

Voluntary or Mandatory Retirement

Philip H. Dybvig

Washington University in Saint Louis

Hong Liu

Washington University in Saint Louis

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preliminary: comments welcome

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Goal of the research

We build a workhorse model of lifetime consumption and investment for answering questions about retirement, pensions, and insurance. Some features of the model include:

- voluntary or mandatory retirement date
- nonnegative wealth constraint on borrowing
- idealized hedging using life insurance
- bequest motive or not
- can handle wage and mortality varying through life
- stochastic wages over time

Specialized assumptions for today's presentation

Today's presentation focuses on a simple version of the model in which much of the solution can be computed directly. This version is easier to understand than the first example we worked out, which required relatively more numerical work and included stochastic wages with age-dependent average wage and mortality profiles (more on this later).

- constant wage (over time and states of nature)
- constant mortality rate
- retirement is either
 - mandatory (fixed)
 - voluntary
- insurance is fairly priced¹ and continuously available
- no bequest

¹In the notation of the paper, $\tau = \delta$

Some notation

- c_t – consumption at time t
- $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$ – felicity of consumption if not retired
- $u(Kc_t)$ – felicity of consumption if retired, $K > 1$
- δ – hazard rate for mortality (constant for this version of the model)
- ρ – pure rate of time discount
- r – riskfree rate
- $dS_t/S_t = \mu dt + \sigma dB_t$ – stock price process
- $\kappa \equiv (\mu - r)/\sigma$ – price of risk
- θ_t – holding in the stock (wealth units)
- τ – voluntary retirement date
- T – mandatory retirement date
- $k < 1$ – retirement income fraction
- W^* – initial wealth

Choice problem: voluntary retirement

Choose the retirement date τ (a stopping time), the adapted consumption process c_t , and the adapted portfolio choice θ_t to maximize

$$E \left[\int_{t=0}^{\tau} e^{-(\rho+\delta)t} u(c_t) dt + \int_{t=\tau}^{\infty} e^{-(\rho+\delta)t} u(Kc_t) dt \right]$$

subject to the budget conditions on the wealth process

$$W_0 = W^*$$

$$dW_t = \begin{cases} (r+\delta)W_t dt + \theta_t(\mu - r)dt + \theta_t\sigma dB_t - c_t dt + w dt & \text{for } t \leq \tau \\ (r+\delta)W_t dt + \theta_t(\mu - r)dt + \theta_t\sigma dB_t - c_t dt + k w dt & \text{for } t \geq \tau \end{cases}$$

$$(\forall t \geq 0) W_t \geq 0$$

standard except voluntary retirement date τ , mortality δ , preference for not working K , and post-retirement income multiplier k

Choice problem: mandatory retirement

Given the mandatory fixed retirement date T , choose the adapted consumption process c_t , and the adapted portfolio choice θ_t to maximize

$$E \left[\int_{t=0}^T e^{-(\rho+\delta)t} u(c_t) dt + \int_{t=T}^{\infty} e^{-(\rho+\delta)t} u(Kc_t) dt \right]$$

subject to the budget conditions on the wealth process

$$W_0 = W^*$$

$$dW_t = \begin{cases} (r+\delta)W_t dt + \theta_t(\mu - r)dt + \theta_t\sigma dB_t - c_t dt + w dt & \text{for } t \leq T \\ (r+\delta)W_t dt + \theta_t(\mu - r)dt + \theta_t\sigma dB_t - c_t dt + k w dt & \text{for } t \geq T \end{cases}$$

$$(\forall t \geq 0) W_t \geq 0$$

standard except mandatory retirement date T , mortality δ , preference for not working K , and post-retirement income multiplier k

Solution technique: all cases use the dual approach

- using a change of variables to marginal utility (or marginal felicity)
- can be viewed as a solution to a convex dual to the original problem
- as in He in Pagès [1993] but with some new twists
- gives a linear differential equation more generally than the one-shot approach of Pliska [1982,1986] and others including Cox and Huang [1989]
- post-retirement, we have the same explicit solution to both problems
- pre-retirement solution for voluntary retirement
 - explicit solution up to one constant
- pre-retirement solution for mandatory retirement
 - in the dual, approximate the fixed maturity by a sequence of random regimes
 - Erlang distribution, a sum of Poissons, stationary each stage
 - from Liu and Loewenstein [2002] and Carr [1998]
 - recursive solution: solve for one constant in each stage

Solution technique: post-retirement

Dual approach: we use a change of variables to the marginal felicity of consumption $z_t = du(Kc_t)/dc_t$. We have

$$d(e^{-(\rho+\delta)t}z(t))/(e^{-(\rho+\delta)t}z(t)) \leq -(r+\delta)dt - \frac{\mu-r}{\sigma}dB_t$$

with equality if $W > 0$ (complementarity slackness for nonnegative wealth). This is in effect looking at the first-order conditions for minimizing cost subject to the utility level.

Primal variables can be computed by taking the appropriate derivatives of the dual value function (just as dual variables can be computed by taking the appropriate derivatives of the primal value function). Therefore, solving the dual gives us an solution of the primal problem in terms of implicit functions.

Solution: post-retirement

$\phi^R(z)$, the dual value function post-retirement, solves the Bellman equation (a linear ODE)

$$\frac{1}{2}\kappa^2 z^2 \phi_{zz}^R - (r + \kappa - \rho - \delta)z \phi_z^R - (\rho + \delta)\phi^R - K^{-b} \frac{z^b}{b} + kwz = 0$$

with smooth-pasting conditions at the zero-wealth boundary:

$$\phi_z^R(\bar{z}_R) = 0 \quad \text{and} \quad \phi_{zz}^R(\bar{z}_R) = 0$$

(much simpler than the primal $J''(0)$ infinite).

The solution is

$$\phi^R(z) = A^R z^{\alpha_1} - \eta^R \frac{z^b}{b} + \xi^R wz$$

with nonnegative wealth boundary

$$\bar{z}_R = \left(\frac{\eta^R(\alpha_1 - b)}{\xi^R(\alpha_1 - 1)w} \right)^\gamma$$

where A^R , α_1 , η^R , and ξ^R are known (but in some cases messy) functions of the parameters.

Solution technique: voluntary retirement

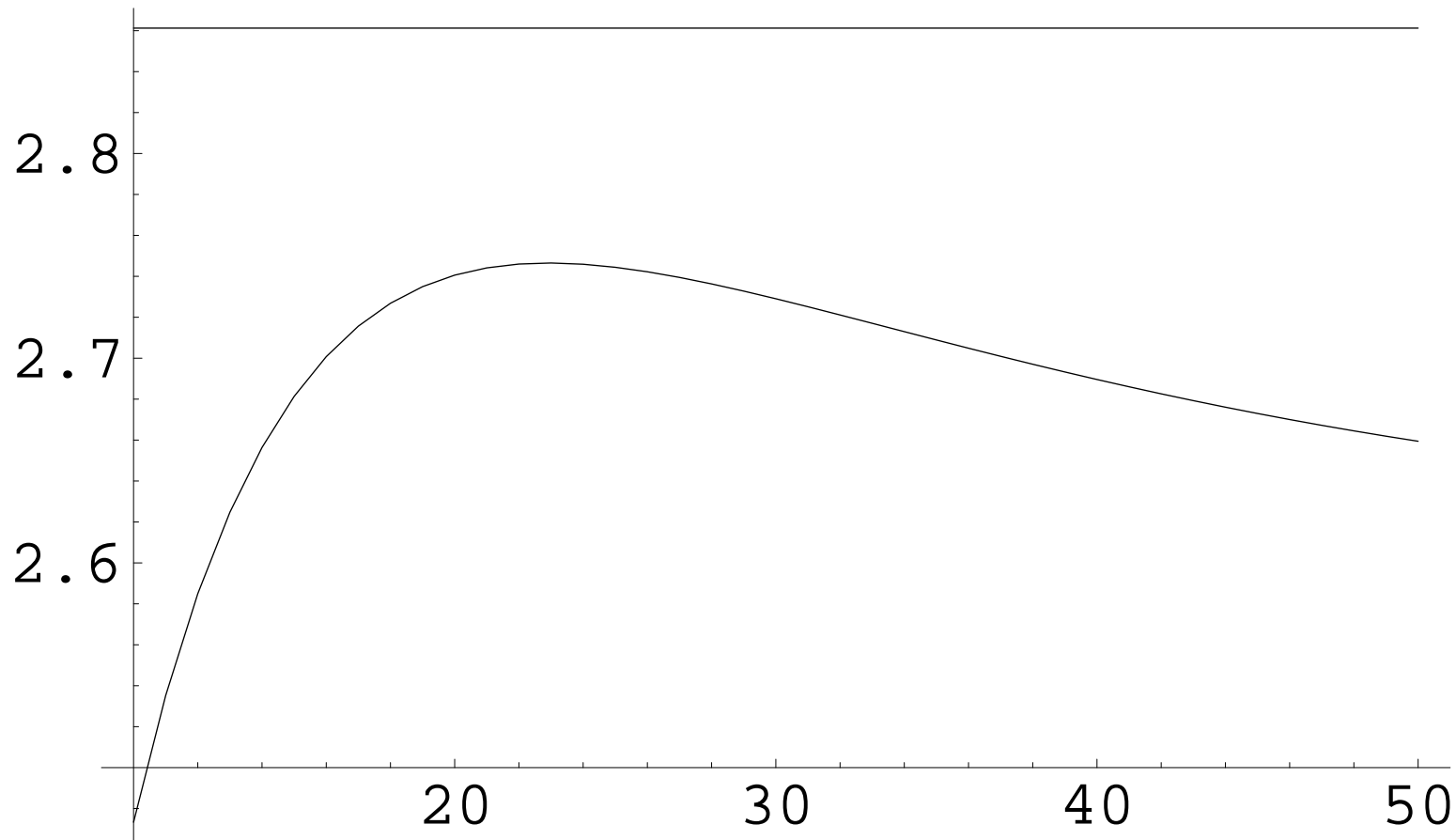
In this case there are two boundaries, a non-negative wealth boundary (where $W = 0$) and a retirement boundary (where W is large). The differential equation for the solution links to the post-retirement solution at the retirement boundary where there is a smooth-pasting condition. Since the case we are looking at has constant hazard rate of mortality, wage, and disdain for working, the solution is stationary and the wealth required for retirement is the same at all times. Using the known form of the post-retirement solution for the value at retirement, we can solve for the solution in the dual up to numerical solution of a single constant. The value function in this case is the sum of terms of 4 known powers of z .

Solution technique: mandatory retirement

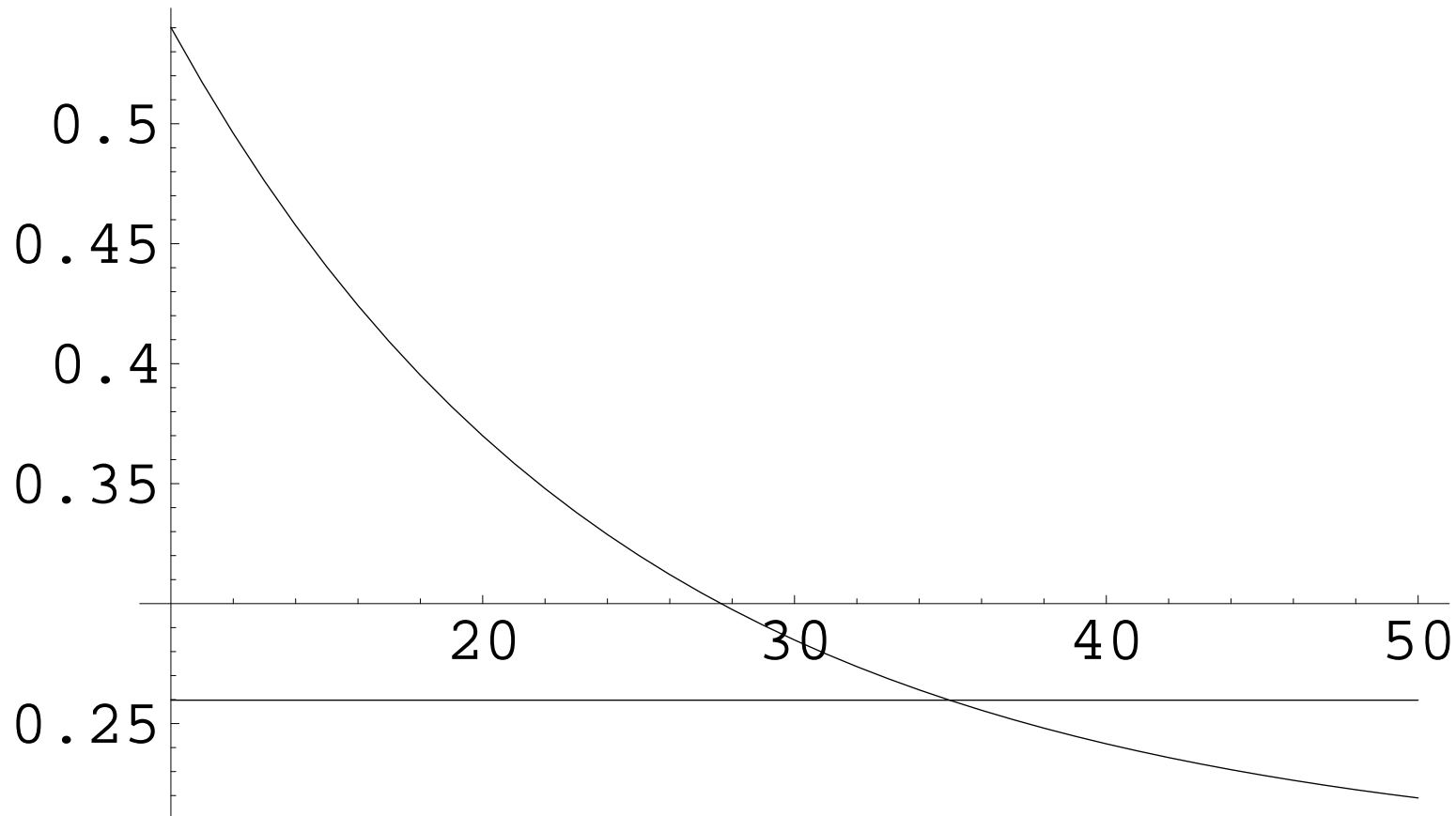
In this case, solution is subtle because the value function is time-dependent. However, we can approximate the solution closely by solving only stationary problems. This uses a device used by Carr [1998] for valuation of American puts and by Liu and Loewenstein [2002] for optimal investment in the presence of trading costs. The idea is that a fixed horizon T is approximated very closely by the sum of n exponential random variables each having mean T/n . Since an exponential distribution has a constant (Poisson) hazard rate, each regime has a stationary problem. Then the solution can be obtained recursively from the last regime before retirement. Because the approximate solution has a possible transition to the next regime at any point in time, the PDE for the value function has a probability weight for the transition to the next regime. Solving each regime requires solving numerically for a single constant. This approach is equivalent to the mathematical technique of the “horizontal method of lines.”

Parameters for the plots

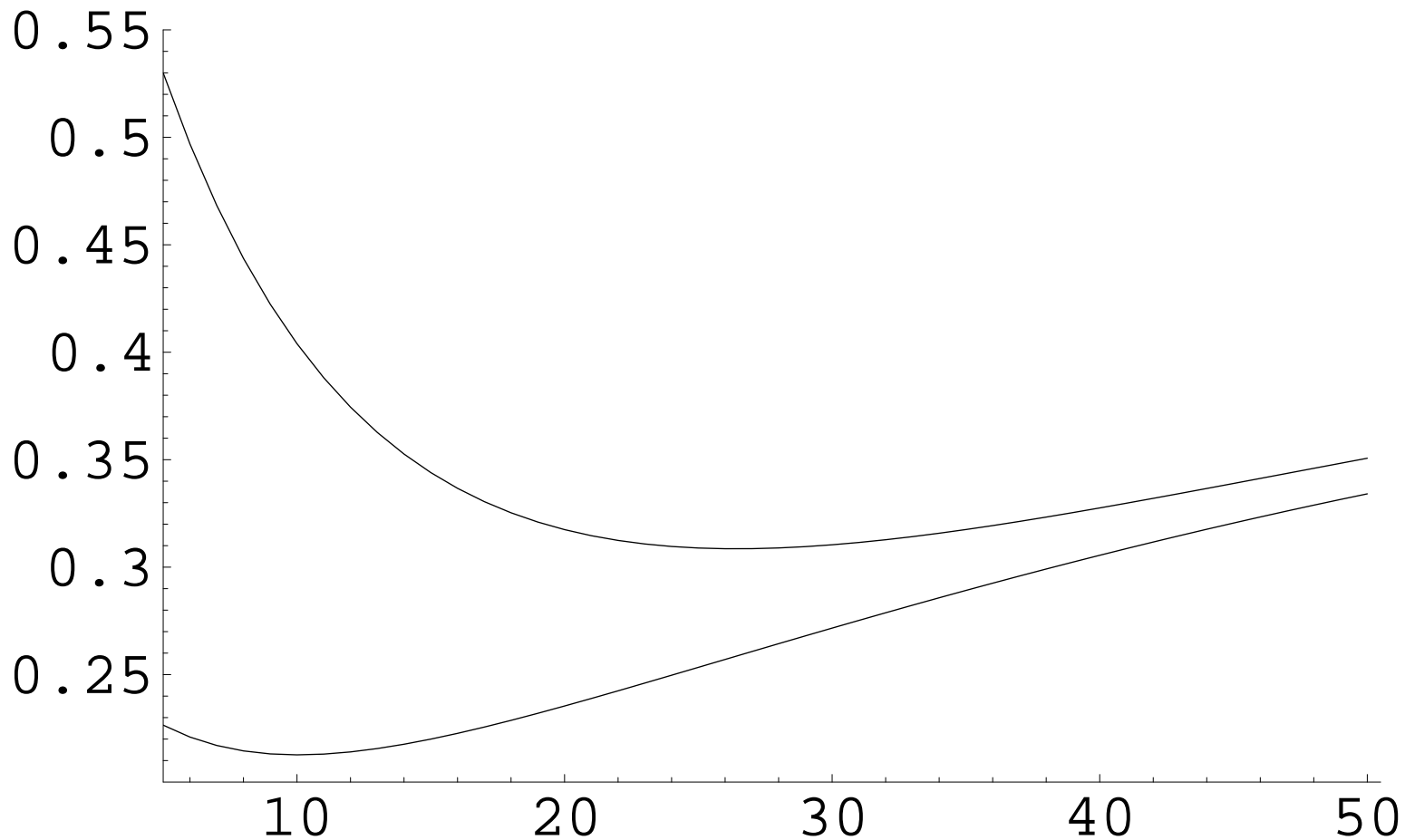
- $r = 1\%/year$ – interest rate
- $\rho = 1\%/year$ – pure rate of time discount
- $\delta = 2\%/year$ – hazard rate of mortality
- K – preference for not working
- $k = 5\%$ – income after retirement
- $\mu = 5\%/year$ – mean stock return
- $\sigma = 22\% / \sqrt{year}$ – standard deviation of stock return
- $w = 1/year$ – wage income (therefore wealth is in units of the multiple of income)
- $n = 50$ – number of regimes used in the approximation for the mandatory retirement case



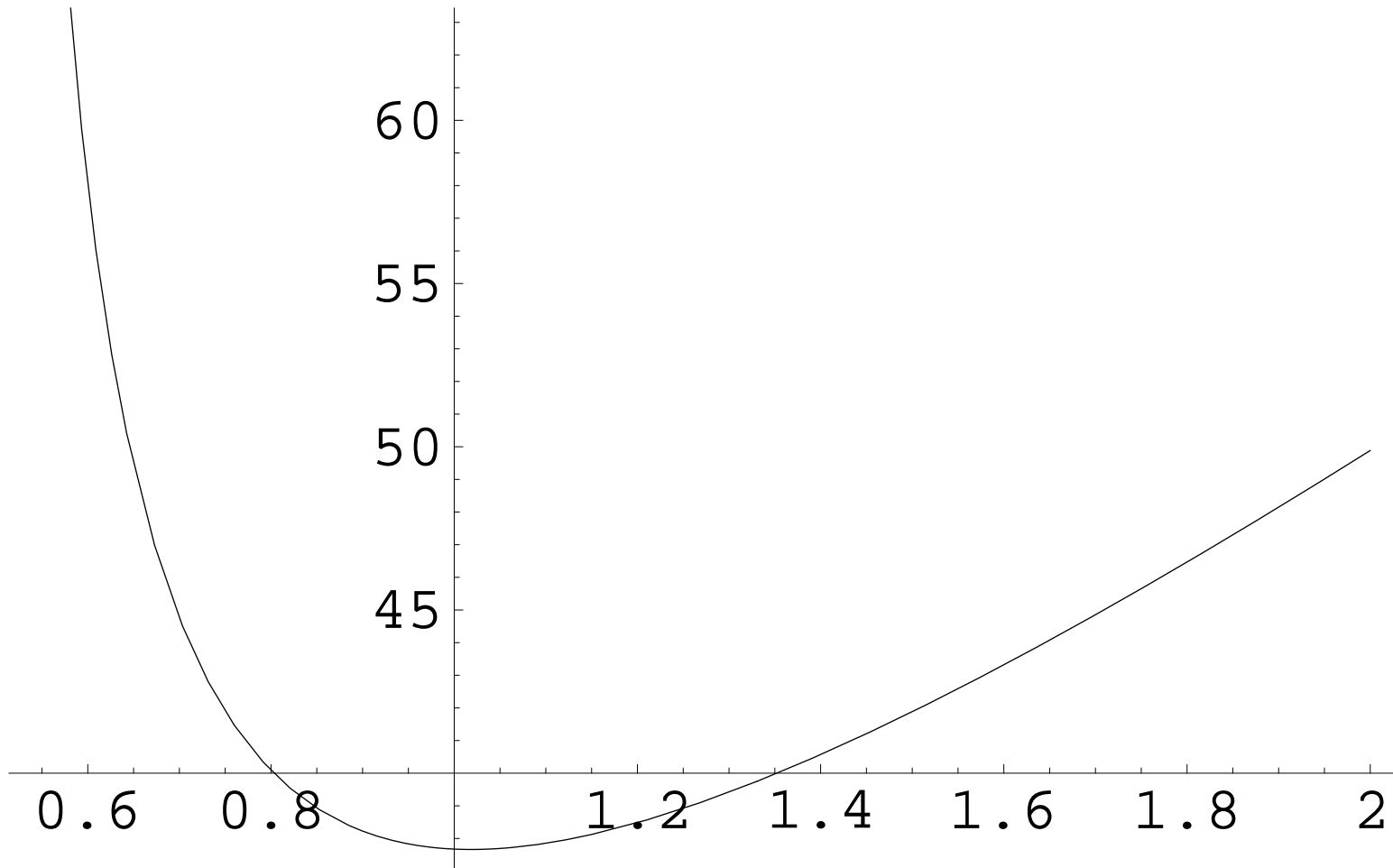
Fraction of wealth ($= \theta/W$) in stocks as a function of years to retirement (wealth 1). The horizontal line is for voluntary retirement (mean time to retirement= 40). The curved line is for mandatory retirement, which implies more caution. Human capital is most of wealth at this W .



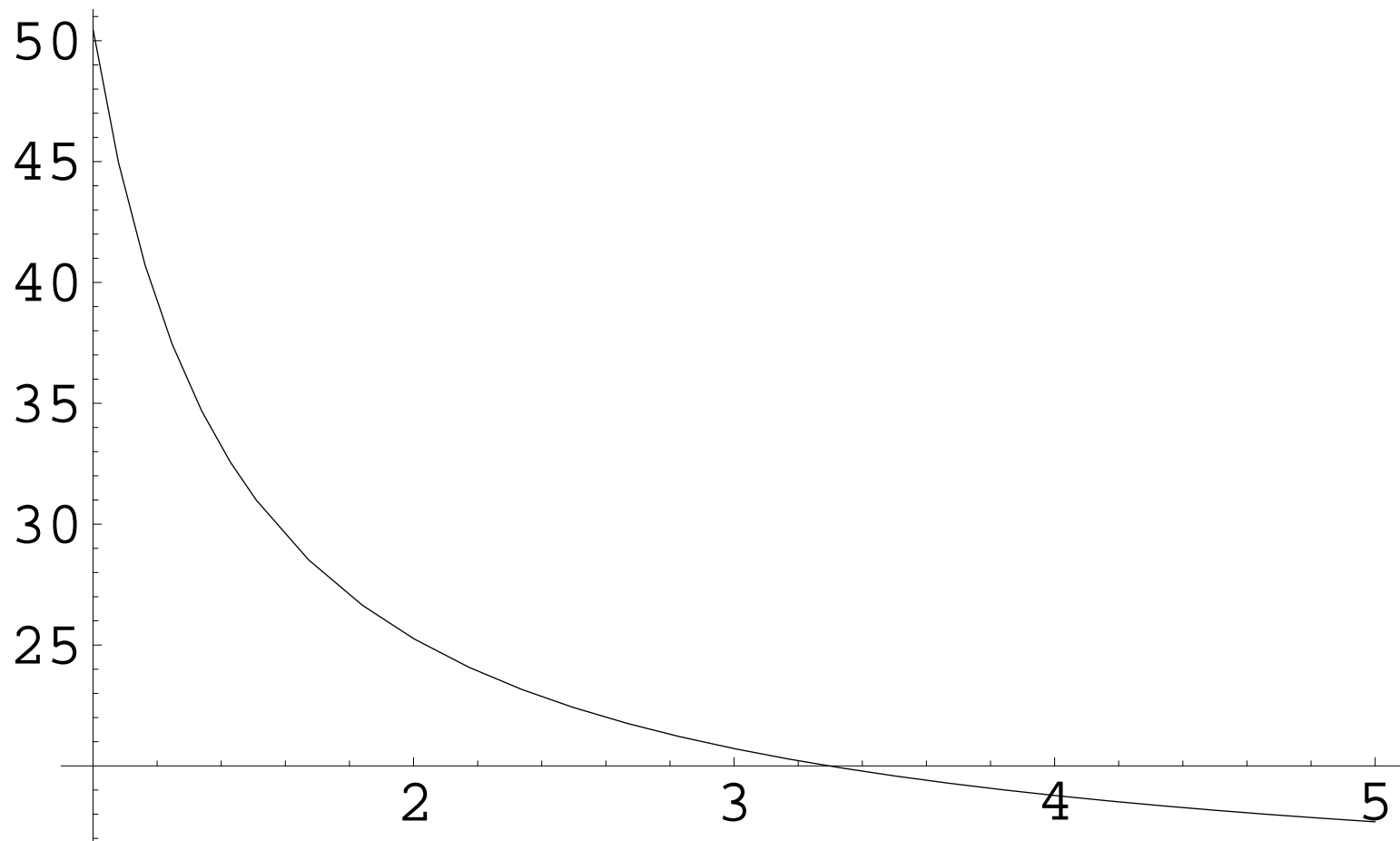
Net savings ($= w - c$) as a function of years to retirement (wealth 1). The horizontal line is for voluntary retirement (mean time to retirement = 40). The curved line is for mandatory retirement. The comparison of the two cases seems to vary with wealth.



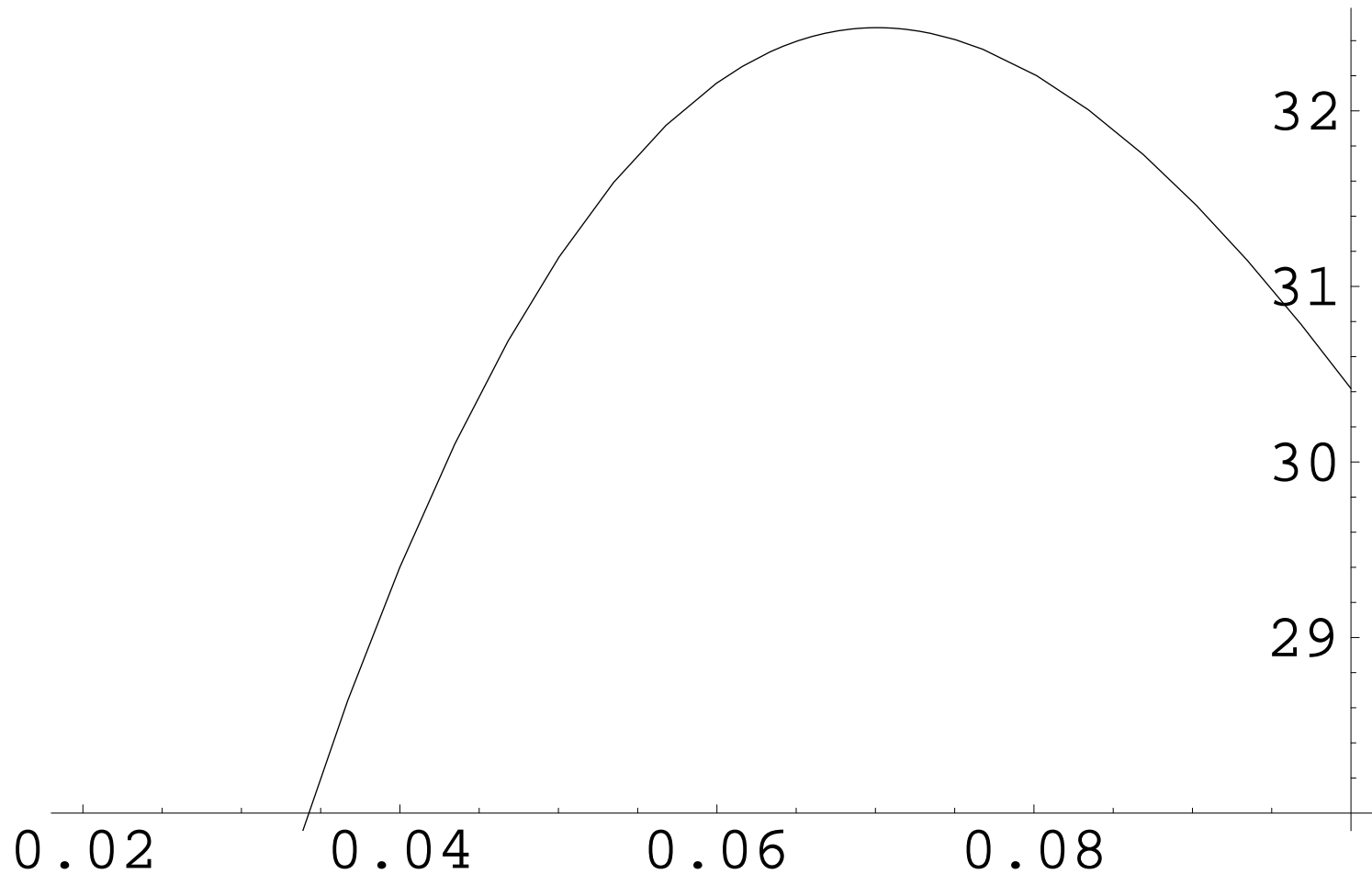
Value of voluntary retirement (in certainty equivalent per unit wealth) as a function of years to retirement for $W = 10$ (upper) and $W = 15$ (lower). Each minimum is near the expected time to voluntary retirement.



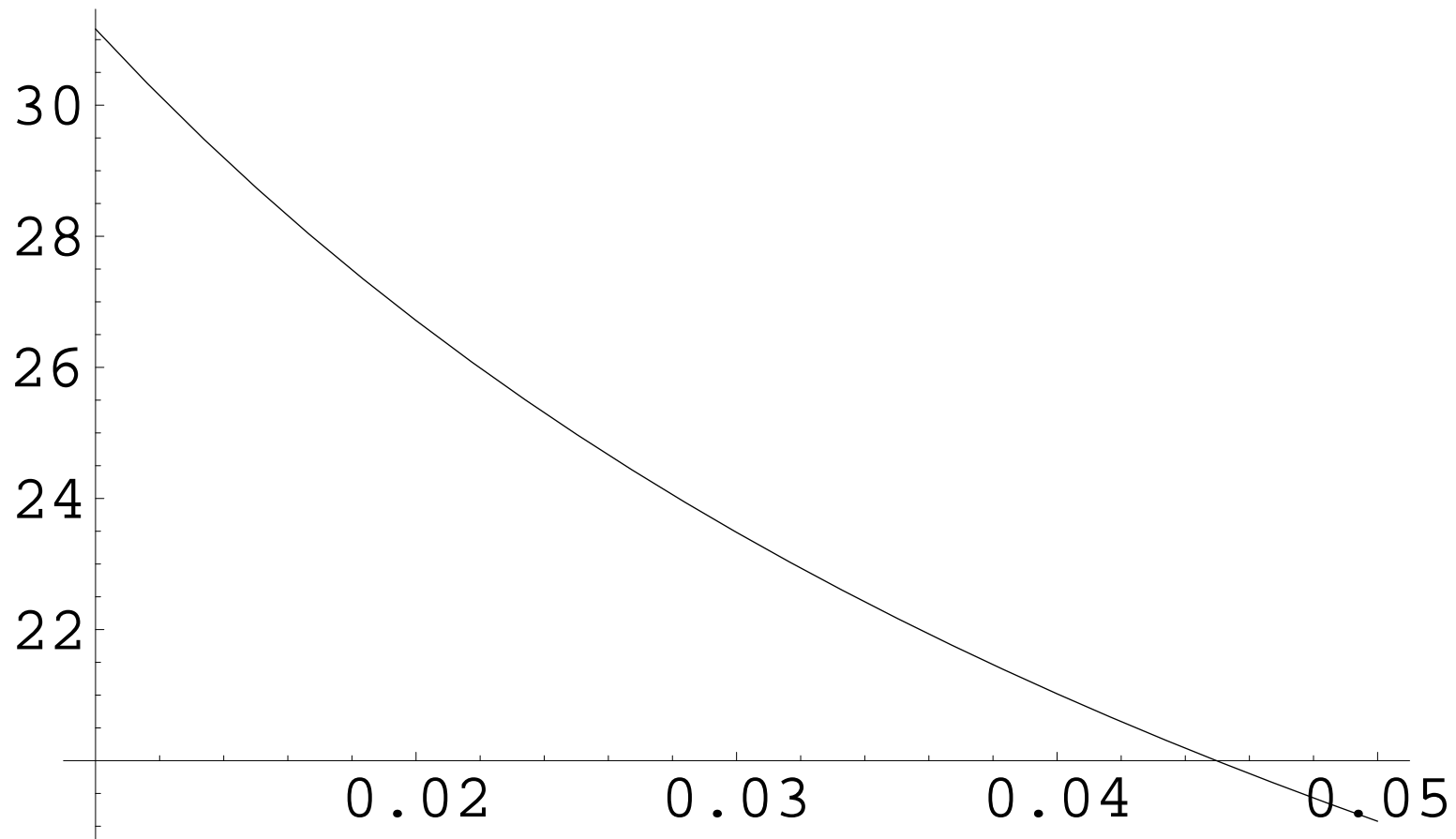
Expected number of years to voluntary retirement as a function of risk aversion γ at $W = 0$. This is probably due to an effective increase in impatience when γ increases.



Critical wealth \bar{W} for voluntary retirement retirement as a function of relative risk aversion γ .

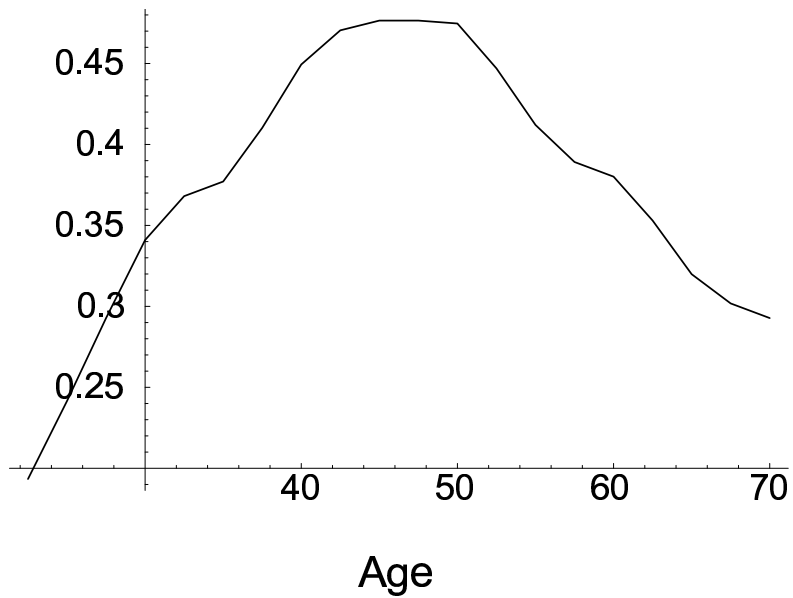


Critical retirement wealth \bar{W} as a function of μ can be thought of as response of leisure (inversely related to \bar{W}) to the price of later consumption (inversely related to μ). As μ increases, first leisure decreases (substitution effect between later consumption and leisure) then increases (income effect).

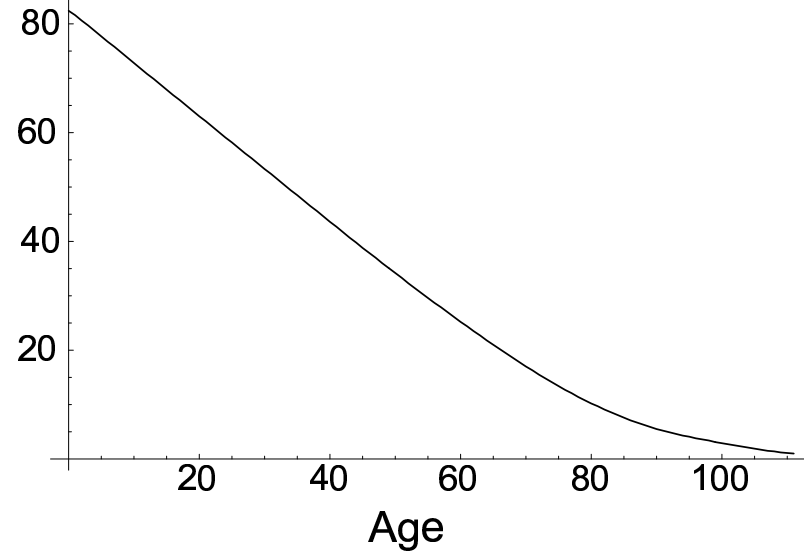


Critical retirement wealth \bar{W} falls as the pure rate of time preference ρ increases. Similarly, but not shown, \bar{W} falls as the mortality hazard rate δ rises.

Disposable Income (\$100,000)

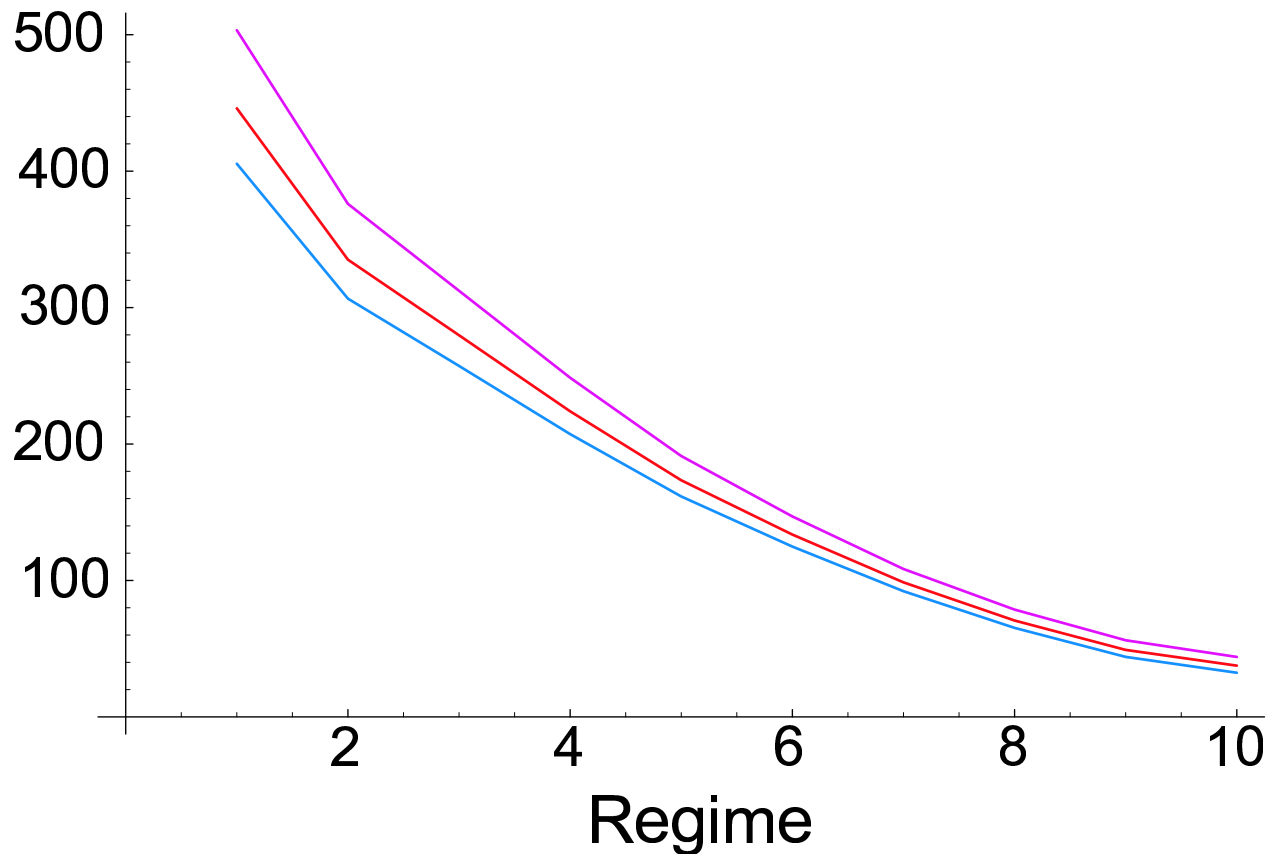


Expected Remaining Lifetime



Our original analysis assumed a lifetime income and mortality profile similar to empirical data.

Wealth threshold for retirement



Retirement is age dependent in the original analysis. We think of this as an intermediate case between the stationary voluntary retirement case in most of these slides (which has a horizontal boundary) and the mandatory retirement case (which has a vertical boundary).

Conclusion

We have constructed a workhorse model for analyzing retirement, pensions, and life insurance. The model is able to accommodate:

- voluntary or mandatory retirement date
- nonnegative wealth constraint on borrowing
- idealized hedging using life insurance
- bequest motive or not
- can handle wage and mortality varying through life
- stochastic wages over time