

Intuitive vs. Computable Topological Aspects of Computational Electromagnetics

P. Robert Kotiuga

prk@bu.edu

Boston University

Intuitive problems, such as checking if a space is contractible, are easily characterized in terms of homotopy groups but, in four or more dimensions, such a characterization is provably computationally intractable. On the other hand, cohomology theory may not be intuitive, but it does provide a formal connection between Maxwell's equations and the lumped parameters occurring in Kirchhoff's laws. Also, cohomological information is efficiently extracted from the data structures used in finite element analysis. A natural question is: Do engineers need to go beyond the linear algebra and sparse matrix techniques associated with homology calculations? It turns out that there are inverse problems involving "near force-free magnetic fields" where the conjectured characterization of the space of solutions, involves computationally intractable topological invariants. Hence, it is imperative to investigate algebraic structures found in the data structures of finite element analysis, which yield topological insights not deducible from cohomological considerations alone.

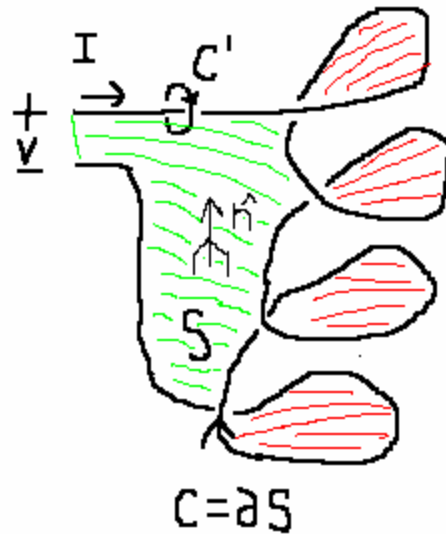
The Hurewicz map is a well-defined map taking representatives of generators of homotopy groups to their homology classes. In this sense, it provides a natural framework for comparing the intuitive but intractable with the computable but less intuitive. The presentation will develop this theme in the context of computational electromagnetics.

- 1) Kotiuga, P. R., *Topology-Based Inequalities and Inverse Problems for Near Force-Free Magnetic Fields*, IEEE Trans. MAG. March 2004.
- 2) Gross, P.W., Kotiuga, P.R., *Electromagnetic Theory and Computation: A Topological Approach*. MSRI Monograph series # 48; Cambridge U. P., 2004. ISBN # 0521801605.
- 3) Suuriniemi, S., Kettunen, L., Kotiuga, P.R., *Techniques for Systematic Treatment of Certain Coupled Problems*. IEEE Trans. MAG-39, (3), May 2003, pp 1737-1740.

Inductance Calculations and Potentials

$$L = \frac{\Phi}{I}$$

$$I = \int_{c'} H \cdot d\ell$$



$R =$ Complement
of wire relative to
 \mathbb{R}^3

$$\Phi = \int_S B \cdot \hat{n} dS = \int_{c=\partial S} A \cdot d\ell$$

$$[c'] \neq 0 \text{ in } H_1(\mathbb{R}, \mathbb{Z}),$$

$$[S] \neq 0 \text{ in } H_2(\mathbb{R}, \partial\mathbb{R}; \mathbb{Z})$$

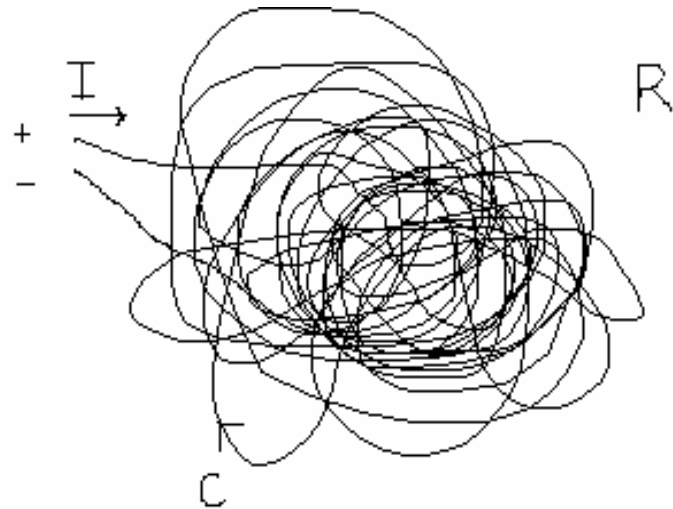
To compute inductance, we use a vector or a scalar potential.

Inductance Calculations and Potentials

Moral: If we use a scalar potential, we need the cut for two reasons:

1. To define a single valued scalar potential
2. To find the flux if we use first principles (as opposed to an energy functional)

Consider a “current carrying ball of wool” or some complicated current carrying surface.



Facts:

1. $C \approx S$ where S is an orientable embedded 2-D manifold with boundary
2. There is an algorithm to compute S as a level set of a harmonic map ($\mathbb{R}^2 \rightarrow S^1$). It has:
 - Time complexity:
 $O(n^2)$ + Complexity of “solving” an elliptic p.d.e.
 - Space complexity:
 $O(n^{4/3})$ # nodes in FE mesh required for a solution to the harmonic map problem.
3. It appears that no exact arithmetic algorithm exists which runs in polynomial time! (belted trees are not embedded manifolds).

Notes: H_* is all we need, but optimality (reordering) requires more sophisticated topological information.

ρ^* vs. H^* Algebraically

- π_i is abelian for $i > 1$, and for $i = 1$ we have a Hurewicz map : $h_i : \pi_i \rightarrow H_i$
- $i = 1$; $h_1 : \pi_1 \rightarrow H_1$ is onto, kernel $h_1 = [\pi_1, \pi_1]$
? $H_1 = \pi_1 / [\pi_1, \pi_1]$ (Poincaré)
- $i = 2$; $h_2 : \pi_2 \rightarrow H_2$ (typically neither 1-1 nor onto)
 - Ker h_2 – spheres which bound 3-manifolds in \mathbb{R}^4 which are not balls.

p^* vs. H^* Algebraically

- $H_2/\text{Image}(h_2)$ –homology classes in R necessarily having handles when realized by orientable embedded manifolds.
- Thurston Norm on $(H_2/\text{Image}(h_2))$? R (uniquely 3-D)

Computation and the Thurston Norm

In general the Thurston norm requires exponential time to compute. However “tight” bonds are computable in polynomial time. This is key for trying to make the “cuts on handles of cuts” idea (Haken Hierarchy) work.

Lower Central Series (l.c.s. (p_1))

- Recall $H_1 = \pi_1 / [\pi_1, \pi_1]$
- Define:

$$\mathbf{p}_1^1 = \mathbf{p}_1$$

$$\mathbf{p}_1^2 = [\mathbf{p}_1^1, \mathbf{p}_1^1] = [\mathbf{p}_1, \mathbf{p}_1]$$

$$\mathbf{p}_1^3 = [\mathbf{p}_1^1, \mathbf{p}_1^2] = [\mathbf{p}_1, [\mathbf{p}_1, \mathbf{p}_1]]$$

⋮

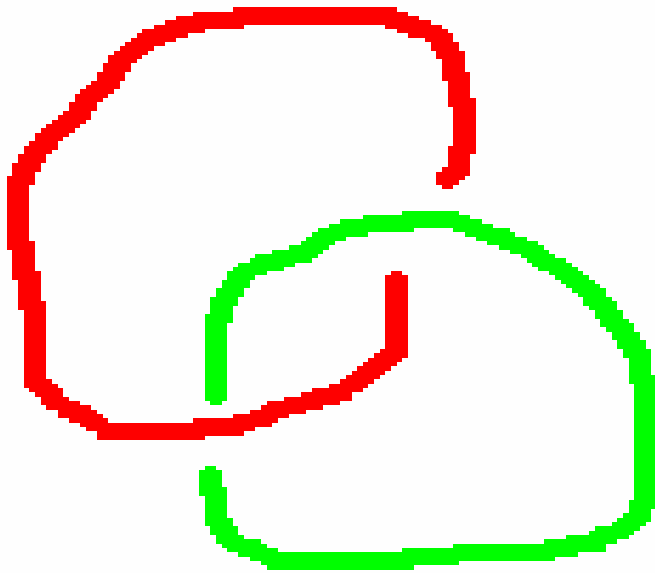
$$\mathbf{p}_1^k = [\mathbf{p}_1^1, \mathbf{p}_1^{(k-1)}]$$

$$\Rightarrow G_k = \mathbf{p}_1^k / \mathbf{p}_1^{(k-1)} \text{ Abelian } (G_1 = H_1)$$

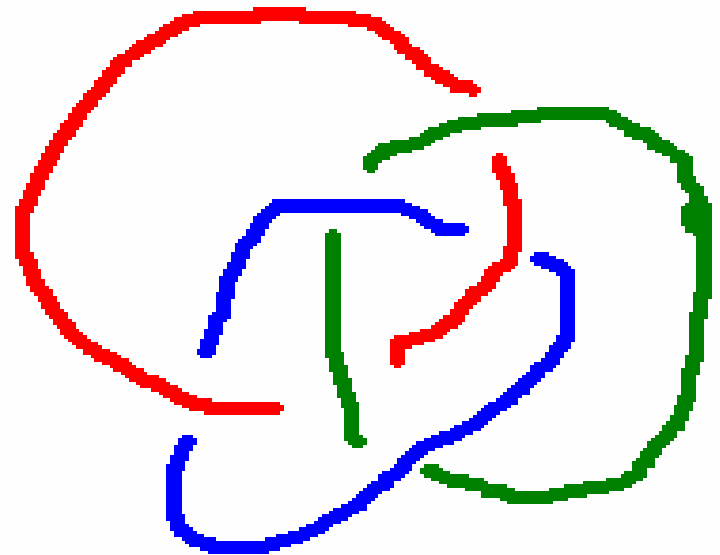
LCS, Continued

- This “data” is equivalent to (in 3-D):
 1. Massey products in the cohomology ring
 2. Differential graded Lie Algebras found in “the minimal models of rational homotopy theory”
 3. Chen’s iterated integrals for computing loop space homology.
- The l.c.s. is computable in polynomial time from FE data structures (to a given depth). Note that π_1 doesn’t seem to have this property
- Remark: l.c.s(π_1) is very effective in detection tangling, which in turn is related to π_2 being trivial.

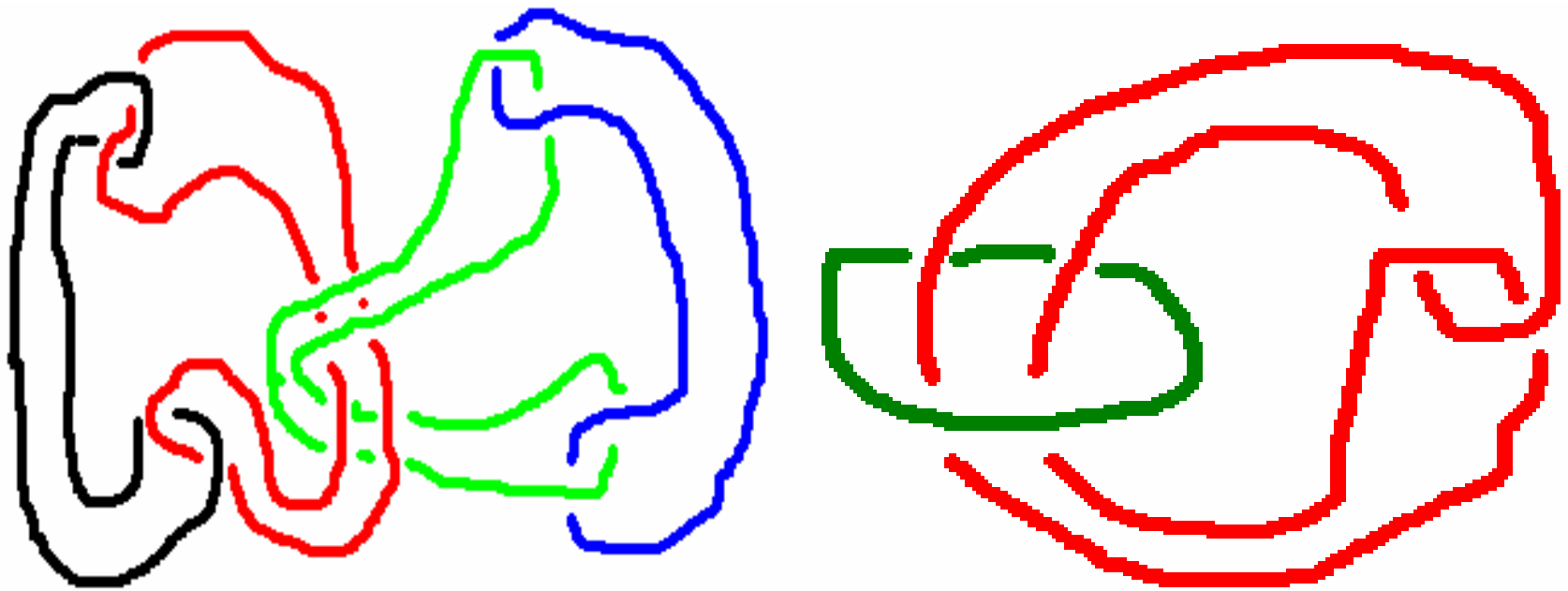
Detected at the
second level:



Detected at the
third level:



Detected at the fourth level:



...Can always find $n+1$ tangled curves which become untangled if any one is removed. This is detected at the $n+1$ st level.

Fact: I.c.s(π_1) being trivial is necessary but not sufficient for π_2 to be trivial.

Tangled (π_2 trivial) but tangling cannot be detected at any level of the I.c.s(π_1)



Problems:

There is no polynomial time algorithm to:

1. Check if π_2 is trivial
2. Check to see if a cut has minimal genus.

Hopes:

1. Obstructions to π_2 being trivial can be found in $I.c.s(\pi_1)$ - a computable measure of $[\pi_1, \pi_1]$ complexity.
2. An “Alexander Norm” can be computed in polynomial time to get a tight bound on the “Thurston Norm” (which counts the genus of the cuts.) (McMullen)
3. Recent work of Shelly Harvey refines this work.