements**

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**Abstract**  
A variant of Murat and Tartar’s div-curl lemma is stated and proved for Nedelec’s edge elements. In the Galerkin finite element setting one can expect control of $L^2$ norms of vectorfields and also sufficient control of their curls in $H^{-1}$. But the divergence is usually just controlled through the integral of the vectorfields against a finite dimensional space of gradients. The proposed lemma is designed to handle this case. The proof uses a uniform norm equivalence related to discrete compactness properties of vector FE spaces and a super-approximation property of scalar FE spaces.
Compensated compactness

- Tool for **homogenisation** (oscillatory material coefficients) and some **non-linear PDE** (e.g. conservation laws).

- Weakly converging sequence of functions $\mathbb{R}^n \rightarrow \mathbb{R}^p$. Compose with $Q : \mathbb{R}^p \rightarrow \mathbb{R}$ quadratic form.

\[
    u_h \rightarrow u \text{ weakly in } L^2(\mathbb{R}^n; \mathbb{R}^p) \tag{1}
\]

\[
    \text{convergence of } Q \circ u_h \rightarrow Q \circ u ? \tag{2}
\]

- Main example: div-curl lemma:

\[
    u_h \rightarrow u, \ u'_h \rightarrow u' \text{ weakly in } L^2(\mathbb{R}^3; \mathbb{R}^3) \tag{3}
\]

\[
    \text{convergence of } u_h \cdot u'_h \rightarrow u \cdot u ? \tag{4}
\]

It is sufficient that:

\[
    (\text{div } u_h) \text{ is precompact in } H^{-1}(\mathbb{R}^3; \mathbb{R}) \tag{5}
\]

\[
    (\text{curl } u'_h) \text{ is precompact in } H^{-1}(\mathbb{R}^3; \mathbb{R}^3) \tag{6}
\]

to ensure that for all $\phi \in C_c^\infty$:

\[
    \int (u_h \cdot u'_h) \phi \rightarrow \int (u \cdot u') \phi. \tag{7}
\]
Finite elements

- $U$ bounded convex and smooth domain in $\mathbb{R}^3$. Uniform curved tetrahedral meshes $\mathcal{T}_h$ with max diameter $h$. Study the limit $h \to 0$.

- $X_h$ lowest order Nédélec edge elements on $\mathcal{T}_h$. A finite dimensional space of vector fields $u \in L^2(U; \mathbb{R}^3)$ s.t. $\text{curl } u \in L^2(U; \mathbb{R}^3)$ (defined a priori as a distribution).

- Vector fields $u_h \in X_h$ have continuous tangential traces across faces of mesh, but normal component may jump.

- Most popular finite element space for electromagnetics:
  - Good approximation properties on non-smooth domains,
  - No spurious eigenvalues for $\text{curl } \text{curl}$ operator.

(contrary to other nodal finite element spaces which have continuous vector fields) (Costabel, Dauge, Boffi, Brezzi et al., Monk, Demkowicz et al.).

- $Y_h$ space of continuous piecewise linear functions on $\mathcal{T}_h$. Then the following sequence (exists and) is exact.

\[
Y_h \xrightarrow{\text{grad}} X_h \xrightarrow{\text{curl}} L^2, \quad (8)
\]

Div-curl lemma for edge elements
Variational formulations and motivation

- Variational formulation of a problem from electromagnetics. Find \( u_h \in X_h \) s.t. \( \forall u'_h \in X_h :\)

\[
\int u_h \cdot u'_h + \int \text{curl } u_h \cdot \text{curl } u'_h = \int f(u_h) u'_h. \tag{9}
\]

- A priori estimates:
  - Energy: typically \( u'_h = u_h \) gives bound on \( \|u_h\|_{L^2} \) and \( \|\text{curl } u_h\|_{L^2} \). In particular \( (\text{curl } u_h) \) is precompact in \( H^{-1} \).
  - Divergence: test with \( u'_h = \text{grad } p'_h \) where \( p'_h \in Y_h \cap H^1_0 \), gives estimate on \( \int u_h \cdot \text{grad } p'_h \).

- Best case: \( \int u_h \cdot \text{grad } p'_h = 0 \) for all \( p'_h \in Y_h \cap H^1_0 \), i.e. \( u_h \) is discrete divergence free.

- Question:
  If \( (u_h) \) is bounded in \( L^2 \) and discrete divergence free, is \( (\text{div } u_h) \) precompact in \( H^{-1} \)?
Div-curl lemma for Edge elements

- It's an open problem (much stronger than discrete compactness property of Kikuchi).

- **Prove:** If $u_h, u'_h \in X_h$, weakly converging in $L^2$ and $u_h$ is discrete div free and $(\text{curl } u'_h)$ is precompact in $H^{-1}$, then for all $\phi \in C_c^\infty(U)$:

  $$ \int (u_h \cdot u'_h) \phi \rightarrow \int (u \cdot u') \phi. $$  

- **Corollary:** if $u_h = v_h + \text{grad } p_h$ with $v_h$ discrete divergence free and $p_h \in Y_h \cap H_0^1$ precompact in $H^1$, then ok. (because each part also converges weakly).

- **Difficulty:** $p_h$ contains less information than $\text{div } u_h$. 

Continuous and discrete Hodge decompositions

\[ L^2 \text{ vectorfields on } U \text{ can be uniquely decomposed as:} \]

\[ u = v + \text{grad } p \text{ with } \text{div } v = 0 \text{ and } p \in H_0^1(U). \quad (11) \]

Dirichlet problem gives \( p \). Neumann also useful.

\[ \text{Discrete analogues: } u_h \in X_h \text{ can be uniquely decomposed as:} \]

\[ u_h = v_h + \text{grad } p_h \text{ with } p \in Y_h \text{ and } v_h \perp \text{grad } Y_h. \quad (12) \]

(Neumann type)

\[ \text{Weak convergence: if } u_h \text{ converges weakly in } L^2 \text{ then } v_h \text{ converges weakly in } L^2 \text{ and } p_h \text{ converges weakly in } H^1. \]
Estimates on Hodge decompositions

- Mixed regularity (div or curl control) can be easily read off Hodge decomposition.

- In $L^2(U; \mathbb{R}^3)$ let $V$ be the orthogonal of $\text{grad} \, H^1(U)$. Divergence free and 0 normal component on $\partial U$.

- Continuous estimate:
  On $V$, $\| \text{curl}(\cdot) \|_{-1}$ is a norm equivalent to the $L^2$ norm. Hard part: $\text{curl} \, L^2 \to H^{-1}$ has closed range (regularity of $\partial U$).

- Discrete analogue. $V_h \subset X_h$ the $L^2$ orthogonal of $\text{grad} \, Y_h$. We have $V_h \not\subset V$. Nevertheless:

- There is $C > 0$ such that $\forall h$:

\[ \forall v_h \in V_h \quad \| v_h \|_0 \leq C \| \text{curl} \, v_h \|_{-1}. \quad (13) \]

Technicalities similar to proof of discrete compactness (Girault).
Application

- Proposition:
Let \((v_h)\) be a sequence with \(v_h \in V_h\).
If \((\text{curl } v_h)\) converges to \(\text{curl } v\) in \(H^{-1}\) for some \(v \in V\),
then \((v_h)\) converges to \(v\) in \(L^2\).

- Proof:
  - Denote by \(P_V\) the \(L^2\) orthogonal projection onto \(V\).
    \((P_V v_h)\) converges to \(v\) in \(L^2\).
  - Denote by \(P_h\) the \(L^2\) orthogonal projection onto \(X_h\).
    \((P_h v)\) converges to \(v\) in \(L^2\) and \(P_h v \in V_h\).
  - Write:

    \[
    \|P_h v - v_h\|_0 \leq C \|\text{curl } P_h v - \text{curl } v_h\|_{-1} \quad (14)
    \]
    \[
    \leq C \|\text{curl } P_h v - \text{curl } P_V v_h\|_{-1} \quad (15)
    \]
    \[
    \leq C \|P_h v - P_V v_h\|_0. \quad (16)
    \]

Hence \((v_h)\) converges to \(v\).
Super-approximation

\begin{itemize}
\item **Proposition:** For any function $\phi \in C^\infty_c$ (smooth and compactly supported) we have:
\[
\lim_{h \to 0} \sup_{p_h \in Y_h} \inf_{\widetilde{p}_h \in \widetilde{Y}_h} \frac{\|\phi p_h - \widetilde{p}_h\|_1}{\|p_h\|_1} = 0. 
\]  
\end{itemize}

\begin{itemize}
\item That is: $Y_h$ is almost stable under multiplication by $\phi$, even though it is not a compact endomorphism of $H^1(U)$.
\end{itemize}

\begin{itemize}
\item **Proof:**
\end{itemize}

Let $\Pi_h$ be the nodal interpolator. For all $p_h \in Y_h$ we have:
\[
\|\Pi_h(\phi p_h) - \phi p_h\|_1 \leq C h \sum_T |\phi p_h|_{T,2} \leq C h \|p_h\|_1, 
\]  
(Leibniz’ rule and second order derivatives of $p_h$ vanish)
Application of super-approximation

- **Proposition:** Suppose vectorfields \( v_h \to v \) weakly in \( L^2 \), with \( v_h \in X_h \) discrete divergence free. Suppose scalar fields \( p_h \to p \) weakly in \( H^1 \) with \( p_h \in Y_h \). Then for all \( \phi \in C_c^\infty \):

\[
\int (v_h \cdot \text{grad} \ p_h)\phi \to \int (v \cdot \text{grad} \ p)\phi. \tag{19}
\]

- **Proof:** Pick \( \phi \in C_c^\infty \). We have:

\[
\int (v_h \cdot \text{grad} \ p_h)\phi = \int v_h \cdot \text{grad} (\phi p_h) - \int (v_h \cdot \text{grad} \ \phi)p_h. \tag{20}
\]

- The first term can be replaced by: \( \int v_h \cdot \text{grad} (\phi p_h - \overline{p}_h) \), for any \( \overline{p}_h \in \overline{Y}_h \). It converges to 0 by superapproximation.
- In the second term \( (v_h \cdot \text{grad} \ \phi) \to v \cdot \text{grad} \ \phi \) weakly in \( L^2 \), whereas \( p_h \to p \) strongly in \( L^2 \) (compact injection \( H^1 \to L^2 \)).
- Therefore:

\[
\int (v_h \cdot \text{grad} \ p_h)\phi \to -\int (v \cdot \text{grad} \ \phi)p = \int (v \cdot \text{grad} \ p)\phi. \tag{21}
\]
Div-Curl lemma for Edge Elements (proof)

\textbf{Lemma:} If $u_h, u'_h \in X_h$, weakly converging in $L^2$ and $u_h = v_h + \text{grad } p_h$ with $v_h$ discrete div free, $p_h \in Y_h \cap H^1_0$ precompact in $H^1$, and $(\text{curl } u'_h)$ is precompact in $H^{-1}$, then for all $\phi \in C^\infty_c(U)$:

$$\int (u_h \cdot u'_h) \phi \rightarrow \int (u \cdot u') \phi. \quad (22)$$

\textbf{Proof:}

- $\text{grad } p_h \rightarrow \text{grad } p$ strongly in $L^2$ hence $\int (\text{grad } p_h \cdot u'_h) \phi \rightarrow \int (\text{grad } p \cdot u') \phi$.

- Write $u' = v' + \text{grad } p'$ with $v' \in V$ and $p' \in H^1$, and $u'_h = v'_h + \text{grad } p'_h$ with $v'_h \in V_h$ and $p'_h \in Y_h$.

- We have $\text{curl } v'_h \rightarrow \text{curl } v$ in $H^{-1}$ hence $v'_h \rightarrow v$ in $L^2$ by uniform norm estimate on discrete Hodge decompositions. Hence:

$$\int (v_h \cdot v'_h) \phi \rightarrow \int (v \cdot v') \phi. \quad (23)$$

- By superapproximation we get:

$$\int (v_h \cdot \text{grad } p'_h) \phi \rightarrow \int (v \cdot \text{grad } p') \phi. \quad (24)$$
Variants and related properties

- Div-curl lemma can be stated for differential forms in arbitrary dimensions. Discrete variants can be obtained for discrete differential forms (Whitney forms), following a procedure similar to the one eposited here.

- Discrete compactness: the injection $H^1 \to L^2$ is compact. Similarly if $v_h \in X_h$ is bounded in $L^2$, has bounded curl in $L^2$, is discrete divergence free and satisfies some boundary condition, then has subsequence converging in $L^2$. Direct consequence of uniform norm equivalence.

- All families of spaces for which the discrete div-curl lemma is true satisfy a local discrete compactness property.

- Reference and Bibliography in preprint:
  
  http://www.math.uio.no/eprint/pure_math/2003/30-03.html