Formal theory of PDEs and simulation of fluid flow

Bijan Mohammadi and Jukka Tuomela

Université de Montpellier 2, France , University of Joensuu, Finland

mohamadi@math.univ-montp2.fr , jukka.tuomela@joensuu.fi

1. Introduction

In many physical models there appear constraints or conserved quantities which make the system essentially overdetermined. However, usually in numerical computations one uses square models (as many equations as unknowns). Hence one may encounter great difficulties in designing numerical methods which take into account constraints which are only implicitly present in the system.

We propose an alternative approach to solve numerically these kind of systems of PDEs which is based on the formal theory of PDEs [6]. The idea is to find the *involutive form* of the system, and use it explicitly in the computations. The involutive form is important because many properties of the system cannot be determined if the system is not involutive. For example one may not be able to say if the system is elliptic or not, if it is not first transformed to the involutive form.

We illustrate our approach by considering a compressible flow problem. The numerical results show that quite straightforward numerical schemes work when using the involutive form while classical approach needs quite elaborate specific schemes.

2. Compressible flow model

We consider the problem of low-Mach number hydro-dynamical flow simulation. The equations are derived from the Navier-Stokes system. Since we are only interested in perturbations to the pressure due to the flow dynamics the assumptions made on the state equation linking the density and temperature are such that the acoustic perturbations are removed. In this way the density can in fact be eliminated from the system, and so after some manipulations we arrive at the system [4]

$$\begin{cases} u_t - \frac{4}{3\mathsf{Re}}T\Delta u - \frac{1}{3\mathsf{Re}}T\nabla\times\nabla\times u + u\nabla u + T\nabla p = 0 \\ -\nabla\cdot u + \frac{1}{\mathsf{RePr}}\Delta T = 0 \\ T_t - T\nabla\cdot u + \langle u, \nabla T\rangle = 0 \end{cases} \tag{1}$$

where u is the velocity field, T is the temperature, p is the pressure, Re is the Reynolds number and Pr is the Prandtl number. Denoting $y = (u, T, p)$ we can write the whole system as $\mathcal{S}_0 y = 0$.

Now this is a second order system which is not involutive: by differentiating and eliminating we can find an *integrability condition* which is analogous to Poisson equation for the pressure in the classical Stokes system. Informally this is the idea of involutivity: find all the integrability conditions (up to given order). Unfortunately the precise definition of involutivity is rather involved so we just refer to [5] and [1] for details.

Let us then introduce the following notation: if A and B are matrices of the same size then

$$A \diamond B = \sum_{i,j} a_{ij} b_{ij}$$

Further let us define the operator

$$\mathcal{S}_1 = \left(\nabla\cdot, \tfrac{4\mathsf{Pr}}{3}\left(\partial_t + \langle u, \nabla\rangle\right), -\tfrac{4}{3\mathsf{Re}}\Delta\right)$$

where $\langle u, \nabla\rangle = \sum u_i \partial_{x_i}$. Then we compute

$$\begin{aligned} -\mathcal{S}_1\mathcal{S}_0 y = & -T\Delta p - \langle\nabla T, \nabla p\rangle - \tfrac{7}{3\mathsf{Re}}\langle\nabla T, \nabla\times\nabla\times u\rangle \\ & + \tfrac{8}{3\mathsf{Re}}\nabla u \diamond d^2 T - (\nabla u)^t \diamond \nabla u \\ & + \left(\tfrac{4\mathsf{Pr}}{3} - 1\right)\left(\nabla\cdot u_t + \langle u, \nabla\nabla\cdot u\rangle\right) - \tfrac{4\mathsf{Pr}}{3}(\nabla\cdot u)^2 \end{aligned}$$

where d^2T is the second differential of T. The initial system (1) together with the equation $-\mathcal{S}_1\mathcal{S}_0 y = 0$ will be called the *completed* system.

3. 1D case

In our model already 1D case is nontrivial, unlike in the incompressible Navier-Stokes system. Hence for simplicity we will show numerical results only in this case. There are two cases of interest: evidently 1D model itself, but also the linearised model. The latter is useful in stability considerations. In the linearised case we will only consider the situation where the reference solution $(\bar{u}, \bar{T}, \bar{p})$ is constant. The linear system is then

$$\mathcal{L}_0 y = \begin{cases} u_t - \frac{4}{3\mathsf{Re}}\bar{T}u_{xx} + \bar{u}u_x + \bar{T}p_x = 0 \\ -u_x + \frac{1}{\mathsf{RePr}}T_{xx} = 0 \\ T_t - \bar{T}u_x + \bar{u}T_x = 0 \\ -\bar{T}p_{xx} + (\frac{4\mathsf{Pr}}{3} - 1)\left(u_{xt} + \bar{u}u_{xx}\right) = 0 \end{cases} \tag{2}$$

The nonlinear system is

$$\mathcal{N}_0 y = \begin{cases} u_t - \frac{4}{3\mathsf{Re}}Tu_{xx} + uu_x + Tp_x = 0 \\ -u_x + \frac{1}{\mathsf{RePr}}T_{xx} = 0 \\ T_t - Tu_x + uT_x = 0 \\ -Tp_{xx} - T_x p_x + (\frac{4\mathsf{Pr}}{3} - 1)\left(u_{xt} + uu_{xx} + (u_x)^2\right) = 0 \end{cases} \tag{3}$$

To solve the above systems numerically we introduce an auxiliary variable z as follows. Let us denote $\tilde{y} = (y, z)$ and define

$$\begin{aligned} \mathcal{L}_1 &= \left(\partial_x, \tfrac{4\mathsf{Pr}}{3}\left(\partial_t + \bar{u}\partial_x\right), -\tfrac{4}{3\mathsf{Re}}\Delta, 1\right) \\ \mathcal{N}_1 &= \left(\partial_x, \tfrac{4\mathsf{Pr}}{3}\left(\partial_t + u\partial_x + u_x\right), -\tfrac{4}{3\mathsf{Re}}\Delta, 1\right) \end{aligned}$$

Note that $\mathcal{N}_1\mathcal{N}_0 = 0$ and $\mathcal{L}_1\mathcal{L}_0 = 0$. Hence $\mathcal{N}_1$ (resp. $\mathcal{L}_1$) is the *compatibility operator* for $\mathcal{N}_0$ (resp. $\mathcal{L}_0$). Then we introduce the linear system

$$\tilde{\mathcal{L}}\tilde{y} = \begin{cases} u_t - \frac{4}{3\mathsf{Re}}\bar{T}u_{xx} + \bar{u}u_x + \bar{T}p_x - z_x = 0 \\ \frac{4\mathsf{Pr}}{3}z_t - u_x + \frac{1}{\mathsf{RePr}}T_{xx} - \frac{4\mathsf{Pr}}{3}\bar{u}z_x = 0 \\ T_t - \bar{T}u_x + \bar{u}T_x - \frac{4}{3\mathsf{Re}}z_{xx} = 0 \\ -\bar{T}p_{xx} + \ (\frac{4\mathsf{Pr}}{3} - 1)\left(u_{xt} + \bar{u}u_{xx}\right) + z = 0 \end{cases} \tag{4}$$

The nonlinear case is given by

$$\tilde{\mathcal{N}}\tilde{y} = \begin{cases} u_t - \frac{4}{3\mathsf{Re}}Tu_{xx} + uu_x + Tp_x - z_x = 0 \\ \frac{4\mathsf{Pr}}{3}z_t - u_x + \frac{1}{\mathsf{RePr}}T_{xx} - \frac{4\mathsf{Pr}}{3}uz_x + \frac{4\mathsf{Pr}}{3}u_x z = 0 \\ T_t - Tu_x + uT_x - \frac{4}{3\mathsf{Re}}z_{xx} = 0 \\ -Tp_{xx} - T_x p_x + (\frac{4\mathsf{Pr}}{3} - 1)\left(u_{xt} + uu_{xx} + (u_x)^2\right) + z = 0 \end{cases} \tag{5}$$

These system are called *augmented systems*. They are obtained by using the formal transpose of the operators $\mathcal{L}_1$ and $\mathcal{N}_1$. Then we compute:

$$\mathcal{L}_1\tilde{\mathcal{L}}\,\tilde{y} = \tfrac{16\mathsf{Pr}^2}{9}z_{tt} + \tfrac{16}{9\mathsf{Re}^2}z_{xxxx} - \left(1 + \tfrac{16\bar{u}^2\mathsf{Pr}^2}{9}\right)z_{xx} + z = 0$$

Hence z by itself is a solution of a well-posed problem. Note that introducing z restores the squareness of the system which in turn makes it possible to use standard software. On the other hand since z should be identically zero for the exact solution, the values of z gives us a way to control the error.

4. Results

We did our computations using Femlab 3.0 [2]. We considered periodic boundary conditions for all variables. Since z is supposed to be zero it is natural to take zero intial conditions for it.

Here we just consider the linearised case. We took zero initial conditions for u and p, and Gaussian "perturbation" for T. In this case it is known that the solution tends to an equilibrium (not stationary) solution, and moreover the L^2–norm of the pressure dependes linearly on Pr.

As seen in Figure 1 our formulation produces the desired result. Figure 2 gives the time evolution of L^2–norm of z for various values of Pr. It is seen that the norm is quite big in the beginning, but is very small when the equilibrium solution is found. Note that taking an equilibrium solution as initial value would keep z small even in the beginning.

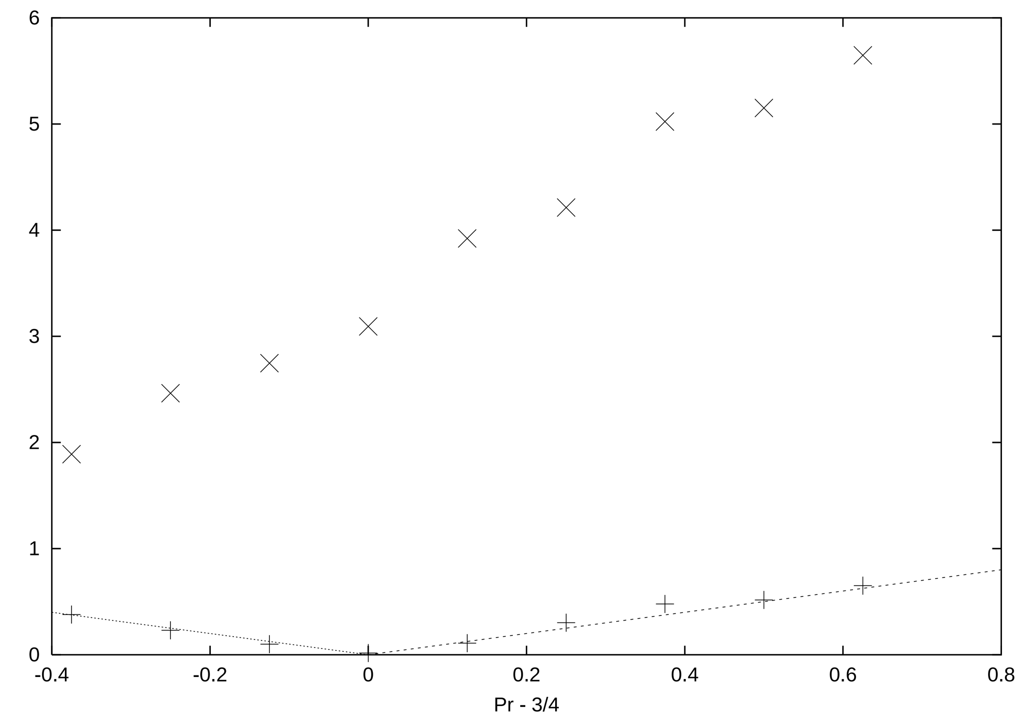

Figure 1: *The L^2–norm of the pressure in the equilibrium solution as a function of Prandtl number. × denotes the results obtained using the initial system (1) and + denotes the results obtained with augmented system (4). The line gives the value given by theory.*

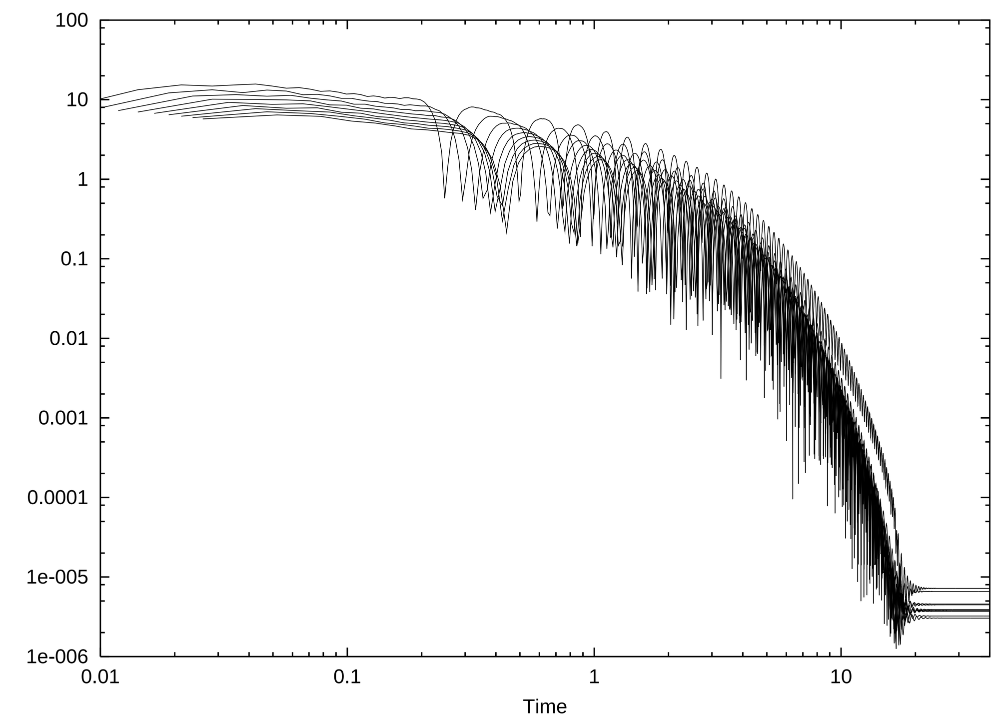

Figure 2: *The L^2–norm of z as a function of time for various values of* Pr.

5. Conclusion

The involutive systems are usually not square which makes their numerical treatment difficult. We proposed above one possible way to deal with this issue: we introduced an auxiliary variable which restored the squareness of the system. This auxiliary variable may be useful in adaptive error control; however, this was not attempted in the computations reported above. All in all the use of involutive systems raises many questions which are still quite open and which will be treated elsewhere. Some results can be found in [3].

References

[1] P.I. Dudnikov and S.N. Samborski, *Linear overdetermined systems of partial differential equations*, PDE VIII (M.A. Shubin, ed.), Encyclopaedia of Mathematical Sciences 65, Springer, 1996, pp. 1–86.

[2] Femlab 3.0 : http://www.comsol.com/products/femlab/

[3] B. Mohammadi and J. Tuomela, *Involutivity and numerical solution of PDE systems*, to appear in the Proceedings of ECCOMAS, Jyväskylä, July 24–28, 2004, Finland.

[4] F. Nicoud, *Conservative high-order finite-difference schemes for low-Mach number flows*, J. Comput. Phys. **158** (2000), no. 1, 71–97.

[5] J. F. Pommaret, *Systems of Partial Differential Equations and Lie Pseudogroups*, Mathematics and its applications, vol. 14, Gordon and Breach Science Publishers, 1978.

[6] D. Spencer, *Overdetermined systems of linear partial differential equations*, Bull. Am. Math. Soc **75** (1969), 179–239.