PROJECTION-BASED INTERPOLATION IN THREE DIMENSIONS

Quasi Optimal p-Interpolation Estimates

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IMA Special Workshop: Hot Topics Workshop: Compatible Spatial Discretizations for PDE, May 11 - May 15, 2004

Plan of presentation:

I Commuting projections interpolation

II Polynomial preserving extension operators

III Poincare-Friedrichs inequalities

IV Polynomial preserving, right inverses of grad, curl and div operators

V Projection-Based Interpolation

VI Optimal p-estimates

Support: Air Force, NSF(NPACI)
Web: http://ticam.utexas.edu/˜leszek

Bibliography:

1 M. Ainsworth and L. Demkowicz "Explicit Polynomial Preserving Trace Liftings on a Triangle", *ICES Report 03-47*.

2 L. Demkowicz, P. Monk, L. Vardapetyan, and W. Rachowicz. "De Rham Diagram for hp Finite Element Spaces", Computers and *Mathematics with Applications*, Vol. 39, 7-8, 29-38, 2000.

3 L. Demkowicz "Projection-Based Interpolation", *ICES Report 04-03*.

4 L. Demkowicz, A. Buffa, "H^1, $\boldsymbol{H}(\text{curl})$ and $\boldsymbol{H}(\text{div})$- Conforming Projection-Based Interpolation in Three Dimensions" *ICES Report 04-24*.

5 J. Gopalakrishnan, L. Demkowicz, "Quasioptimality of Some Spectral Mixed Methods" *Journal of Computational and Applied Mathematics*, Vol. 167, Issue 1 , 1 May 2004, Pages 163-182

I Commuting projections

$$\begin{tikzcd}
\mathbb{R} \arrow[r] \arrow[d, "id"'] & H^1 \arrow[r, "\boldsymbol{\nabla}"] \arrow[d, "P^{grad}"] & \boldsymbol{H}(\mathrm{curl}) \arrow[r, "\boldsymbol{\nabla}\times"] \arrow[d, "P^{curl}"] & \boldsymbol{H}(\mathrm{div}) \arrow[r, "\boldsymbol{\nabla}\circ"] \arrow[d, "P^{div}"] & L^2 \arrow[r] \arrow[d, "P"'] & \mathbf{0} \\
\mathbb{R} \arrow[r] & \mathcal{P}^{p+1}_{p_e+1,p_f+1} \arrow[r, "\boldsymbol{\nabla}"] & \mathbf{P}^{p}_{p_e,p_f} \arrow[r, "\boldsymbol{\nabla}\times"] & \mathbf{P}^{p-1}_{p_f-1,p_e} \arrow[r, "\boldsymbol{\nabla}\circ"] & \mathcal{P}^{p-2}_{p_f} \arrow[r] & \mathbf{0}\,.
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r] \arrow[d, "id"] & H^{\frac{1}{2}+\epsilon} \arrow[r, "\boldsymbol{\nabla}"] \arrow[d, "P^{-\frac{1}{2}+\epsilon,grad}"] & \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(\mathrm{curl}) \arrow[r, "\boldsymbol{\nabla}\times"] \arrow[d, "P^{-\frac{1}{2}+\epsilon,curl}"] & H^{-\frac{1}{2}+\epsilon} \arrow[r] \arrow[d, "P^{-\frac{1}{2}+\epsilon}"] & \mathbf{0} \\
\mathbb{R} \arrow[r] & \mathcal{P}^{p_f+1}_{p_e+1} \arrow[r, "\boldsymbol{\nabla}"] & \mathbf{P}^{p_f}_{p_e} \arrow[r, "\boldsymbol{\nabla}\times"] & \mathcal{P}^{p_f-1} \arrow[r] & \mathbf{0}\,.
\end{tikzcd}$$

- Operator P^{grad} : $H^1 \ni u \rightarrow P^{grad}u \in \mathcal{P}^{p}_{p_f,p_e}$,

$$\begin{cases} \|\boldsymbol{\nabla}(P^{grad}u - u)\| \rightarrow \min \\ (P^{grad}u - u, 1) = 0 \end{cases}$$

- Operator P^{curl} : $\boldsymbol{H}(\mathrm{curl}) \ni \boldsymbol{E} \rightarrow P^{curl}\boldsymbol{E} \in \boldsymbol{P}^{p}_{p_f,p_e}$,

$$\begin{cases} \|\boldsymbol{\nabla} \times (P^{curl}\boldsymbol{E} - \boldsymbol{E})\| \rightarrow \min \\ (P^{curl}\boldsymbol{E} - \boldsymbol{E}, \boldsymbol{\nabla}\phi) = 0 \qquad \forall \phi \in \mathcal{P}^{p+1}_{p_f+1,p_e+1} \end{cases}$$

- Operator P^{div} : $\boldsymbol{H}(\mathrm{div}) \ni \boldsymbol{F} \rightarrow P^{div}\boldsymbol{F} \in \boldsymbol{P}^{p}_{p_f}$,

$$\begin{cases} \|\boldsymbol{\nabla} \circ (P^{div}\boldsymbol{F} - \boldsymbol{F})\| \rightarrow \min \\ (P^{div}\boldsymbol{F} - \boldsymbol{F}, \boldsymbol{\nabla} \times \boldsymbol{\phi}) = 0 \quad \forall \boldsymbol{\phi} \in \boldsymbol{P}^{p+1}_{p_f+1,p_e+1} \,. \end{cases}$$

- Operator $P^{-\frac{1}{2}+\epsilon,grad}\ :\ H^{\frac{1}{2}+\epsilon} \ni u \to P^{-\frac{1}{2}+\epsilon,grad}u \in \mathcal{P}^p_{p_e}$,

$$\begin{cases} \|\boldsymbol{\nabla}(P^{-\frac{1}{2}+\epsilon,grad}u - u)\|_{-\frac{1}{2}+\epsilon} \to \min \\ (P^{-\frac{1}{2}+\epsilon,grad}u - u, 1) = 0 \end{cases}$$

- Operator $P^{-\frac{1}{2}+\epsilon,curl}\ :\ \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(\mathrm{curl}) \ni \boldsymbol{E} \to P^{-\frac{1}{2}+\epsilon,curl}\boldsymbol{E} \in \boldsymbol{P}^p_{p_e}$,

$$\begin{cases} \|\boldsymbol{\nabla} \times (P^{-\frac{1}{2}+\epsilon,curl}\boldsymbol{E} - \boldsymbol{E})\|_{-\frac{1}{2}+\epsilon} \to \min \\ (P^{-\frac{1}{2}+\epsilon,curl}\boldsymbol{E} - \boldsymbol{E}, \boldsymbol{\nabla}\phi)_{-\frac{1}{2}+\epsilon} = 0 \qquad\qquad \forall \phi \in \mathcal{P}^{p+1}_{p_e+1} \end{cases}$$

- standard $H^{-\frac{1}{2}+\epsilon}$-projection: $H^{-\frac{1}{2}+\epsilon} \ni v \to P^{-\frac{1}{2}+\epsilon}v \in \mathcal{P}^p$,

$$\|P^{-\frac{1}{2}+\epsilon}v - v\|_{-\frac{1}{2}+\epsilon} \to \min\ .$$

II Polynomial preserving extension operators

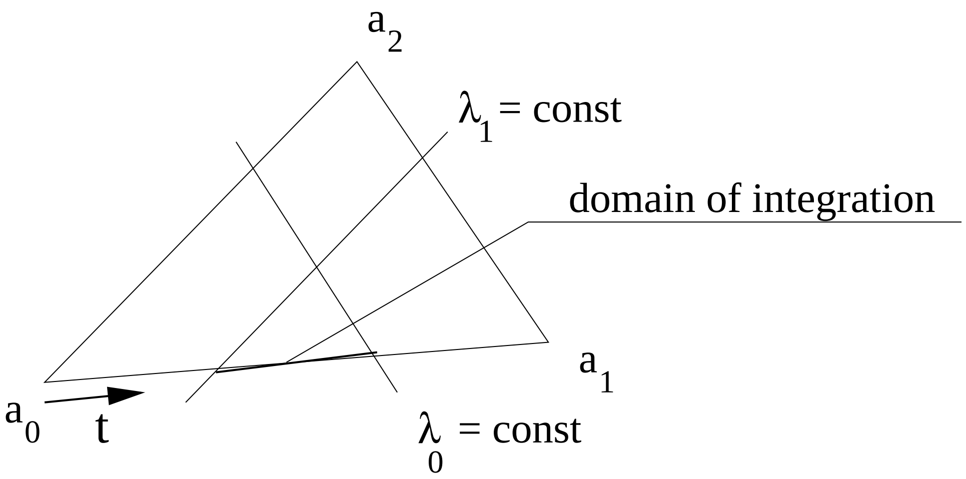
Commuting, polynomial preserving, continuous extension operators for a triangular face f

$$\begin{tikzcd}
H^{\frac{1}{2}+\epsilon}(f) \arrow[r, "\boldsymbol{\nabla}"] \arrow[d, shift left, "Tr^{grad}"] & \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(\mathrm{curl}, f) \arrow[d, shift left, "Tr^{curl}"] \\
H^{\epsilon}(\partial f) \arrow[r, "\frac{\partial}{\partial s}"] \arrow[u, shift left, "E^{grad}"] & H^{-1+\epsilon}(\partial f) \arrow[u, shift left, "E^{curl}"]
\end{tikzcd}$$

Here $\epsilon \in (0, \frac{1}{2}]$.

$$\begin{tikzcd}
\mathcal{P}^{p}_{p_e}(f) \arrow[r, "\boldsymbol{\nabla}"] \arrow[d, shift left, "Tr^{grad}"] & \boldsymbol{P}^{p-1}_{p_e-1}(f) \arrow[d, shift left, "Tr^{curl}"] \\
\mathcal{P}^{p_e}(\partial f) \arrow[r, "\frac{\partial}{\partial s}"] \arrow[u, shift left, "E^{grad}"] & \mathcal{P}^{p_e-1}(\partial f) \arrow[u, shift left, "E^{curl}"]
\end{tikzcd}$$

H^1 extension from one edge to triangle (Babuška):



$$\tilde{\phi}(\lambda_1, \lambda_2) = \int_0^1 \phi(\lambda_1 + \xi\lambda_2)\, d\xi = \frac{1}{\lambda_2}\int_{\lambda_2}^{\lambda_1+\lambda_2} \phi(t)\, dt\ .$$

Edge to edge operators:

$$\tilde{\phi}(\lambda_1, \lambda_2) = \int_0^1 \phi(\lambda_1 + \xi\lambda_2)\, d\xi = \frac{1}{\lambda_2}\int_{\lambda_2}^{\lambda_1+\lambda_2} \phi(t)\, dt\ .$$

$$\phi \rightarrow A^{right}\phi = \tilde{\phi}|_{\lambda_0=0}$$

$$\phi \rightarrow A^{left}\phi = \tilde{\phi}|_{\lambda_1=0}$$

The three edge problem:

Given f_i defined on edge $\lambda_i = 0, i = 0, 1, 2,$

$$\begin{cases} \text{Find } \phi_i \text{ such that} \\ \phi_i + A^{right}\phi_{i-1} + A^{left}\phi_{i+1} = f_i \ . \end{cases}$$

Multiply by $\omega^i, \omega^3 = 1$, to decouple,

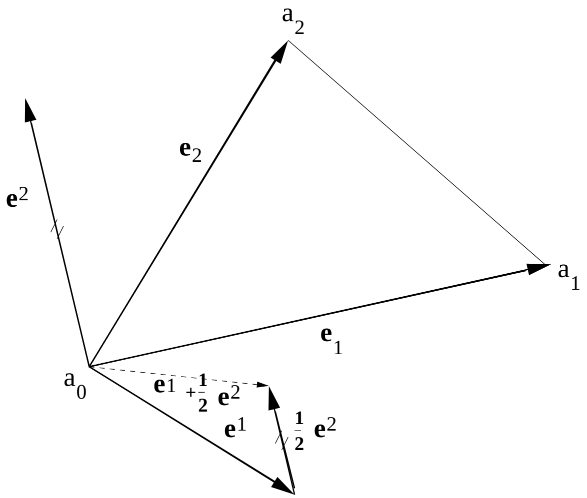
$$\begin{cases} \text{Find } \phi_i\omega^i \text{ such that} \\ \phi_i\omega^i + \omega A^{right}\left(\phi_{i-1}\omega^{i-1}\right) + \omega^{-1}A^{left}\left(\phi_{i+1}\omega^{i+1}\right) = f_i\omega^i \ . \end{cases}$$

Define then,

$$E^{grad}(f_1, f_2, f_3) = \sum_i \tilde{\phi}_i \ .$$

H(curl) **extension from one edge to triangle:** Differentiate the H^1-extension formula

$$\boldsymbol{\nabla}\tilde{\phi} = \int_0^1 \phi'(\lambda_1 + \xi\lambda_2)(\boldsymbol{e}^1 + \xi\boldsymbol{e}^2)\,d\xi\ .$$



Define,

$$\tilde{\boldsymbol{\psi}} = \int_0^1 \psi(\lambda_1 + \xi\lambda_2)(\boldsymbol{e}^1 + \xi\boldsymbol{e}^2)\,d\xi\ .$$

where $\boldsymbol{\psi} = \psi\bar{\boldsymbol{e}}^1$.

Notice that $\boldsymbol{\nabla} \times \tilde{\boldsymbol{\psi}} = 0$! Consequently, the corresponding three edge problem for the $H(\text{curl})$ extension will have a solution, if the boundary data are curl free,

$$\sum_i \int_{e_i} g_i \, ds = 0 \, .$$

Given *arbitrary* g_i, use the *stable* decomposition into a constant and a curl free component,

$$g_i = g_{i,0} + c \, ,$$

extend constant c using Whitney shape functions, and solve the three edge problem for curl-free data $g_{i,0}$. Alternatively, identify the corresponding potential on the boundary,

$$\frac{\partial f}{\partial s} = g_0 \, ,$$

extend f_i's using the H^1-extension operator, and take its gradient. The commutativity property makes the the H^1- and $H(\text{curl})$- three edge problems equivalent; it is sufficient to solve just one of them, the solution of other one comes then "for free".

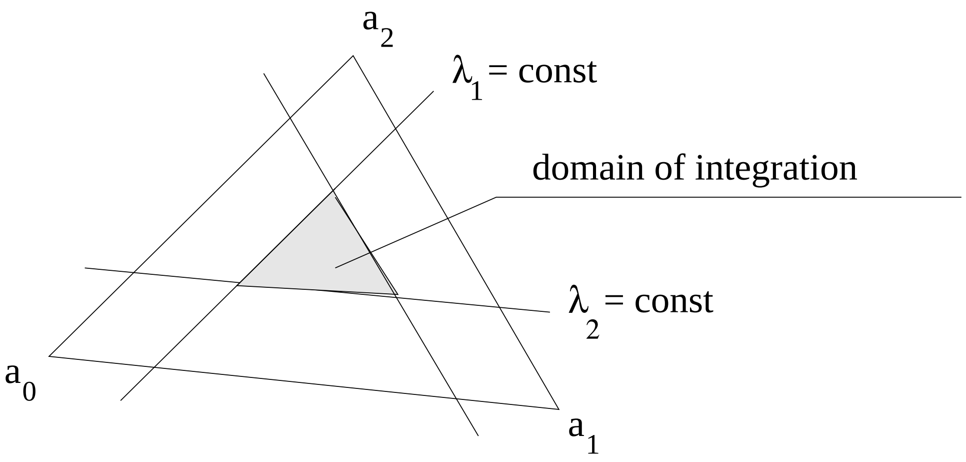
Commuting, polynomial preserving, continuous extension operators for a tetrahedron T (conjectured)

$$\begin{tikzcd}
H^1(T) \arrow[r, "\boldsymbol{\nabla}"] \arrow[d, shift left, "Tr^{grad}"] & \boldsymbol{H}(\mathrm{curl}, T) \arrow[r, "\boldsymbol{\nabla}\times"] \arrow[d, shift left, "Tr^{curl}"] & \boldsymbol{H}(\mathrm{div}, T) \arrow[d, shift left, "Tr^{div}"] \\
H^{\frac{1}{2}}(\partial T) \arrow[r, "\boldsymbol{\nabla}_{\partial T}"] \arrow[u, shift left, "E^{grad}"] & \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl}, \partial T) \arrow[r, "\boldsymbol{\nabla}_{\partial T}\times"] \arrow[u, shift left, "E^{curl}"] & H^{-\frac{1}{2}}(\partial T) \arrow[u, shift left, "E^{div}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{P}^{p}_{p_f,p_e}(T) \arrow[r, "\boldsymbol{\nabla}"] \arrow[d, shift left, "Tr^{grad}"] & \boldsymbol{P}^{p-1}_{p_f-1,p_e-1}(T) \arrow[r, "\boldsymbol{\nabla}\times"] \arrow[d, shift left, "Tr^{curl}"] & \boldsymbol{P}^{p-2}_{p_f-2}(T) \arrow[d, shift left, "Tr^{div}"] \\
\mathcal{P}^{p_f}_{p_e}(\partial T) \arrow[r, "\boldsymbol{\nabla}_{\partial T}"] \arrow[u, shift left, "E^{grad}"] & \boldsymbol{P}^{p_f-1}_{p_e-1}(\partial T) \arrow[r, "\boldsymbol{\nabla}_{\partial T}\times"] \arrow[u, shift left, "E^{curl}"] & \mathcal{P}^{p_f-2}(\partial T) \arrow[u, shift left, "E^{div}"]
\end{tikzcd}$$

3D extension operators from a face f to tetrahedron T

$$\begin{aligned}\tilde{\phi}(\lambda_1, \lambda_2, \lambda_3) &= 2\int_0^1 \int_0^{1-\xi_2} \phi(\lambda_1 + \xi_1\lambda_3, \lambda_2 + \xi_2\lambda_3)\, d\xi_1 d\xi_2 \\ &= \frac{2}{\lambda_3^2} \int_{\lambda_2}^{\lambda_2+\lambda_3} \int_{\lambda_1}^{\lambda_1+\lambda_2+\lambda_3-\mu_2} \phi(\mu_1, \mu_2)\, d\mu_1 d\mu_2\ .\end{aligned}$$



$\boldsymbol{H}$(curl)-extension:

$$\tilde{\boldsymbol{\psi}} = 2\int_0^1 \int_0^{1-\xi_2} \left\{ \psi_1(\boldsymbol{e}^1 + \xi_1 \boldsymbol{e}^3) + \psi_2(\boldsymbol{e}^2 + \xi_2 \boldsymbol{e}^3) \right\} \, d\xi_1 d\xi_2 \, ,$$

where

$$\boldsymbol{\psi} = \psi_1 \bar{\boldsymbol{e}}^1 + \psi_2 \bar{\boldsymbol{e}}^2,$$

with $\bar{\boldsymbol{e}}^i$ denoting projections of dual vectors $\boldsymbol{e}^i$ onto face f.

$\boldsymbol{H}$(div)-extension:

$$\begin{aligned}\tilde{\boldsymbol{\chi}} = 2\int_0^1 \int_0^{1-\xi_2} & \{\chi(\lambda_1 + \xi_1\lambda_3, \lambda_2 + \xi_2\lambda_3) \\ & (\boldsymbol{e}^1 \times \boldsymbol{e}^2 - \xi_1 \, \boldsymbol{e}^2 \times \boldsymbol{e}^3 - \xi_2 \, \boldsymbol{e}^3 \times \boldsymbol{e}^1)\} \, d\xi_1 d\xi_2 \, ,\end{aligned}$$

where,

$$\boldsymbol{\chi} = \chi \, \bar{\boldsymbol{e}}^1 \times \bar{\boldsymbol{e}}^2 \, .$$

III Poincare - Friedrichs inequalities

Poincare's inequalities:
There exist $C > 0$ such that,

$$\|u\|_{0,T} \leq C\|\boldsymbol{\nabla} u\|_{0,T} ,$$

for every function $u \in H^1(K)$ belonging to either of the two families:
Case 1: $(u,1)_{0,T} = 0$,
Case 2: $u = 0$ on ∂K.

Poincare's inequalities for fractional spaces defined on a face f:
There exist $C > 0$ such that

$$\|u\|_{0,f} \leq C|u|_{\frac{1}{2}+\epsilon,f} \approx C\|\boldsymbol{\nabla} u\|_{-\frac{1}{2}+\epsilon,f} ,$$

for every function $u \in H^{\frac{1}{2}+\epsilon}(f)$ belonging to either of the two families:
Case 1: $(u,1)_{0,f} = 0$,
Case 2: $u = 0$ on ∂f.

Discrete Friedrichs Inequalities:
There exist $C > 0$ such that,

$$\|\boldsymbol{E}\|_{0,T} \leq C\|\boldsymbol{\nabla} \times \boldsymbol{E}\|_{0,T} \ ,$$

for every (discrete divergence free) polynomial $\boldsymbol{E}$ belonging to either of the two families,

Case 1:

$$\boldsymbol{E} \in \boldsymbol{P}^{p}_{p_f,p_e}(T), \qquad (\boldsymbol{E}, \boldsymbol{\nabla}\phi)_{0,T} = 0, \quad \forall \phi \in \mathcal{P}^{p+1}_{p_f+1,p_e+1} \ .$$

Case 2:

$$\boldsymbol{E} \in \boldsymbol{P}^{p}_{-1,-1}(T), \qquad (\boldsymbol{E}, \boldsymbol{\nabla}\phi)_{0,T} = 0, \quad \forall \phi \in \mathcal{P}^{p+1}_{-1,-1} \ .$$

Discrete Friedrich's Inequality for fractional spaces defined on face f:
There exist $C > 0$ such that,

$$\|\boldsymbol{E}\|_{-\frac{1}{2}+\epsilon} \leq C\|\boldsymbol{\nabla} \times \boldsymbol{E}\|_{-\frac{1}{2}+\epsilon,f} \ ,$$

for every discrete divergence free polynomial $\boldsymbol{E} \in \boldsymbol{P}^{p_f}_{p_e}(T)$, i.e.,

$$(\boldsymbol{E}, \boldsymbol{\nabla}\phi)_{-\frac{1}{2}+\epsilon,f} = 0, \quad \forall \phi \in \mathcal{P}^{p_f+1}_{p_e+1} \ .$$

Discrete Friedrichs Inequality for $\boldsymbol{H}(\mathrm{div})$ space

There exist $C > 0$ such that,

$$\|\boldsymbol{H}\|_{0,T} \leq C\|\boldsymbol{\nabla} \circ \boldsymbol{E}\|_{0,T} \ ,$$

for every discrete curl free polynomial $\boldsymbol{H}$ belonging to either of the two families:

Case 1: $\boldsymbol{H} \in \boldsymbol{P}^{p}_{p_f}(T)$ and,

$$(\boldsymbol{H}, \boldsymbol{\nabla} \times \boldsymbol{\phi})_{0,T} = 0, \quad \forall \boldsymbol{\phi} \in \boldsymbol{P}^{p+1}_{p_f+1,p_e+1} \ .$$

Case 2: $\boldsymbol{H} \in \boldsymbol{P}^{p}_{-1}(T)$ and,

$$(\boldsymbol{H}, \boldsymbol{\nabla} \times \boldsymbol{\phi})_{0,T} = 0, \quad \forall \boldsymbol{\phi} \in \boldsymbol{P}^{p+1}_{-1,-1} \ .$$

IV Polynomial preserving, right inverses of gradient, curl and div operators

$$\begin{tikzcd}
H^1(T) \arrow[r, "\boldsymbol{\nabla}"] & \boldsymbol{H}(\mathbf{curl}, T) \arrow[r, "\boldsymbol{\nabla}\times"] & \boldsymbol{H}(\mathrm{div}, T) \arrow[r, "\boldsymbol{\nabla}\circ"] & L^2(T) \\
\mathcal{P}^{p+1}_{p_e+1,p_f+1} & \mathbf{P}^{p}_{p_e,p_f} \arrow[l, "G"'] & \mathbf{P}^{p-1}_{p_f-1,p_e} \arrow[l, "K"'] & \mathcal{P}^{p-2}_{p_f} \arrow[l, "D"']
\end{tikzcd}$$

$$\begin{tikzcd}
H^{\frac{1}{2}+\epsilon}(f) \arrow[r, "\boldsymbol{\nabla}"] & \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(\mathrm{curl}, f) \arrow[r, "\boldsymbol{\nabla}\times"] & H^{-\frac{1}{2}+\epsilon}(f) \\
\mathcal{P}^{p_f+1}_{p_e+1} & \mathbf{P}^{p_f}_{p_e} \arrow[l, "G"'] & \mathcal{P}^{p_f-1} \arrow[l, "K"']
\end{tikzcd}$$

Characterization of commuting projections:

$$\begin{aligned}
P^{div}\boldsymbol{F} &= P_0^{div}(\boldsymbol{F} - DP(\boldsymbol{\nabla}\circ\boldsymbol{F})) + DP(\boldsymbol{\nabla}\circ\boldsymbol{F}) \\
P^{curl}\boldsymbol{E} &= P_0^{curl}(\boldsymbol{E} - KP_0^{div}(\boldsymbol{\nabla}\times\boldsymbol{E})) + KP_0^{div}(\boldsymbol{\nabla}\times\boldsymbol{E}) \\
P^{grad}\boldsymbol{F} &= P_0^{grad}(u - GP_0^{curl}(\boldsymbol{\nabla}u)) + GP_0^{curl}(\boldsymbol{\nabla}u)\,.
\end{aligned}$$

$P_0^{grad}, P_0^{curl}, P_0^{div}$ denote the L^2 projections on subspaces of polynomials $\mathcal{P}^{p+1}_{p_e+1,p_f+1}, \mathbf{P}^{p}_{p_e,p_f}, \mathcal{P}^{p-2}_{p_f}$ of zero gradient, curl or div, respectively.

V Projection-Based Interpolation

H^1-conforming:

$$\begin{cases} u_1(a) = u(a) \\ \|u - \Pi u\|_{\epsilon,e} \to \min \\ \|\boldsymbol{\nabla}_f(u - \Pi u)\|_{-\frac{1}{2}+\epsilon,f} \to \min \\ \|\boldsymbol{\nabla}(u - \Pi u)\|_{0,K} \to \min \end{cases}$$

$\boldsymbol{H}$(curl)-conforming:

$$\begin{cases} \displaystyle\int_e \boldsymbol{E}_t - \Pi^{curl}\boldsymbol{E}_t = 0 \\ \| \displaystyle\int (\boldsymbol{E}_t - \Pi^{curl}\boldsymbol{E}_t)\|_{0,\epsilon} \to \min \\ \begin{cases} \|\mathrm{curl}_f(\boldsymbol{E}_t - \Pi^{curl}\boldsymbol{E}_t)\|_{-\frac{1}{2}+\epsilon,f} \to \min \\ (\boldsymbol{E}_t - \Pi^{curl}\boldsymbol{E}_t, \boldsymbol{\nabla}_f\phi)_{-\frac{1}{2}+\epsilon,f} = 0, \quad \forall\phi \in \mathcal{P}^{p_f+1}_{-1} \end{cases} \\ \begin{cases} \|\boldsymbol{\nabla} \times (\boldsymbol{E} - \Pi^{curl}\boldsymbol{E})\|_{0,T} \to \min \\ (\boldsymbol{E} - \Pi^{curl}\boldsymbol{E}, \boldsymbol{\nabla}\phi)_{0,T} = 0, \quad \forall\phi \in \mathcal{P}^{p+1}_{p_f+1,p_e+1} \end{cases} \end{cases}$$

$\boldsymbol{H}(\mathrm{div})$-conforming:

$$\begin{cases} \|\boldsymbol{F}_n - \Pi^{div}\boldsymbol{F}_n\|_{-\frac{1}{2}+\epsilon,f} \to \min \\ \begin{cases} \|\boldsymbol{\nabla} \circ (\boldsymbol{F} - \Pi^{div}\boldsymbol{F})\|_{0,T} \to \min \\ (\boldsymbol{F} - \Pi^{div}\boldsymbol{F}, \boldsymbol{\nabla} \times \boldsymbol{\phi})_{0,T} = 0, \quad \forall \boldsymbol{\phi} \in \mathcal{P}^{p+1}_{p_f+1} \end{cases} \end{cases}$$

Commuting Projection and (Projection Based) Interpolation Operators for a tetrahedron and a triangular face

$$\begin{tikzcd}
\mathbb{R} \arrow[r] \arrow[d, "id"] & H^{\frac{3}{2}+\epsilon}(T) \arrow[r, "\boldsymbol{\nabla}"] \arrow[d, "\Pi", "P^1"'] & \boldsymbol{H}^{\epsilon}(\mathrm{curl},T) \cap \boldsymbol{H}^{\frac{1}{2}+\epsilon}(T) \arrow[r, "\boldsymbol{\nabla}\times"] \arrow[d, "\Pi^{\mathrm{curl}}", "P^{curl}"'] & \boldsymbol{H}^{\epsilon}(\mathrm{div},T) \arrow[r, "\boldsymbol{\nabla}\circ"] \arrow[d, "\Pi^{\mathrm{div}}", "P^{div}"'] & L^2 \arrow[d, "P"] \\
\mathbb{R} \arrow[r] & \mathcal{P}^{p+1}_{p_e+1,p_f+1} \arrow[r, "\boldsymbol{\nabla}"] & \mathbf{P}^{p}_{p_e,p_f} \arrow[r, "\boldsymbol{\nabla}\times"] & \mathbf{P}^{p-1}_{p_f-1,p_e} \arrow[r, "\boldsymbol{\nabla}\circ"] & \mathcal{P}^{p-2}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r] \arrow[d, "id"] & H^{\frac{1}{2}+\epsilon}(f) \arrow[r, "\boldsymbol{\nabla}"] \arrow[d, "\Pi", "P^{\frac{1}{2}+\epsilon}"'] & \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(\mathrm{curl},f) \arrow[r, "\boldsymbol{\nabla}\times"] \arrow[d, "\Pi^{\mathrm{curl}}"] & H^{-\frac{1}{2}+\epsilon}(f) \arrow[r] \arrow[d, "P"] & \mathbf{0} \\
\mathbb{R} \arrow[r] & \mathcal{P}^{p_f+1}_{p_e+1} \arrow[r, "\boldsymbol{\nabla}"] & \mathbf{P}^{p_f}_{p_e} \arrow[r, "\boldsymbol{\nabla}\times"] & \mathcal{P}^{p_f-2} \arrow[r] & \mathbf{0}
\end{tikzcd}$$

VI ϵ-(sub) optimal p estimates

Step 1: Comparison with the commuting projections on the element level

The interpolation error is bounded by the projection error and the interpolation error on the element boundary,

$$\begin{aligned}
\|u - \Pi u\|_{1,T} &\leq \|u - P^{grad} u\|_{1,T} + \|P^{grad} u - \Pi u\|_{1,T} \\
&\preceq \|u - P^{grad} u\|_{1,T} + \|E\|_{L(H^{\frac{1}{2}}(\partial T), H^1(T))} \|P^{grad} u - \Pi u\|_{\frac{1}{2},\partial T} \\
&\preceq \|u - P^{grad} u\|_{1,T} + \|u - P^{grad} u\|_{\frac{1}{2},\partial T} + \|u - \Pi u\|_{\frac{1}{2},\partial T} \\
&\preceq \|u - P^{grad} u\|_{1,T} + \|u - \Pi u\|_{\frac{1}{2},\partial T} \, ,
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\boldsymbol{E} - \Pi^{curl} \boldsymbol{E}\|_{curl,T} &\preceq \|\boldsymbol{E} - P^{curl} \boldsymbol{E}\|_{curl,T} + \|\boldsymbol{n} \times (\boldsymbol{E} - \Pi^{curl} \boldsymbol{E})\|_{-\frac{1}{2},curl,\partial T} \\
\|\boldsymbol{F} - \Pi^{div} \boldsymbol{F}\|_{div,T} &\preceq \|\boldsymbol{F} - P^{div} \boldsymbol{F}\|_{div,T} + \|\boldsymbol{n} \circ (\boldsymbol{F} - \Pi^{div} \boldsymbol{F})\|_{-\frac{1}{2},\partial T} \, .
\end{aligned}$$

Step 2: Localization to faces

$$\begin{aligned}
\|u\|_{\frac{1}{2},\partial T} &\leq \|u\|_{\frac{1}{2}+\epsilon,\partial T} &&\preceq \sum_f \|u|_f\|_{\frac{1}{2}+\epsilon,f} \\
\|\boldsymbol{E}\|_{-\frac{1}{2},\mathrm{curl},\partial T} &\leq \|\boldsymbol{E}\|_{-\frac{1}{2}+\epsilon,\mathrm{curl},\partial T} &&\preceq \sum_f \|\boldsymbol{E}|_f\|_{-\frac{1}{2}+\epsilon,\mathrm{curl},f} \\
\|F\|_{-\frac{1}{2},\partial T} &\leq \|F\|_{-\frac{1}{2}+\epsilon,\partial T} &&\preceq \sum_f \|F|_f\|_{-\frac{1}{2}+\epsilon,f}\,.
\end{aligned}$$

Step 3: Comparison with the commuting projections on the face level

The face interpolation error is bounded by the face projection error and the interpolation error on the face boundary,

$$\begin{aligned}
\|u - \Pi u\|_{\frac{1}{2}+\epsilon,f} &\preceq \|u - P^{grad}u\|_{\frac{1}{2}+\epsilon,f} + \|u - \Pi u\|_{\epsilon,\partial f} \\
\|\boldsymbol{E} - \Pi^{curl}\boldsymbol{E}\|_{-\frac{1}{2}+\epsilon,curl,f} &\preceq \|\boldsymbol{E} - P^{-\frac{1}{2}+\epsilon,curl}\boldsymbol{E}\|_{-\frac{1}{2}+\epsilon,curl,f} \\
&\quad + \|E_t - \Pi^{curl}\boldsymbol{E}\|_{-1+\epsilon,\partial f} \ .
\end{aligned}$$

Step 4: Estimation of the interpolation error on edges

$$\begin{aligned}
\|u - \Pi u\|_{\epsilon,\partial f} &\preceq \sum_{e\in f} \|u - \Pi u\|_{\epsilon,e} \\
&\preceq \sum_{e\in f} p_e^{r-\epsilon} \|u - \underbrace{u_1}_{\text{linear interpolant}} \|_{r,\epsilon} \\
&\preceq \sum_{e\in f} \|u\|_{r,\epsilon}
\end{aligned}$$

Here $r > \frac{1}{2}$. Similarly for the $H(\text{curl})$ estimate.

Final estimates

For every ϵ, there exists a constant $C = O(\epsilon^{-\frac{1}{2}})$ such that,

$$\begin{aligned} \|u - \Pi u\|_{1,T} &\leq Cp^{-(r-\epsilon)}\|u\|_{1+r,T} \\ \|\boldsymbol{E} - \Pi^{curl}\boldsymbol{E}\|_{0,curl,T} &\leq Cp^{-(r-\epsilon)}\|\boldsymbol{E}\|_{r,curl,T} \\ \|\boldsymbol{F} - \Pi^{div}\boldsymbol{F}\|_{0,div,T} &\leq Cp^{-(r-\epsilon)}\|\boldsymbol{F}\|_{r,div,T} \end{aligned}$$