

A Discrete Theory of Connections on Principal Bundles

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Control and Dynamical Systems

Introduction

■ Motivation and Background

- Motivated by the desire to develop a theory of **Discrete Lagrangian Reduction**.
- **Discrete Connections** are necessary to represent the Discrete Lagrange-Poincaré operator, which is a coordinate representation of the reduced Euler-Lagrange operator that describes the reduced dynamics of a G -invariant discrete Lagrangian system.
- Constructed by considering a splitting of the **Discrete Atiyah sequence** of a principal bundle.
- Recovers continuous connections in a natural manner.
- Computational applications include Discrete Levi-Civita connections and curvatures for Discrete Riemannian manifolds.

Principal and Associated Bundles

■ Definitions

- A **Principal Bundle** is a manifold Q with a free *left* action $G \times Q \rightarrow Q$ of a Lie group G , such that the natural projection $\pi : Q \rightarrow Q/G$ is a submersion.

- An **Associated Bundle** \tilde{M} with fibre M by definition,

$$\tilde{M} = Q \times_G M = (Q \times M)/G.$$

In particular, two associated bundles arise when considering the continuous and discrete Atiyah sequence of a principal bundle:

- $\tilde{\mathfrak{g}}$, where the action of G on $Q \times \mathfrak{g}$ is given by $g(q, \xi) = (gq, Ad_g \xi)$, and $\pi_{\mathfrak{g}} : \tilde{\mathfrak{g}} \rightarrow Q/G$ is given by $\pi_{\mathfrak{g}}([q, \xi]_G) = \pi(q)$.
- \tilde{G} , where the action of G on $Q \times G$ is given by $g(q, h) = (gq, ghg^{-1})$, and $\pi_G : \tilde{G} \rightarrow Q/G$ is given by $\pi_G([q, g]_G) = \pi(q)$.

Discrete Atiyah Sequence

■ Discrete Atiyah Sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \tilde{G} & \xrightarrow{i} & (Q \times Q)/G & \xrightarrow{(\pi, \pi)} & S \times S \longrightarrow 0 \\
 & & \parallel & & \downarrow \alpha_{\mathcal{A}_d} & & \parallel \\
 & & 1_{\tilde{G}} & & & & 1_{S \times S} \\
 & & \parallel & & & & \parallel \\
 0 & \longrightarrow & \tilde{G} & \xrightarrow{i_1} & \tilde{G} \oplus (S \times S) & \xrightarrow{\pi_2} & S \times S \longrightarrow 0 \\
 & & \parallel & & \downarrow \pi_1 & & \parallel \\
 & & & & & & & & \parallel \\
 & & & & & & & & i_2
 \end{array}$$

■ Maps

- $i : \tilde{G} \rightarrow (Q \times Q)/G$, where,

$$i([q, g]_G) = [q, gq]_G.$$

- $(\pi, \pi) : (Q \times Q)/G \rightarrow S \times S$, where,

$$(\pi, \pi)([q_0, q_1]_G) = (\pi q_0, \pi q_1).$$

Discrete Connection Form

■ Definition

A **Discrete Connection Form** is a continuous map,

$$\mathcal{A}_d : Q \times Q \rightarrow G,$$

such that,

- \mathcal{A}_d is G -equivariant.

$$\mathcal{A}_d \circ L_g = I_g \circ \mathcal{A}_d.$$

This is the discrete analogue of the statement, $\mathcal{A} \circ L_g = Ad_g \circ \mathcal{A}$.

- \mathcal{A}_d induces a splitting of the Discrete Atiyah sequence.

$$\mathcal{A}_d(i_q(g)) = g.$$

This is the discrete analogue of the statement, $\mathcal{A}(\xi_Q) = \xi$.

Discrete Horizontal and Vertical Spaces

Horizontal Space

$$\text{Hor}_q = \{(q, q') \in Q \times Q \mid \mathcal{A}_d(q, q') = e\}.$$

This is the discrete analogue of the statement $\text{Hor}_q = \{v_q \in TQ \mid \mathcal{A}(v_q) = 0\}$.

Vertical Space

$$\text{Ver}_q = \{(q, q') \in Q \times Q \mid (\pi, \pi)(q, q') = e_{S \times S}\} = \{i_q(g) \mid g \in G\}.$$

This is the discrete analogue of the statement $\text{Ver}_q = \{v_q \in TQ \mid \pi_*(v_q) = 0\} = \{\xi_Q \mid \xi \in \mathfrak{g}\}$.

Horizontal Component

$$\text{hor}(q_0, q_1) = (\cdot, \cdot)_{q_0}^h \circ (\pi, \pi)(q_0, q_1) = (q_0, (\mathcal{A}_d(q_0, q_1))^{-1} q_1).$$

Vertical Component

$$\text{ver}(q_0, q_1) = i_{q_0} \circ (\pi_1, \mathcal{A}_d)(q_0, q_1) = (q_0, \mathcal{A}_d(q_0, q_1)q_0).$$

Decomposition

$$\text{hor}(q_0, q_1) \cdot \text{ver}(q_0, q_1) = (q_0, \mathcal{A}_d(q_0, q_1)(\mathcal{A}_d(q_0, q_1))^{-1} q_1) = (q_0, q_1).$$

Isomorphism between $(Q \times Q)/G$ and $(S \times S) \oplus \tilde{G}$

Discrete Atiyah Sequence

$$0 \longrightarrow \tilde{G} \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{(\pi_1, \mathcal{A}_d)} \end{array} (Q \times Q)/G \begin{array}{c} \xrightarrow{(\pi, \pi)} \\ \xleftarrow{(\cdot, \cdot)^h} \end{array} S \times S \longrightarrow 0$$

Lemma

The map $\alpha_{\mathcal{A}_d} : (Q \times Q)/G \rightarrow (S \times S) \oplus \tilde{G}$ defined by,

$$\alpha_{\mathcal{A}_d}([q_0, q_1]_G) = (\pi q_0, \pi q_1) \oplus [q_0, \mathcal{A}_d(q_0, q_1)]_G,$$

is a well-defined bundle isomorphism.

The inverse of $\alpha_{\mathcal{A}_d}$ is given by,

$$\alpha_{\mathcal{A}_d}^{-1}((x_0, x_1) \oplus [q, g]_G) = [(e, g) \cdot (x_0, x_1)_q^h]_G,$$

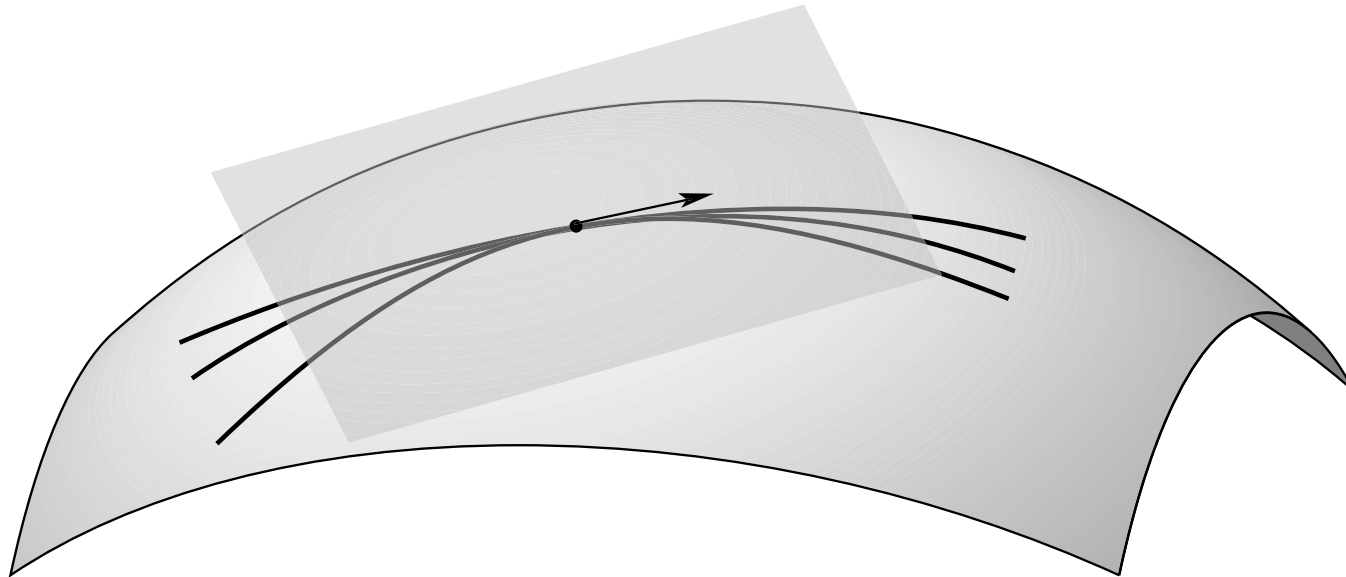
for any $q \in Q$ such that $\pi q = x_0$.

Continuous Connections from Discrete Connections

- Given a discrete G -valued connection 1-form $\mathcal{A}_d : Q \times Q \rightarrow G$, we associate with it a continuous \mathfrak{g} -valued connection 1-form $\mathcal{A} : TQ \rightarrow \mathfrak{g}$ by the following construction,

$$\mathcal{A}([q(\cdot)]) = [\mathcal{A}_d(q(0), q(\cdot))],$$

where $[\cdot]$ denotes the **equivalence class of curves** associated with a tangent vector.



Applications to Discrete Lagrangian Reduction

■ Discrete Lagrange–Poincaré operator

- From the **Reduced Discrete Euler–Lagrange** operator,

$$[EL_d(L_d)]_G : Q^3/G \rightarrow T^*Q/G,$$

we obtain the **Discrete Lagrange–Poincaré** operator,

$$\mathcal{LP}_d(l_d) : (Q/G)^3 \times_{Q/G} (\tilde{G} \oplus \tilde{G}) \rightarrow T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*,$$

by using a **discrete connection** to identify

$$Q^3/G \quad \text{with} \quad (Q/G)^3 \times_{Q/G} (\tilde{G} \oplus \tilde{G}),$$

and a **continuous connection** to identify

$$T^*Q/G \quad \text{with} \quad T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*.$$

Discrete Levi-Civita Connection

■ Discrete Riemannian manifold

- Cartan's perspective: Bundle of oriented orthonormal frames over a manifold as a principal $SO(n)$ bundle.
- Semidiscretize by discretizing manifold with a simplicial complex, but keeping the group $SO(n)$ continuous.
- Associate with each n -simplex a metric tensor g .

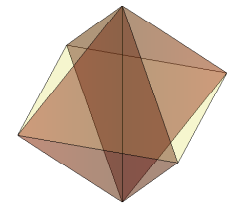
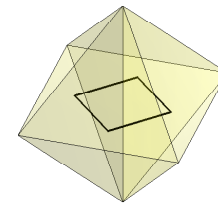
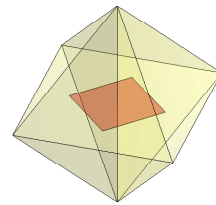
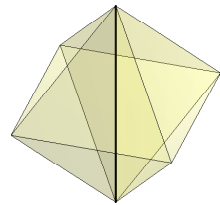
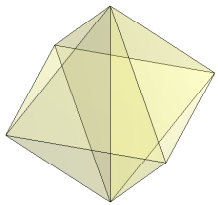
■ Constructing the Levi-Civita connection

- The Levi-Civita connection is a $SO(n)$ -valued **dual one-form**.
- This element of $SO(n)$ transforms the frame associated with a n -simplex into the frame associated with an adjacent n -simplex, and is assigned to the codimension-one face common to both simplices.

Discrete Levi-Civita Connection

Curvature

- Curvature is a dual two-form, and is associated with the dual of a codimension-two simplex, given by $\star\sigma^{n-2}$.



Simplicial
Complex,
 K

$n - 2$ primal
simplex,
 σ^{n-2}

2 dual
cell,
 $\star\sigma^{n-2}$

1 dual
chain,
 $\partial \star \sigma^{n-2}$

$n - 1$ primal
chain,
 $\star\partial \star \sigma^{n-2}$

- The curvature \mathcal{B} of the discrete Levi-Civita connection is given by,

$$\langle \mathcal{B}, \star\sigma^{n-2} \rangle = \langle \mathcal{A}, \star\sigma^{n-2} \rangle = \langle A, \partial \star \sigma^{n-2} \rangle.$$

Conclusion

■ Summary

- Constructed a Discrete Atiyah sequence.
- Described equivalent representations of a discrete connection.
- Continuous connections as limits of discrete connections.
- Applications to Discrete Lagrangian Reduction.

■ Future Directions

- Further examine the links with groupoid theory.
- Computationally efficient methods of constructing discrete connections that limit to a prescribed continuous connection.
- Discrete Connections for Semidiscrete Principal Bundles.