New Mimetic Discretizations of Diffusion-type Problems on Polygonal Meshes

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Properties of the discretization

- locally conservative;
- symmetric, i.e. $\mathcal{G}_e^h = \left(\mathcal{D}_e^h\right)^*$;
- can accommodate full diffusion tensor;
- solution algorithm results in a SPD matrix;
- exact for linear solutions;
- second order accurate for cell-centered pressures on general polygonal meshes including AMR, non-matching and non-convex meshes;
- at least first order accurate for fluxes;
- easy to extend to 3D polyhedral meshes;
- can be extended to some other PDEs.

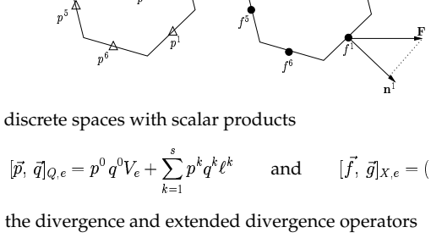
Model problem

Consider the Dirichlet boundary value problem:

$$\begin{aligned} \mathrm{div}\ \mathbf{F} &= b\,, \\ \mathbf{F} &= -\mathbf{K}\ \mathrm{grad}\ p. \end{aligned}$$

Support-operators discretization on element

- define degrees of freedom for p and $\mathbf{F}$:



$f^i = \mathbf{F}\cdot\mathbf{n}^i$

- equip the discrete spaces with scalar products

$$[\vec{p},\,\vec{q}]_{Q,e} = p^0\,q^0V_e + \sum_{k=1}^{s} p^kq^k\ell^k \qquad \text{and} \qquad [\vec{f},\,\vec{g}]_{X,e} = (M_e\vec{f},\,\vec{g});$$

- discretize the divergence and extended divergence operators

$$\mathcal{DIV}_e^h\,\vec{f} = \frac{1}{V_e}\sum_{k=1}^{s} f^k\,\ell^k, \qquad \mathcal{D}_e^h\,\vec{f} = (\mathcal{DIV}_e^h\,\vec{f},\,-f^1,-f^2,\dots,-f^s)^T;$$

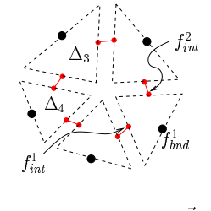
- derive the discrete flux operator from the Green formula

$$[\vec{f},\,\mathcal{G}_e^h\,\vec{p}]_{X,e} = [\vec{p},\,\mathcal{D}_e^h\,\vec{f}]_{Q,e}, \qquad \forall\vec{f},\ \forall\vec{p}.$$

The discrete equations for element e are

$$\begin{aligned} \mathcal{DIV}_e^h\ \vec{f} &= b_e, \\ \vec{f} &= \mathcal{G}_e^h\ \vec{p}. \end{aligned}$$

Derivation of matrix M_e



1. Split the element e into T triangles Δ_t;
2. Build and assemble mass matrices for the triangles. Let $\hat{M}_e$ be the global mass matrix.
3. Eliminate interior fluxes by solving the constraint optimization problem for a given vector $\vec{f}_{bnd} \equiv \vec{f}$:

$$(M_e\vec{f}_{bnd},\ \vec{f}_{bnd}) \equiv \min_{\vec{f}_{int}} \left(\hat{M}_e \begin{bmatrix}\vec{f}_{bnd}\\ \vec{f}_{int}\end{bmatrix},\ \begin{bmatrix}\vec{f}_{bnd}\\ \vec{f}_{int}\end{bmatrix}\right)$$

subject to

$$\mathcal{DIV}_t^h\,\vec{f}_t = \mathcal{DIV}_e^h\,\vec{f}_{bnd},$$

where $\vec{f}_1 = (f^1_{bnd},\,f^2_{int},\,f^1_{int})^T$ and $\mathcal{DIV}_t^h$ is the divergence for Δ_t, $1 \le t \le T$.

Interface conditions

For each mesh edge $\partial e_i \cap \partial e_j$, we impose 2 continuity conditions:

$$f_i^k = -f_j^l \quad \text{and} \quad p_i^k = p_j^l.$$

References

1. Kuznetsov Y, Lipnikov K, Shashkov M (2004) Mimetic finite difference method on polygonal meshes for diffusion-type problems. LAUR 03-7608, submitted to *Comp. Geosciences*.

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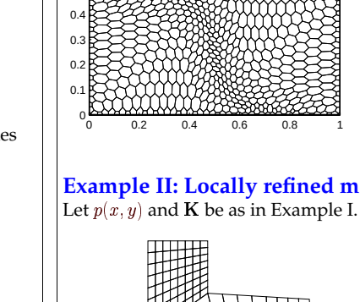
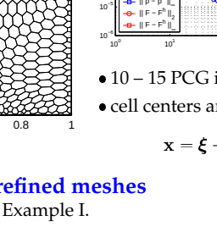
Example I: Polygonal meshes

Let $p(x,\,y) = x^3y^2 + x\cos(xy)\sin(x)$ be the exact solution and

$$\mathbf{K}(x,\,y) = \begin{pmatrix} (x+1)^2+y^2 & -xy \\ -xy & (x+1)^2 \end{pmatrix}.$$



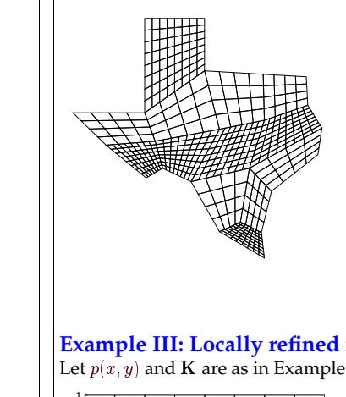


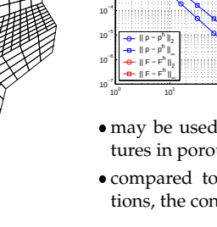
- 10 – 15 PCG iterations for $\epsilon = 10^{-12}$;
- cell centers are given by the mapping

$$\mathbf{x} = \boldsymbol{\xi} + 0.1\ \sin(2\pi\xi_x)\sin(2\pi\xi_y).$$

Example II: Locally refined meshes

Let $p(x,y)$ and $\mathbf{K}$ be as in Example I.

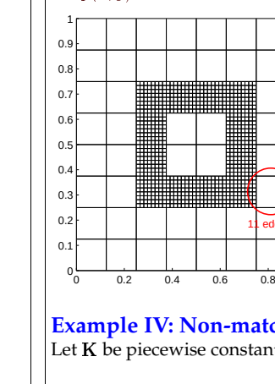


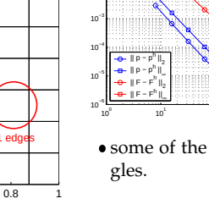


- may be used for modeling layered structures in porous media;
- compared to other low-order discretizations, the convergence rate is optimal.

Example III: Locally refined meshes (strong refinement)

Let $p(x,y)$ and $\mathbf{K}$ are as in Example I.



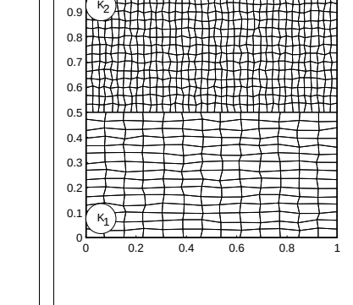


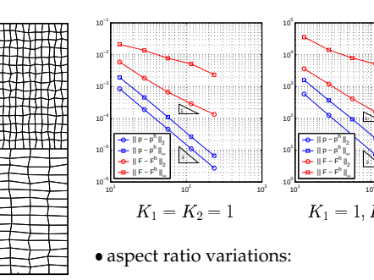
- some of the polygons have a few 180° angles.

Example IV: Non-matching meshes

Let $\mathbf{K}$ be piecewise constant and the exact solution be

$$p(x,\,y) = \begin{cases} \dfrac{7}{16} - \dfrac{K_2}{12K_1} + \dfrac{2K_2}{3K_1}y^3, & y < 0.5, \\ y - y^4, & y \ge 0.5. \end{cases}$$





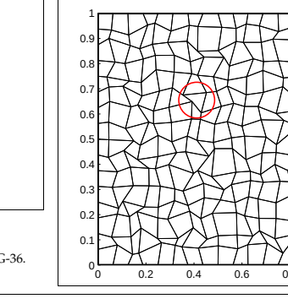
$K_1 = K_2 = 1$ $\qquad$ $K_1 = 1, K_2 = 10^6$

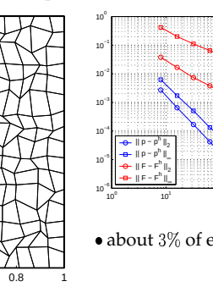
- aspect ratio variations:

$$167 < \max_{cells} \frac{\max_k \ell^k}{\min_k \ell^k} < 2024.$$

Example V: Meshes with non-convex polygons

Let $p(x,y)$ and $\mathbf{K}$ be as in Example I.





- about 3% of elements ane non-convex.