FINITE VOLUME ELEMENT METHOD FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATION WITH NONSMOOTH INITIAL DATA

RAJEN K. SINHA* RAYTCHO D. LAZAROV† AND RICHARD E. EWING‡

Abstract

A semidiscrete finite volume element(FVE) approximation to parabolic integro differential equation(PIDE) is analyzed in a two-dimensional convex polygonal domain. More precisely, for homogeneous equation, an elementary energy technique is used to derive optimal error estimate in L^2 and H^1 norms for positive time when the given initial function is in H_0^1.

Key words. Parabolic equation, integro-differential equation, optimal error estimate, smooth and nonsmooth initial data.

AMS subject classifications. 65M12, 65M60, 65N40.

1 Introduction

In this presentation, we consider the semidiscrete FVE method for solving initial-boundary value problems which arise naturally in many applications, such as in nonlocal reactive flows in porous media (Cushman and Glinn [7] and Dagan [8]) and heat conduction through materials with memory (Renardy *et al.*[19]) are described by the equations of the form

$$\begin{aligned} u_t - \nabla \cdot (\mathcal{A}\nabla u) &= -\int_0^t \nabla \cdot (\mathcal{B}\nabla u(s))ds + f(x,t) \ \text{ in } \ \Omega \times (0,T], \quad (1.1)\\ u &= 0 \ \text{ on } \ \partial\Omega \times (0,T], \\ u(\cdot,0) &= u_0 \ \text{ in } \ \Omega, \end{aligned}$$

where $u_t = \partial u/\partial t$.

*Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati - 781039, India (rajen@iitg.ernet.in).

†Department of Mathematics, Texas A&M University, College Station, TX 77843-3404 (lazarov@math.tamu.edu).

‡Institute for Scientific Computation, Texas A&M University, College Station, TX 77843-3404 (ewing@isc.tamu.edu).

1.1 Basic assumptions

We assume that $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain with boundary $\partial\Omega$ and $T < \infty$. Here, $\mathcal{A} = \{a_{i,j}(x)\}_{2\times 2}$ is a symmetric and uniformly positive definite matrix in Ω and $\mathcal{B} = \{b_{i,j}(x,t,s)\}$ is a 2×2 matrix. The coefficients $a_{ij}(x)$, $b_{ij}(x;t,s)$ are assumed to be smooth for our purpose.

Remark. One very important characteristic of these models is that they all express the conservation of a certain quantity (mass, momentum, heat, etc.) in any moment for any subdomain. This in many application is the most desirable feature of the approximation method when it comes to numerical solution of the corresponding initial-boundary value problem.

2 Finite volume element approximations

Let T_h be a quasi-uniform triangulation of Ω. Let

$$N_h = \{p \ : \ p \text{ is a vertex of element } K \in T_h \text{ and } p \in \bar{\Omega}\}.$$

Further, we denote $N_h^0 = N_h \cap \Omega$. For a vertex $x_i \in N_h$, let $\Pi(i)$ be the index set of those vertices that, along with x_i, are in some element of T_h.

Based on the triangulation T_h, we now introduce a dual mesh T_h^* as follows: In each element $K \in T_h$ consisting of vertices x_i, x_j and x_k, select a point $q \in K$, and select a point x_{ij} by straight lines $\gamma_{ij,K}$. Then for a vertex x_i we let V_i be the polygon whose edges are $\gamma_{ij,K}$ in which x_i is a vertex of the element K. We call this V_i a *control volume* centered at x_i. Further, we note that $\cup_{x_i \in N_h} V_i = \bar{\Omega}$. Thus, the dual mesh T_h^* is then defined as the collection of these *control volumes*. A *control volume* centered at a vertex x_i is given in Figure 1.

We call the control volume mesh T_h^* regular or quasi-uniform if there exists a positive constant $C > 0$ such that

$$C^{-1}h^2 \leq \text{meas}(V_i) \leq Ch^2 \quad \text{for all} \quad V_i \in T_h^*,$$

where h is the maximum diameter of all elements $K \in T_h$.

There are various ways to introduce a regular dual mesh T_h^* depending on the choices of the point q in an element $K \in T_h$ and the points x_{ij} on its edges. We choose q to be the barycenter of an element $K \in T_h$, and the points x_{ij} are chosen to be the midpoints of the edges of K.

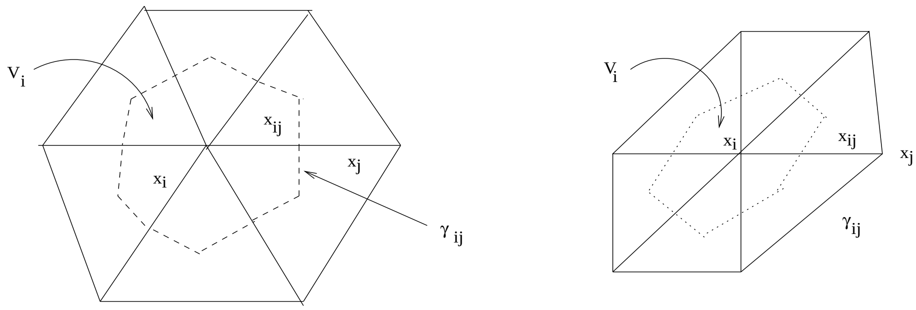

Figure 1: *Control volumes with barycenter as internal point and interface γ_{ij} of V_i and V_j.*

For the purpose of finite volume element approximation, let S_h be the standard linear finite element space defined on the triangulation T_h,

$$S_h = \{v \in C(\Omega) \ : \ v|_K \ \text{is linear for all } K \in T_h \ \text{and} \ v|_{\partial\Omega} = 0\},$$

and its dual volume element space S_h^*,

$$S_h^* = \{v \in L^2(\Omega) \ : \ v|_V \ \text{is constant for all } V \in T_h^* \ \text{and} \ v|_{\partial\Omega} = 0\}.$$

Obviously, $S_h = \text{span}\{\phi_i(x) \ : \ x_i \in N_h^0\}$ and $S_h^* = \text{span}\{\chi_i(x) \ : \ x_i \in N_h^0\}$, where ϕ_i are the standard nodal basis functions associated with the node x_i, and χ_i are the characteristic functions of the volume V_i. Let $I_h \ : \ C(\Omega) \to S_h$ and $I_h^* \ : \ C(\Omega) \to S_h^*$ be the usual interpolation operators, i.e.,

$$I_h u = \sum_{x_i \in N_h} u_i \phi_i(x) \quad \text{and} \quad I_h^* u = \sum_{x_i \in N_h} u_i \chi_i(x),$$

where $u_i = u(x_i)$.

The finite volume element approximation is then defined to be the function $u_h : \bar{J} \to S_h$ such that

$$\begin{aligned} (u_{ht}, I_h^*\chi) + A(u_h, I_h^*\chi) &= \int_0^t B(t, s; u_h(s), I_h^*\chi) ds \\ &+ \ (f, I_h^*\chi) \ \ \forall \chi \in S_h \end{aligned} \tag{2.2}$$

with $u_h(0) = \tilde{P}_h u_0$, where $\tilde{P}_h u_0$ is the L^2-projection of u_0 onto S_h defined by

$$(\tilde{P}_h u_0, I_h^*\chi) = (u_0, I_h^*\chi), \quad \forall \chi \in S_h. \tag{2.3}$$

The bilinear forms $A(\cdot,\cdot)$ and $B(t, s; \cdot, \cdot)$ in (2.2) are defined by

$$A(u, v) = -\sum_{x_i \in N_h} v_i \int_{\partial V_i} \mathcal{A}(x) \nabla u \cdot \mathbf{n} dS_x$$

$$B(t, s; u, v) = -\sum_{x_i \in N_h} v_i \int_{\partial V_i} \mathcal{B}(x, t, s) \nabla u \cdot \mathbf{n} dS_x$$

for $(u,v)\in((H_0^1\cap H^2)\cup S_h)\times S_h^*$, where $\mathbf{n}$ is the outer-normal vector of the involved integration domain. Note that when $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$ the bilinear forms $A(\cdot,\cdot)$ and $B(t,s;\cdot,\cdot)$ are given by

$$A(u,v)=\int_\Omega \mathcal{A}(x)\nabla u\cdot\nabla v dx; B(t,s;u(s),v)=\int_\Omega \mathcal{B}(x,t,s)\nabla u(s)\cdot\nabla v dx. \quad (2.4)$$

In order to describe features of the bilinear forms defined in (2.4) and (2.2), we define some discrete norms on S_h and S_h^*,

$$|u_h|_{0,h}^2=(u_h,u_h)_{0,h},\quad |u_h|_{1,h}^2=\sum_{x_i\in N_h}\sum_{x_j\in\Pi(i)}\text{meas}(V_i)((u_{hi}-u_{hj})/d_{ij}^2,$$
$$\|u_h\|_{1,h}^2=|u_h|_{0,h}^2+|u_h|_{1,h}^2,\qquad |||u_h|||=(u_h,I_h^*u_h),$$

where $(u_h,v_h)_{0,h}=\sum_{x_i\in N_h}\text{meas}(V_i)u_{hi}v_{hi}=(I_h^*u_h,I_h^*v_h)$ and $d_{ij}=d(x_i,x_j)$ is the distance between x_i and x_j.

The discrete norms $|\cdot|_{0,h}$ and $\|\cdot\|_{1,h}$ are equivalent to the usual norms $\|\cdot\|$ and $\|\cdot\|_1$, respectively on S_h.

3 Motivation and main results

In the recent years, the numerical methods for this type of problems (1.1) by means of FVE methods are the subject of much interest (cf. Ewing, Lazarov and Lin [10] and [11]). It is due to certain conservation features of FVE methods that are desirable in many applications. More recently, Ewing *et al.* in [10] and [11], have studied FVE approximation to such problem in the framework of the standard Petrov-Galerkin formulation. They have obtained L^2-error estimate of the form(cf. page 305 in [11])

$$\begin{aligned}\|u(t)-u_h(t)\| &\le Ch^2(\|u_0\|_{3,p}+\|u(t)\|_{3,p}\\ &\quad+\int_0^t(\|u(s)\|_{3,p}+\|u_t(s)\|_{3,p})ds),\quad p>1,\end{aligned}\qquad (3.5)$$

where u and u_h represent the solution of (1.1) and its FVE approximation, respectively. Note that the estimate (3.5) is optimal with respect to the approximation property, but its regularity requirement on the exact solution seems to be too high when compared with that for finite element methods. This is primarily due to the fact that the bounds in the L^2-norm of a new variant of the Ritz-Volterra projection(*so called* Petrov-Volterra projection introduced by Ewing *et al.* in [10]-[11]) are not optimal with respect to the regularity of the solution. For homogeneous problem with smooth initial data, we show optimal order L^2-error estimate which is optimal with respect to the order of convergence as well as its regularity of the solution. The main result for smooth initial data is given in the following theorem.

Theorem 3.1 *Let u and u_h, respectively, satisfy (1.1) and (2.2) with $f \equiv 0$. Then for $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_h(0) = \tilde{P}_h u_0$, we have*

$$\|e(t)\| \leq Ch^2 \|u_0\|_2.$$

Our main concern is to prove optimal error estimates for homogeneous equation ($f = 0$) with nonsmooth initial data. This is motivated by the fact that the solution of a homogeneous linear parabolic equation have the so-called *smoothing property*. That is, the solution is sufficiently smooth for positive time t, even when the initial data are not. In quantitative form, this may be expressed by the inequality

$$\|u(t)\|_\alpha \leq Ct^{-\alpha/2} \|u_0\|, \quad t \in J \tag{3.6}$$

which is valid for any $\alpha \geq 0$. However, this is not the case with parabolic integro-differential equations as they have a *limited smoothing property*. This fact is proved by Thomée and Zhang in [21], where the authors have shown that the inequality (3.6) is valid only for $\alpha \leq 2$. Since the smoothing property plays a significant role in studying the error analysis in the semidiscrete solution, an attempt has been made to achieve optimal order of convergence in L^2 and H^1 norms for the FVE method when the given initial function u_0 is only in $H_0^1(\Omega)$. More important, our analysis uses only energy technique. The main results are presented in the following theorem.

Theorem 3.2 *Let u and u_h, respectively, satisfy (1.1) and (2.2) with $f \equiv 0$. Then for $u_0 \in H_0^1(\Omega)$ and $u_h(0) = \tilde{P}_h u_0$, we have*

$$\|e(t)\|_1 \ \leq \ Ct^{-1/2} h \|u_0\|_1, \quad t \in (0, T].$$

Theorem 3.3 *Let u and u_h, respectively, satisfy (1.1) and (2.2) with $f \equiv 0$. Then for $u_0 \in H_0^1(\Omega)$ and $u_h(0) = \tilde{P}_h u_0$, we have*

$$\|e(t)\| \ \leq \ Ct^{-1} h^2 \|u_0\|_1, \quad t \in (0, T].$$

For the literature on the theoretical framework and the basic tools for the analysis of the finite volume element methods for elliptic and parabolic problems, we refer to [1],[2], [6], [4], [5], [9], [12], [13], [16], [17] and references therein.

4 Error estimate for nonsmooth initial data

Below, we state some a priori bounds for the solution u satisfying (1.1) under appropriate regularity assumption on the initial function u_0.

Lemma 4.1 *Let u satisfy (1.1) with $f = 0$, and let $0 \leq i, j, k \leq 2$. If $0 \leq k + 2j - i \leq 2$, then*

$$t^i \left\| \frac{\partial^j u}{\partial t^j}(t) \right\|_k^2 \leq C \|u_0\|_{k+2j-i}^2.$$

Further, if $0 \leq k + 2j - i - 1 \leq 2$, then

$$\int_0^t s^i \left\| \frac{\partial^j u}{\partial s^j}(s) \right\|_k^2 ds \leq C \|u_0\|_{k+2j-i-1}^2.$$

As usual we split the error $e(t) = u(t) - u_h(t)$ as

$$e(t) = (W_h u - u_h) - (W_h u - u) = \theta - \rho,$$

where the Ritz-Volterra projection $W_h : H_0^1(\Omega) \to S_h$ defined by

$$A(W_h u - u, I_h^* \chi) = \int_0^t B(t, s; (W_h u - u)(s), I_h^* \chi) ds, \quad \forall\, \chi \in S_h. \qquad (4.7)$$

4.1 Estimates for the Ritz-Volterra projection

The following L^2 and H^1 error estimates for ρ and its temporal derivative are established.

Theorem 4.1 *Let ρ satisfy (4.7). Then we have*

$$\|\rho(t)\|_1 \leq Ch \left(\|u(t)\|_2 + \int_0^t \|u(s)\|_2 ds \right),$$

$$\|\rho_t(t)\|_1 \leq Ch \left(\|u(t)\|_2 + \|u_t(t)\|_2 + \int_0^t \|u(s)\|_2 ds \right).$$

Theorem 4.2 *Let ρ satisfy (4.7). Then we have*

$$\|\rho(t)\| \leq Ch^2 \left(\|u(t)\|_2 + \int_0^t \|u(s)\|_2 ds \right),$$

$$\|\rho_t(t)\| \leq Ch^2 \left(\|u(t)\|_2 + \|u_t(t)\|_2 + \int_0^t \|u(s)\|_2 ds \right).$$

Remark. (i) The estimates in Theorem 4.2 are optimal with respect to the order of convergence as well as the regularity requirement on the solution. This improves upon the earlier result of [10] and [11] by requiring less regularity on the solution.

(ii) As a consequence of Theorems 4.1 and 4.2, error estimates associated with the Ritz projection $R_h : H_0^1 \to S_h$ defined by

$$A(R_h u - u, I_h^* \chi) = 0, \quad \forall\, \chi \in S_h$$

can easily be obtained. Thus, we immediately have

$$\|R_h u - u\| + h\|R_h u - u\|_1 \leq C h^j \|u\|_j, \quad u \in H_0^1(\Omega) \cap H^j(\Omega), \quad j = 1, 2. \tag{4.8}$$

4.2 Estimates for θ

It is easy to verify that θ satisfies an error equation of the form

$$(\theta_t, I_h^* \chi) + A(\theta, I_h^* \chi) = \int_0^t B(t, s; \theta(s), I_h^* \chi) ds - (\rho_t, I_h^* \chi), \quad \forall \chi \in S_h. \tag{4.9}$$

Further, with $\hat{\theta}(t) = \int_0^t \theta(s) ds$, we obatin an error equation in $\hat{\theta}$ as

$$\begin{aligned}(\hat{\theta}_t, I_h^* \chi) \;\; + \;\; A(\hat{\theta}, I_h^* \chi) &= \int_0^t B(s, s; \hat{\theta}(s), I_h^* \chi) ds \\ &- \int_0^t \int_0^s B_\tau(s, \tau; \hat{\theta}(\tau), I_h^* \chi) d\tau ds - (\rho, I_h^* \chi), \quad \chi \in S_h. \end{aligned} \tag{4.10}$$

In the following lemma, we have obtained the estimates of θ and $\hat{\theta}$.

Lemma 4.2 *Let $\hat{\theta}$ satisfy (4.10) and $u_h(0) = \tilde{P}_h u_0$, where $\tilde{P}_h$ is defined by (2.3). Then there is a positive constant C independent of h such that*

$$\|\hat{\theta}(t)\|^2 + \int_0^t \|\hat{\theta}(s)\|_1^2 ds \leq C \int_0^t \|\rho(s)\|^2 ds.$$

Lemma 4.3 *Let θ satisfy (4.9) and $u_h(0) = \tilde{P}_h u_0$. Then there is a positive constant C independent of h such that*

$$t\|\theta(t)\|^2 + \int_0^t s\|\theta(s)\|_1^2 ds \leq C \int_0^t \{\|\rho(s)\|^2 + s^2 \|\rho_s(s)\|^2\} ds.$$

Lemma 4.4 *Let θ satisfy (4.9) and $u_h(0) = \tilde{P}_h u_0$. Then there is a positive constant C independent of h such that*

$$\int_0^t s^2 \|\theta_s(s)\|^2 ds + t^2 \|\theta(t)\|_1^2 ds \leq C \int_0^t \{\|\rho(s)\|^2 + s^2 \|\rho_s(s)\|^2\} ds.$$

Theorem 3.2 follows from Theorem 4.1, Lemma 4.4, Lemma 4.1 and the triangle inequality.

Combine Theorem 4.2, Lemma 4.3 and Lemma 4.1 to prove Theorem 3.3.

Remark. From the proof of Theorems 3.2-3.3, it is clear that we can choose $u_h(0)$ as the L^2 projection of u_0 into S_h defined by (2.3) instead of the elliptic projection $R_h u_0$ or Ritz-Volterra projection $W_h u_0$. Note that the result presented in Theorems 3.1-3.3 are optimal with respect to the approximation property as well as the regularity of the solution.

References

[1] Z. Cai, *On the finite volume element method*, Numer. Math., 58(1991), pp. 713-735.

[2] Z. Cai and S. McCormick, *On the accuracy of the finite volume element method for diffusion equations on composite grids*, SIAM J. Numer. Anal. 27(1990), pp. 636-655.

[3] J. R. Cannon and Y. Lin, *Nonclassical H^1 projections and Galerkin methods for nonlinear parabolic integro-differential equations*, Calcolo 25 (1988), pp. 187-201.

[4] P. Chatzipantelidis, *Finite volume methods for elliptic PDE's: A new approach*, Mathematical Modelling and Numerical Analysis, 36(2002), 307-324.

[5] P. Chatzipantelidis, R.D. Lazarov, and V. Thomee, *Error estimates for the finite volume element method for parabolic equations in convex polygonal domains*, to appear in J. Numer. Meth. PDEs.

[6] S.H. Chou and Q. Li, *Error estimates in L^2, H^1 and L^∞ in covolume methods for elliptic and parabolic problems: A unified approach*, Math. Comp., 69(2000), 103-120.

[7] J. H. Cushman and T. R. Glinn, *Nonlocal dispersion in media with continuously evolving scales of heterogeneity*, Tansport in Porous Media, 13(1993), pp. 123-138.

[8] G. Dagan, *The significance of heterogeneity of evolving scales to transport in porous formations*, Water Resour. Res., 30 (1994), pp. 3327-3336.

[9] R. E. Ewing, T. Lin, Y. Lin, *On the accuracy of the finite volume element method based on piecewise linear polynomials*, SIAM J. Numer. Anal., 39(2002), pp. 1865-1888.

[10] R.E. Ewing, R.D. Lazarov and Y. Lin, *Finite Volume element approximations of nonlocal in time one-dimensional flows in porous media*, Computing, 64(2000), pp. 157-182.

[11] R.E. Ewing, R.D. Lazarov and Y. Lin, *Finite Volume element approximations of nonlocal reactive flows in porous media*, Numer. Methods Partial Differential Equations, 16(2000), pp. 285-311.

[12] W. Hackbusch, On first and second order box schemes, Computing, 41(1989), 277-296.

[13] H. Jianguo and X. Shitong, *On the finite volume element method for general self-adjoint elliptic problems*, SIAM J. Numer. Anal. 35(1998), pp. 1762-1774.

[14] Y. Lin, V. Thomée and L. B. Wahlbin, *Ritz-Volterra projections to finite element spaces and applications to integro-differential and related equations*, SIAM J. Numer. Anal. 28 (1991), pp. 1047-1070.

[15] M. Luskin and R. Rannacher, *On the smoothing property of the Galerkin method for parabolic equations*, SIAM J. Numer. Anal. 19 (1981), pp. 93-113.

[16] I.D. Mishev, *Finite Volume and Finite Volume Element Methods for Non-Symmetric Problems*, Ph.D. Thesis, Technical Report ISC-96-04-MATH, Institute for Scientific Computation, Texas A&M University, College Station, TX, 1997.

[17] I.D. Mishev, *Finite Volume Methods on Voronoi meshes*, Numer. Methods Partial Differential Equations, 16(1998), pp. 193-212.

[18] A. K. Pani and R. K. Sinha, *Error estimates for semidiscrete Galerkin approximation to a time dependent parabolic integro-differential equation with nonsmooth dada*, Calcolo, 37(2000), pp. 181-205.

[19] M. Renardy, W. Hrusa and J. Nohel, *Mathematical Problems in Viscoelasticity*, Pitman Monographs and Surveys in Pure Appl. Math. No. 35 (1987), New York, Wiley.

[20] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Lecture Notes in Math. 1054 (1984), Springer-Verlag.

[21] V. Thomée and N.-Y. Zhang, *Error estimates for semidiscrete finite element methods for parabolic integro-differential equations*, Math. Comp. 53 (1989), pp. 121-139.