Discontinuous Galerkin Methods for Maxwell's Equations in Frequency–Domain

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1 Maxwell's Equations

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, find the electric field $\mathbf{u}$ such that

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{u}) - \omega^2(\varepsilon - i\omega^{-1}\sigma)\mathbf{u} = \mathbf{j} \quad \text{in } \Omega, \qquad (1)$$

$$\mathbf{n} \times \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma, \qquad (2)$$

where $\omega > 0$, μ, ε and σ are the temporal frequency, magnetic permeability, electric permittivity, and electric conductivity, respectively.
Assumption: μ and ε constant (for simplicity).

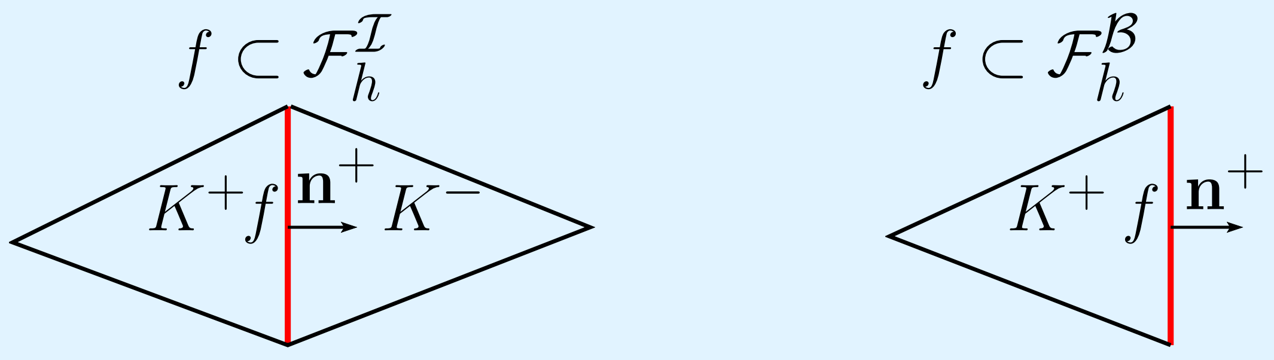
2 DG Discretizations

Main Advantages of DG Methods:

- DG methods are locally conservative;
- Greater robustness means that a wide range of problems may be treated within the same unified framework;
- Finite element meshes with hanging nodes and local spaces of different orders can easily be handled.

2.1 Notation

- $\mathcal{T}_h = \{K\}$ tetrahedral mesh of granularity h;
- $\mathcal{F}_h^{\mathcal{I}}/\mathcal{F}_h^{\mathcal{B}}$ is the set of all interior/boundary faces;
- Trace operators:



$$[\![\mathbf{v}]\!]_T = \mathbf{n}^+ \times \mathbf{v}^+ + \mathbf{n}^- \times \mathbf{v}^-, \qquad [\![\mathbf{v}]\!]_T = \mathbf{n} \times \mathbf{v},$$

$$[\![\mathbf{v}]\!]_N = \mathbf{v}^+ \cdot \mathbf{n}^+ + \mathbf{v}^- \cdot \mathbf{n}^-,$$

$$[\![q]\!]_N = q^+\mathbf{n}^+ + q^-\mathbf{n}^-, \qquad [\![q]\!]_N = q\,\mathbf{n},$$

$$\{\!\{\mathbf{v}\}\!\} = (\mathbf{v}^+ + \mathbf{v}^-)/2, \qquad \{\!\{\mathbf{v}\}\!\} = \mathbf{v},$$

$$\{\!\{q\}\!\} = (q^+ + q^-)/2, \qquad \{\!\{q\}\!\} = q;$$

- Finite Element Space:

$$S_h^\ell = \{v \in L^2(\Omega) : v|_K \in \mathcal{P}^\ell(K)\ \forall K \in \mathcal{T}_h\}, \quad \ell \geq 1.$$

- DG norms:

$$\|\mathbf{v}\|_{\mathbf{V}(h)}^2 = \|\varepsilon^{\frac12}\mathbf{v}\|_{0,\Omega}^2 + \|\mu^{-\frac12}\nabla_h \times \mathbf{v}\|_{0,\Omega}^2 + \|\mu^{-\frac12}h^{-\frac12}[\![\mathbf{v}]\!]_T\|_{0,\mathcal{F}_h}^2,$$

$$\|q\|_{Q(h)}^2 = \|\varepsilon^{\frac12}\nabla_h q\|_{0,\Omega}^2 + \|\varepsilon^{\frac12}h^{-\frac12}[\![q]\!]_N\|_{0,\mathcal{F}_h}^2,$$

$$\|\mathbf{v}\|_{DG}^2 = \|\mu^{-\frac12}\nabla_h \times \mathbf{v}\|_{0,\Omega}^2 + \|\mu^{-\frac12}\mathtt{h}^{-\frac12}[\![\mathbf{v}]\!]_T\|_{0,\mathcal{F}_h}^2 + \|\mu^{-\frac12}\nabla_h \cdot \mathbf{v}\|_{0,\Omega}^2 + \|\mu^{-\frac12}\mathtt{h}^{-\frac12}[\![\varepsilon\mathbf{v}]\!]_N\|_{0,\mathcal{F}_h}^2.$$

2.2 Discrete Curl-Curl Operator

The interior penalty discretization is defined by:

$$a_h(\mathbf{u},\mathbf{v}) = (\mu^{-1}\nabla_h \times \mathbf{u}, \nabla_h \times \mathbf{v})_\Omega + \int_{\mathcal{F}_h} \mathtt{a}\,\mu^{-1}[\![\mathbf{u}]\!]_T \cdot [\![\overline{\mathbf{v}}]\!]_T\,ds - \int_{\mathcal{F}_h} \left([\![\mathbf{u}]\!]_T \cdot \{\!\{\mu^{-1}\nabla_h \times \overline{\mathbf{v}}\}\!\} + [\![\overline{\mathbf{v}}]\!]_T \cdot \{\!\{\mu^{-1}\nabla_h \times \mathbf{u}\}\!\}\right)\,ds,$$

where the *interior penalty stabilization function* $\mathtt{a}$ is given by

$$\mathtt{a} := \alpha\,h^{-1}, \quad \alpha > \alpha_{\min} \geq 0.$$

3 Low-Frequency Problem

Neglecting $\omega^2\varepsilon$ in (1) and assuming $\sigma \equiv 0$ gives:

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{u}) = \mathbf{j} \quad \text{in } \Omega,$$

$$\nabla \cdot (\varepsilon\mathbf{u}) = 0 \quad \text{in } \Omega, \quad \mathbf{n} \times \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma.$$

Key Problem: The design of numerical schemes which incorporate the divergence free constraint.

3.1 Regularized DG Formulation

Find $\mathbf{u}_h \in (S_h^\ell)^3$ such that

$$a_h(\mathbf{u}_h,\mathbf{v}) + r_h(\mathbf{u}_h,\mathbf{v}) = (\mathbf{j},\mathbf{v})_\Omega \quad \forall \mathbf{v} \in (S_h^\ell)^3,$$

where the form $r_h(\cdot,\cdot)$ is defined by

$$r_h(\mathbf{u},\mathbf{v}) = (\mu^{-1}\nabla_h \cdot \mathbf{u}, \nabla_h \cdot \mathbf{v})_\Omega - \int_{\mathcal{F}_h^{\mathcal{I}}} [\![\varepsilon\mathbf{u}]\!]_N\,\{\!\{\mu^{-1}\nabla_h \cdot (\varepsilon\overline{\mathbf{v}})\}\!\}\,ds - \int_{\mathcal{F}_h^{\mathcal{I}}} [\![\varepsilon\overline{\mathbf{v}}]\!]_N\,\{\!\{\mu^{-1}\nabla_h \cdot (\varepsilon\mathbf{u})\}\!\}\,ds + \int_{\mathcal{F}_h^{\mathcal{I}}} \mathtt{d}\mu^{-1}[\![\varepsilon\mathbf{u}]\!]_N \cdot [\![\varepsilon\overline{\mathbf{v}}]\!]_N\,ds,$$

and $\mathtt{d} := \delta\,h^{-1}, \quad \delta > \delta_{\min} \geq 0.$

Theorem 1 *([4]) Assuming* $\mathbf{u} \in H^{s+1}(\mathcal{T}_h)^3$, $s > 1/2$, *we have*

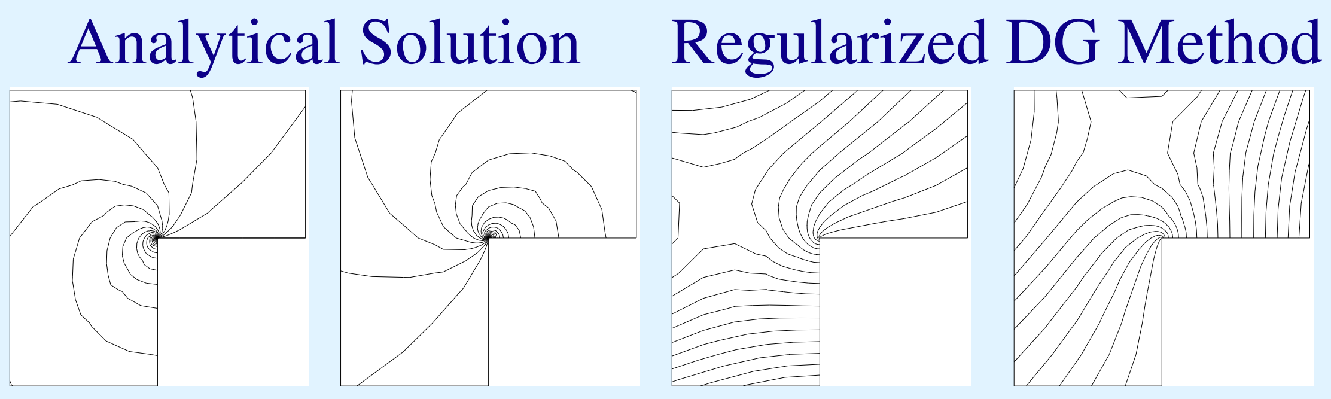
$$\|\mathbf{u} - \mathbf{u}_h\|_{DG} \leq C\,h^{\min\{s,\ell\}}\,\|\mathbf{u}\|_{s+1,\mathcal{T}_h}.$$

Regularity assumptions are *not* minimal
⇒ DG method may *not* converge for nonsmooth problems.

3.2 Example

Let $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$ and select appropriate data so that $\mathbf{u}$ is given, in terms of the polar coordinates (r,ϑ), by

$$\mathbf{u}(x,y) = \nabla(r^{2/3}\sin(2\vartheta/3)) \quad \Rightarrow \quad \mathbf{u} \in H^{2/3-\varepsilon}(\Omega)^2,\ \varepsilon > 0.$$


Analytical Solution

Regularized DG Method

3.3 Mixed DG Formulation

Introducing the Lagrange multiplier $p \in H_0^1(\Omega)$ gives:

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{u}) - \varepsilon\nabla p = \mathbf{j} \quad \text{in } \Omega,$$

$$\nabla \cdot (\varepsilon\mathbf{u}) = 0 \quad \text{in } \Omega, \quad \mathbf{n} \times \mathbf{u} = \mathbf{0}, \quad p = 0 \quad \text{on } \Gamma.$$

Mixed DG methods: Find $(\mathbf{u}_h, p_h)$ in $(S_h^\ell)^3 \times S_h^m$ such that

$$a_h(\mathbf{u}_h,\mathbf{v}) + s_h(\mathbf{u}_h,\mathbf{v}) + b_h(\overline{\mathbf{v}},p_h) = (\mathbf{j},\mathbf{v})_\Omega,$$

$$b_h(\mathbf{u}_h,\overline{q}) - c_h(p_h,q) = 0$$

for all $(\mathbf{v},q) \in (S_h^\ell)^3 \times S_h^m$, where

$$s_h(\mathbf{u},\mathbf{v}) = \int_{\mathcal{F}_h^{\mathcal{I}}} \mathtt{b}[\![\mathbf{u}]\!]_N[\![\overline{\mathbf{v}}]\!]_N\,ds, \quad c_h(p,q) = \int_{\mathcal{F}_h} \mathtt{c}\varepsilon[\![p]\!]_N \cdot [\![\overline{q}]\!]_N\,ds,$$

$$b_h(\mathbf{v},p) = -(\varepsilon\mathbf{v}, \nabla_h p) + \int_{\mathcal{F}_h} \{\!\{\varepsilon\mathbf{v}\}\!\} \cdot [\![p]\!]_N\,ds,$$

and

$$\mathtt{b} := \beta\,h, \quad \beta \geq 0, \qquad \mathtt{c} := \gamma\,h^{-1}, \quad \gamma > 0.$$

- Method I (stabilized): $\beta > 0$ and $m = \ell$.
- Method II (non-stabilized): $\beta = 0$ and $m = \ell + 1$.

Theorem 2 *([6, 5]) Assuming that*

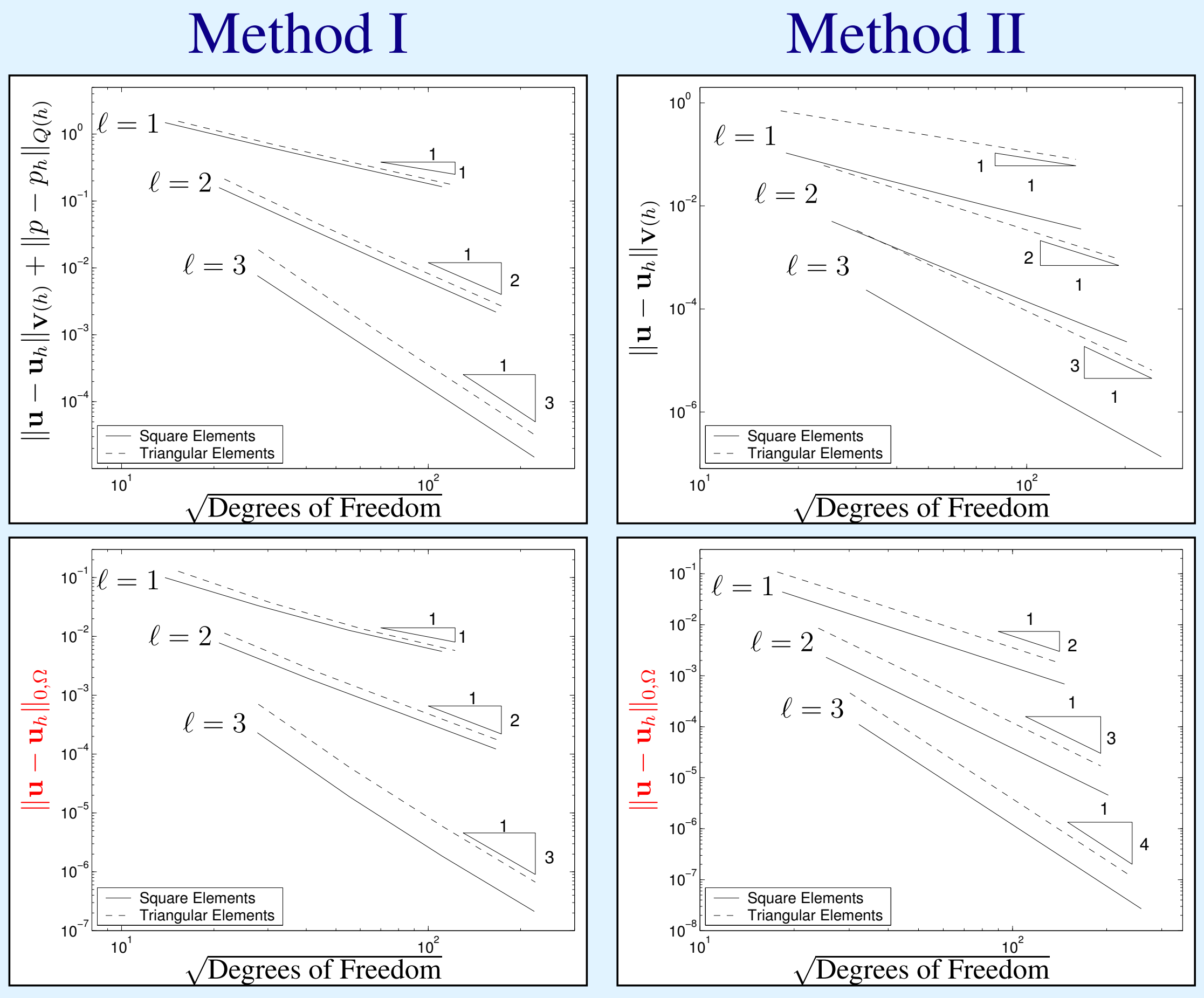
$$\mathbf{u} \in H^s(\mathcal{T}_h)^3, \quad \nabla \times \mathbf{u} \in H^s(\mathcal{T}_h)^3, \quad p \in H^{s+1}(\mathcal{T}_h), \quad s > 1/2,$$

both methods satisfy the following a priori error bound:

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}(h)} + \|p - p_h\|_{Q(h)} \leq C\,h^{\min\{s,\ell\}}\left[\|\varepsilon\mathbf{u}\|_{s,\mathcal{T}_h} + \|\mu^{-1}\nabla \times \mathbf{u}\|_{s,\mathcal{T}_h} + \|p\|_{s+1,\mathcal{T}_h}\right].$$

3.4 Example (Smooth Solution)

We set $\Omega = (-1,1)^2$ and select appropriate data so that $\mathbf{u} = (-\exp(x)(y\cos(y) + \sin(y)), \exp(x)y\sin(y))$ and $p = \sin(\pi(x-1)/2)\sin(\pi(y-1)/2)$.


Method I

Method II

⇒ Method I is *suboptimal* in the $L^2(\Omega)$ norm.

3.5 A Posteriori Error Estimation

Theorem 3 *([3]) Assuming that* $\nabla \cdot \mathbf{j} = 0$ *holds, so that* $p \equiv 0$, *Method II satisfies the following a posteriori error bound:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}(h)} + \|p - p_h\|_{Q(h)} \leq C\left(\sum_{K \in \mathcal{T}_h} \eta_K^2\right)^{1/2},$$

where the elemental error indicator η_K *is given by*

$$\eta_K^2 = h_K^{2\sigma}\|\mathbf{j} - \nabla \times (\mu^{-1}\nabla \times \mathbf{u}_h) + \varepsilon\nabla p_h\|_{0,K}^2 + h_K^{2\sigma-1}\|\widehat{\tau}_K(\mathbf{u}_h) - \tau_K(\mathbf{u}_h)\|_{0,\partial K}^2 + h_K^{-1}\|\mu^{\frac12}[\![\mathbf{u}_h]\!]_T\|_{0,\partial K}^2 + h_K\|[\![\varepsilon\mathbf{u}_h]\!]_N\|_{0,\partial K\setminus\Gamma}^2 + h_K^2\|\nabla \cdot (\varepsilon\mathbf{u}_h)\|_{0,K}^2 + \|\varepsilon^{\frac12}\nabla p_h\|_{0,K}^2 + h_K^{-1}\|\varepsilon^{\frac12}[\![p_h]\!]_N\|_{0,\partial K}^2,$$

$\sigma \in (1/2,1]$ *denotes the parameter within the embeddings* $H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega) \hookrightarrow H^\sigma(\Omega)^3$ *and* $H(\mathrm{curl};\Omega) \cap H_0(\mathrm{div};\Omega) \hookrightarrow H^\sigma(\Omega)^3$, *and* $\widehat{\tau}_K(\mathbf{v})$ *is the* numerical flux*:*

$$\widehat{\tau}_K(\mathbf{v}) = \begin{cases} \mathbf{n}_K \times (\{\!\{\mu^{-1}\nabla \times \mathbf{v}\}\!\} - \mu^{-1}\mathtt{a}\,[\![\mathbf{v}]\!]_T) & \text{on } \partial K \setminus \Gamma, \\ \mathbf{n}_K \times (\mu^{-1}\nabla \times \mathbf{v} - \mu^{-1}\mathtt{a}\,(\mathbf{n}_K \times \mathbf{v})) & \text{on } \partial K \cap \Gamma. \end{cases}$$

4 High-Frequency Problem

Renaming $(\varepsilon - i\omega^{-1}\sigma)$ by ε, problem (1)–(2) becomes

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{u}) - \omega^2\varepsilon\mathbf{u} = \mathbf{j} \quad \text{in } \Omega, \quad \mathbf{n} \times \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma.$$

Here, we assume that ω^2 is not an eigenvalue of the underlying Maxwell eigenproblem.

4.1 Direct DG Method

Find $\mathbf{u}_h \in (S_h^\ell)^3$ such that

$$a_h(\mathbf{u}_h,\mathbf{v}) - \omega^2(\varepsilon\mathbf{u}_h,\mathbf{v})_\Omega = (\mathbf{j},\mathbf{v})_\Omega \quad \forall \mathbf{v} \in (S_h^\ell)^3.$$

Theorem 4 *([2]) Assuming that* $\mathbf{u} \in H^s(\mathcal{T}_h)^3$ *and* $\nabla \times \mathbf{u} \in H^s(\mathcal{T}_h)^3$, $s > 1/2$, *for* $0 < h \leq h_0$, *we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}(h)} \leq C\,h^{\min\{s,\ell\}}\left[\|\varepsilon\mathbf{u}\|_{s,\mathcal{T}_h} + \|\mu^{-1}\nabla \times \mathbf{u}\|_{s,\mathcal{T}_h}\right].$$

Additionally, given that $\mathbf{u} \in H^{\ell+1}(\mathcal{T}_h)^3$ *and* Ω *is convex,*

$$\|\varepsilon^{\frac12}(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega} \leq C\,h^{\ell+1}\,\|\mathbf{u}\|_{\ell+1,\mathcal{T}_h}, \quad \text{for } 0 < h \leq h_1.$$

4.2 Mixed DG Method

Writing $\mathbf{u} = \mathbf{w} + \nabla\varphi$, with $\varphi \in H_0^1(\Omega)$ and $\mathbf{w} \in H_0(\mathrm{curl};\Omega)$, $\nabla \cdot \mathbf{w} = 0$, setting $p := \omega^2\varphi$ and renaming $\mathbf{w}$ by $\mathbf{u}$ gives:

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{u}) - \omega^2\varepsilon\mathbf{u} - \varepsilon\nabla p = \mathbf{j} \quad \text{in } \Omega,$$

$$\nabla \cdot (\varepsilon\mathbf{u}) = 0 \quad \text{in } \Omega, \quad \mathbf{n} \times \mathbf{u} = \mathbf{0}, \quad p = 0 \quad \text{on } \Gamma.$$

Mixed DG Method: find $(\mathbf{u}_h, p_h)$ in $(S_h^\ell)^3 \times S_h^{\ell+1}$, such that

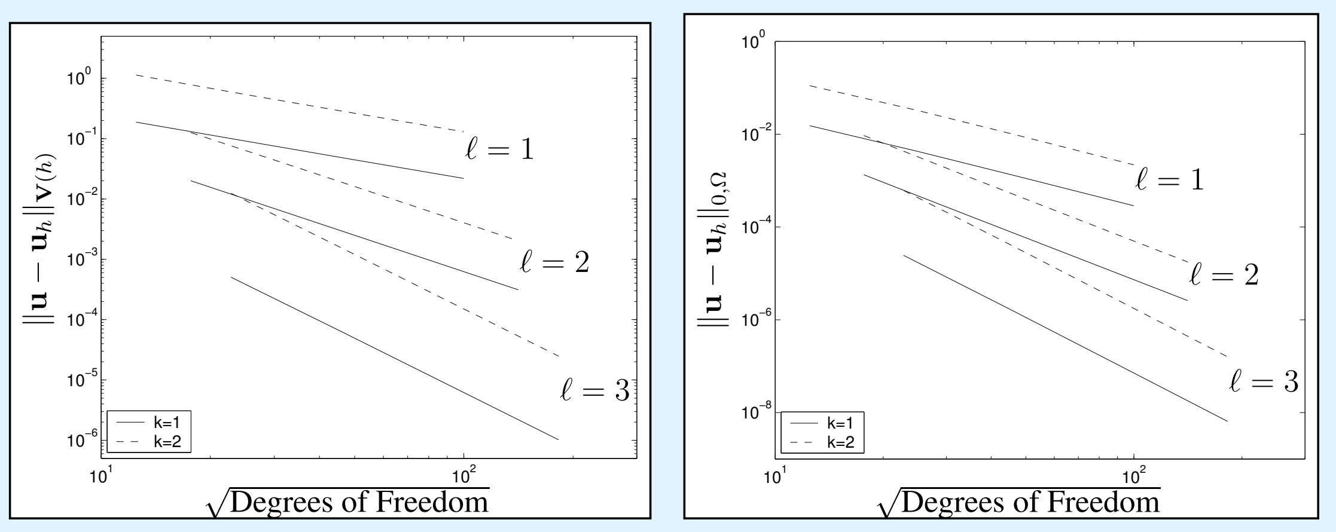
$$a_h(\mathbf{u}_h,\mathbf{v}) - \omega^2(\varepsilon\mathbf{u}_h,\mathbf{v})_\Omega + b_h(\overline{\mathbf{v}},p_h) = (\mathbf{j},\mathbf{v})_\Omega,$$

$$b_h(\mathbf{u}_h,\overline{q}) - c_h(p_h,q) = 0$$

for all $(\mathbf{v},q) \in (S_h^\ell)^3 \times S_h^{\ell+1}$.
⇒ Optimal energy and $L^2(\Omega)$–norm error bounds (see [1]).
Main Advantage: Mixed formulation applicable to the full PDE system in *both* low- and high-frequency regimes.

4.3 Example - Mixed DG Method

We set $\mu = \mu_0$, $\varepsilon = \varepsilon_0$, $\Omega = (-1,1)^2$, $\mathbf{j} = \mathbf{0}$, and select appropriate boundary conditions so that $\mathbf{u} = (\sin(ky), \sin(kx))$ and $p = 0$, where $k = \omega\sqrt{\mu_0\varepsilon_0}$ is the wave number.



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