Variance Reduction for Monte Carlo Methods to Evaluate Option Prices under Multi-factor Stochastic Volatility Models

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References:

Variance Reduction for Monte Carlo Simulation in a Stochastic Volatility Environment

Quantitative Finance 2002

Pricing Asian Options with Stochastic Volatility

Quantitative Finance 2003

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Model Under Risk Neutral

$$
\begin{aligned}
dS_t &= rS_t dt + f(Y_t, Z_t) S_t dW_t^{(0)} \\
dY_t &= \left(\alpha (m_f - Y_t) - \nu_f \sqrt{2\alpha}\, \Lambda_f(Y_t, Z_t) \right) dt \\
&\quad + \nu_f \sqrt{2\alpha} \left(\rho_1 dW_t^{(0)} + \sqrt{1-\rho_1^2}\, dW_t^{(1)} \right) \\
dZ_t &= \left(\delta (m_s - Z_t) - \nu_s \sqrt{2\delta}\, \Lambda_s(Y_t, Z_t) \right) dt \\
&\quad + \nu_s \sqrt{2\delta} \left(\rho_2 dW_t^{(0)} + \rho_{12} dW_t^{(1)} + \sqrt{1-\rho_2^2-\rho_{12}^2}\, dW_t^{(2)} \right)
\end{aligned}
$$

- α **large**: Y_t fast mean reverting on time scale $1/\alpha << 1$
- δ **small**: Z_t slow varying on time scale $1/\delta >> 1$
- Λ_f and Λ_s: market prices of volatility risk
- $|\rho_1| < 1, |\rho_2| < 1, |\rho_{12}| < \sqrt{1-\rho_2^2}$: correlation coefficients

Options

European Options:

$$P(t, S_t, Y_t, Z_t) = \mathbb{E}^\star \left\{ e^{-r(T-t)} H(S_T) \mid S_t, Y_t, Z_t \right\}$$

Asian Call Options (fixed or floating strike):

$$P_t = \mathbb{E}^\star \left\{ e^{-r(T-t)} (A_T - K_1 S_T - K)^+ \mid \mathcal{F}_t \right\}$$

- Arithmetic Average Asian Option (AAO)

$$A_T = \frac{1}{T} \int_0^T S_t dt$$

- Geometric Average Asian Option (GAO)

$$A_T = \exp\left(\frac{1}{T} \int_0^T \ln S_t dt \right)$$

More general sampling functions: $dt \to d\lambda(t)$

$$\mathbf{dV_t} = \mathbf{b(t, V_t)dt + a(t, V_t)d\eta_t}$$

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad V_t = \begin{pmatrix} S_t \\ Y_t \\ Z_t \end{pmatrix} \qquad \eta_t = \begin{pmatrix} W_t^{(0)} \\ W_t^{(1)} \\ W_t^{(2)} \end{pmatrix}$$

$$b(t, v) = \begin{pmatrix} rx \\ \alpha(m_f - y) - \nu_f \sqrt{2\alpha}\, \Lambda_f(y, z) \\ \delta(m_s - z) - \nu_s \sqrt{2\delta}\, \Lambda_s(y, z) \end{pmatrix}$$

$$a(t, v) = \begin{pmatrix} f(y, z)x & 0 & 0 \\ \nu_f \sqrt{2\alpha}\, \rho_1 & \nu_f \sqrt{2\alpha}\, \sqrt{1 - \rho_1^2} & 0 \\ \nu_s \sqrt{2\delta}\, \rho_2 & \nu_s \sqrt{2\delta}\, \rho_{12} & \nu_s \sqrt{2\delta}\, \sqrt{1 - \rho_2^2 - \rho_{12}^2} \end{pmatrix}$$

European Monte Carlo

The price $P(t,x,y,z)$ of an European option at time t is given by

$$P(t,v) \quad = \quad \mathbb{E}^{\star}\left\{e^{-r(T-t)}H(S_T) \mid V_t = v\right\}$$

A basic Monte Carlo approximation is based on calculating the sample mean

$$P(t,x,y,z) \quad \approx \quad \frac{1}{N}\sum_{k=1}^{N} e^{-r(T-t)}H(S_T^{(k)})$$

over N independent realizations of V starting at time t from v.

Importance Sampling

Introduce the martingale

$$Q_t = \exp\left(\int_0^t h(s,V_s)d\eta_s + \frac{1}{2}\int_0^t ||h(s,V_s)||^2 ds\right)$$

and define a new probability $\tilde{I\!P}$ equivalent to $I\!P^\star$ by

$$\frac{d\tilde{I\!P}}{dI\!P^\star} = (Q_T)^{-1}$$

By Girsanov Theorem, under this new measure $\tilde{I\!P}$, the process

$$\tilde{\eta}_t = \eta_t + \int_0^t h(s,V_s)ds$$

is a standard Brownian motion.

Monte Carlo under $\tilde{\mathbb{P}}$

$$P(t,v) = \tilde{\mathbb{E}}\left\{ e^{-r(T-t)} H(S_T) Q_T \mid V_t = v \right\}$$

where

$$Q_T = \exp\left(\int_0^T h(s,V_s)\, d\tilde{\eta}_s - \frac{1}{2}\int_0^T ||h(s,V_s)||^2 ds \right)$$

and the dynamics of our model becomes

$$dV_t = \{ b(t,V_t) - a(t,V_t) h(t,V_t) \} dt + a(t,V_t)\, d\tilde{\eta}_t$$

where $\tilde{\eta}_t$ is a standard Brownian motion.

$$P(t,x,y,z) \approx \frac{1}{N} \sum_{k=1}^{N} e^{-r(T-t)} H(S_T^{(k)}) Q_T^{(k)}$$

Choice of h

Applying Ito's formula to $P(t, V_t)Q_t \quad \Longrightarrow$

$$H(V_T)Q_T = P(t,v) + \int_t^T Q_s(a' \nabla_v P + P\, h)(s, V_s)d\tilde{\eta}_s$$

Variance of the payoff $H(V_T)Q_T$

$$\mathrm{Var}_{\tilde{\mathbb{P}}}\{H(V_T)Q_T\} = \tilde{\mathbb{E}}\left\{\int_t^T Q_s^2 ||a' \nabla_v P + P\, h||^2 ds\right\}.$$

Indeed, if the quantity P to be computed was known, one could obtain a zero variance by choosing

$$h = -\frac{1}{P}(a' \nabla_v P)$$

Our strategy is to use approximations to the exact value P.

Pricing Equation

$\delta << 1$ and $\varepsilon = 1/\alpha << 1$

$$P^{\varepsilon,\delta}(t,x,y,z) \quad = \quad I\!E^\star\left\{ e^{-r(T-t)} H(S_T) \mid S_t = x, Y_t = y, Z_t = z \right\}$$

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3 \right) P^{\varepsilon,\delta} = 0$$

$$P^{\varepsilon,\delta}(T,x,y,z) = H(x)$$

$$\mathcal{L}_0 = (m_f - y)\frac{\partial}{\partial y} + \nu_f^2 \frac{\partial^2}{\partial y^2} \qquad \mathcal{M}_1 = \nu_s\sqrt{2}\left(-\Lambda_s \frac{\partial}{\partial z} + \rho_2 f x \frac{\partial^2}{\partial x \partial z} \right)$$

$$\mathcal{L}_1 = \nu_f\sqrt{2}\left(\rho_1 f x \frac{\partial^2}{\partial x \partial y} - \Lambda_f \frac{\partial}{\partial y} \right) \qquad \mathcal{M}_2 = (m_s - z)\frac{\partial}{\partial z} + \nu_s^2 \frac{\partial^2}{\partial z^2}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2} f^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot \right) \qquad \mathcal{M}_3 = 2\nu_f \nu_s \left(\rho_1\rho_2 + \rho_{12}\sqrt{1-\rho_1^2} \right) \frac{\partial^2}{\partial y \partial z}$$

European Options Approximations

Combination of **singular** and **regular** **perturbations** $\Longrightarrow$

$$P^{\varepsilon,\delta}(t,x,y,z) \approx \mathbf{P_{BS}}(\mathbf{t},\mathbf{x};\mathbf{T},\bar{\sigma}) + (T-t)\left(V_0^\delta \frac{\partial P_{BS}}{\partial \sigma} + V_1^\delta x \frac{\partial^2 P_{BS}}{\partial x \partial \sigma}\right)$$
$$+(T-t)\left(V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2}\right)\right)$$

Leading order Black-Scholes price $\mathbf{P_{BS}}(\mathbf{t},\mathbf{x};\bar{\sigma}(\mathbf{z}))$:

$$\mathcal{L}_{BS}(\overline{\sigma}(z))P_{BS} = 0$$
$$P_{BS}(T,x;\bar{\sigma}(z)) = H(x)$$

at the z-dependent **effective volatility** $\overline{\sigma}(z)$:

$$\overline{\sigma}^2(z) \quad = \quad \langle f^2(\cdot,z)\rangle$$

where the brackets denote the **average** with respect to the invariant distribution $\mathcal{N}(m_f,\nu_f^2)$.

The **small** parameters $(V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon)$ are given by

$$V_0^\delta = -\frac{\nu_s\sqrt{\delta}}{\sqrt{2}}\langle \mathbf{\Lambda_s}\rangle\overline{\sigma}' \qquad V_1^\delta = \rho_{\mathbf{2}}\frac{\nu_s\sqrt{\delta}}{\sqrt{2}}\langle f\rangle\overline{\sigma}'$$

$$V_2^\varepsilon = \frac{\nu_f\sqrt{\varepsilon}}{\sqrt{2}}\left\langle \mathbf{\Lambda_f}\frac{\partial\phi}{\partial y}\right\rangle \qquad V_3^\varepsilon = -\rho_{\mathbf{1}}\frac{\nu_f\sqrt{\varepsilon}}{\sqrt{2}}\left\langle f\frac{\phi}{\partial y}\right\rangle$$

$\overline{\sigma}' = d\bar{\sigma}/dz$, and $\phi(y,z)$ is a solution of the **Poisson equation**

$$\mathcal{L}_0\phi(y,z) \quad = \quad f^2(y,z) - \overline{\sigma}^2(z).$$

Accuracy:

- **Smooth** payoffs: error $= \mathcal{O}(\varepsilon + \delta)$
- **Calls** (kinks): error $= \mathcal{O}(\varepsilon\log|\varepsilon| + \delta)$
- **Digitals** (jumps): error $= \mathcal{O}(\varepsilon^{2/3}\log|\varepsilon| + \delta)$

Corrections Equations

$$P_1^\varepsilon(t,x,z) \quad = \quad (T-t)\left(V_2^\varepsilon x^2\frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 P_{BS}}{\partial x^2}\right)\right)$$

solves

$$\mathcal{L}_{BS}(\bar\sigma)P_1^\varepsilon + \left(V_2^\varepsilon x^2\frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 P_{BS}}{\partial x^2}\right)\right) = 0, \qquad P_1^\varepsilon(T,x,z) = 0$$

$$P_1^\delta(t,x,z) \quad = \quad (T-t)\left(V_0^\delta\frac{\partial P_{BS}}{\partial \sigma} + V_1^\delta x\frac{\partial^2 P_{BS}}{\partial x\partial\sigma}\right)$$

solves

$$\mathcal{L}_{BS}(\bar\sigma)P_1^\delta + 2\left(V_0^\delta\frac{\partial P_{BS}}{\partial \sigma} + V_1^\delta x\frac{\partial^2 P_{BS}}{\partial x\partial\sigma}\right) = 0, \qquad P_1^\delta(T,x) = 0$$

(for European options: $\frac{\partial P_{BS}}{\partial\sigma} = (T-t)\sigma x^2\frac{\partial^2 P_{BS}}{\partial x^2}$)

Computation of h using $\mathbf{P_{BS}}$ only

$$h(t,x,y,z) = \frac{-1}{P_{BS}(t,x;\bar\sigma(z))} a' \, \nabla P_{BS}(t,x;\bar\sigma(z))$$

$$= \frac{-1}{P_{BS}(t,x;\bar\sigma(z))} \begin{pmatrix} f(y,z)x & \frac{\rho_1 \nu_f \sqrt{2}}{\sqrt{\varepsilon}} & \rho_2 \nu_s \sqrt{2\delta} \\ 0 & \nu_f \frac{\sqrt{2}}{\sqrt{\varepsilon}} \sqrt{1-\rho_1^2} & \rho_{12} \nu_s \sqrt{2\delta} \\ 0 & 0 & \nu_s \sqrt{2\delta} \sqrt{1-\rho_2^2-\rho_{12}^2} \end{pmatrix} \begin{pmatrix} \frac{\partial P_{BS}}{\partial x} \\ \frac{\partial P_{BS}}{\partial y} \\ \frac{\partial P_{BS}}{\partial z} \end{pmatrix}$$

$$= \frac{-\frac{\partial P_{BS}}{\partial x}}{P_{BS}(t,x;\bar\sigma(z))} \begin{pmatrix} f(y,z)x \\ 0 \\ 0 \end{pmatrix} - \nu_s \sqrt{2\delta} \frac{\bar\sigma'(z) \frac{\partial P_{BS}}{\partial \sigma}}{P_{BS}(t,x;\bar\sigma(z))} \begin{pmatrix} \rho_2 \\ \rho_{12} \\ \sqrt{1-\rho_2^2-\rho_{12}^2} \end{pmatrix}$$

where we have used that **$\mathbf{P_{BS}}$ does not depend on y.**

The **Vega** is given by $\frac{\partial P_{BS}}{\partial \sigma} = x\sqrt{T-t}\,\mathcal{N}'(d_1(x,z))$.

Computation of h using $\tilde{\mathbf{P}} = \mathbf{P_{BS}} + \mathbf{P_1^\varepsilon} + \mathbf{P_1^\delta}$

$$\tilde{h}(t,x,y,z) = \frac{-1}{\tilde{P}(t,x,z)} a' \, \nabla \tilde{P}(t,x,z)$$

$$= \frac{-1}{\tilde{P}(t,x,z)} \begin{pmatrix} f(y,z)x & \frac{\rho_1 \nu_f \sqrt{2}}{\sqrt{\varepsilon}} & \rho_2 \nu_s \sqrt{2\delta} \\ 0 & \nu_f \frac{\sqrt{2}}{\sqrt{\varepsilon}} \sqrt{1-\rho_1^2} & \rho_{12} \nu_s \sqrt{2\delta} \\ 0 & 0 & \nu_s \sqrt{2\delta} \sqrt{1-\rho_2^2-\rho_{12}^2} \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{P}}{\partial x} \\ \frac{\partial \tilde{P}}{\partial y} \\ \frac{\partial \tilde{P}}{\partial z} \end{pmatrix}$$

$$\approx \frac{-\frac{\partial \tilde{\mathbf{P}}}{\partial x}}{\tilde{\mathbf{P}}(t,x,z)} \begin{pmatrix} f(y,z)x \\ 0 \\ 0 \end{pmatrix} - \nu_s \sqrt{2\delta} \frac{\overline{\sigma}'(z) \frac{\partial P_{BS}}{\partial \sigma}}{P_{BS}(t,x;\bar{\sigma}(z))} \begin{pmatrix} \rho_2 \\ \rho_{12} \\ \sqrt{1-\rho_2^2-\rho_{12}^2} \end{pmatrix}$$

where we have used again that $\tilde{\mathbf{P}}$ **does not depend on** y**,** and neglected terms of higher order.

Numerical Results (European Calls)

r	m_f	m_s	ν_f	ν_s	ρ_1	ρ_2	ρ_{12}	Λ_f	Λ_s	$f(y,z)$
10%	-0.8	-0.8	0.5	0.8	-0.2	-0.2	0	0	0	$\exp(y+z)$

$\$S_0$	Y_0	Z_0	$\$K$	T years
55	-1	-1	50	1

α	δ	P^{MC}	P_{BS}	$\tilde{P}$	$P^{IS}(P_{BS})$	$P^{IS}(\tilde{P})$
100	0.01	.02411 (10.93)	10.779	11.069	.00400 (11.13)	.00099 (11.03)
50	0.05	.02299 (11.03)	10.779	11.208	.00070 (11.03)	.00070 (10.99)
20	**0.1**	**.02260** (11.09)	10.779	11.449	**.00216** (11.09)	**.00128** (11.00)
5	**1**	**.03274** (11.50)	10.779	12.20	**.00384** (12.03)	**.00243** (11.60)

5000 realizations (Euler scheme with time step $\Delta t = 0.005$).

Two-Step Variance Reduction for Asian Options

$$P(t, S_t, Y_t, Z_t, I_t) \quad = \quad \mathbb{E}^\star \left\{ e^{-r(T-t)} \left(\frac{I_T}{T} - K_1 S_T - K \right)^+ \mid S_t, Y_t, Z_t, I_t \right\}$$

where $I_t = \int_0^t S_u du$ (or $dI_t = S_t dt$).

Basic monte Carlo for AAO's

$$P \quad \approx \quad P^{MC} = \frac{e^{-r(T-t)}}{N} \sum_{k=1}^{N} \left(\frac{I_T^{(k)}}{T} - K_1 S_T^{(k)} - K \right)^+$$

- Unlike the case of European options, the approximated prices of AAOs **do not have close-form solutions** (F-Han 2003).
- **Importance sampling** using approximated AAO prices requires numerical PDE solutions $\Longrightarrow$ **not efficient**.
- $\Longrightarrow$ two-step variance reduction strategy by combining **control variates and importance sampling**.

Control Variates for Arithmetic Average Asian Options

- **Constant volatility**: Boyle-Broadie-Glasserman proposed a variance reduction method for AAOs based on using **GAOs as control variates**:

$$\mathbf{P_A^{CV}} = \mathbf{P_A^{MC}} + \lambda(\mathbf{P_G^{MC}} - \mathbf{P_G})$$

- P_G^{MC} is the unbiased Monte Carlo estimator of the GAO price computed using the same run as for P_A^{MC}.
- The company price P_G, i.e. the counterpart geometric average Asian option, has an **analytic solution**.
- The parameter λ is chosen to minimize the sample variance. For Asian options, λ **is often chosen equal to -1.**

Two-Step Strategy in the Stochastic Volatility Case

- Implement **GAOs control variates**.
- **No close-form** solutions for GAOs.
- **Evaluate GAOs by Monte Carlo** using the **importance sampling** variance reduction technique presented and tested in the European options case.

$$P_G(t, V_t, L_t) \quad = \quad \mathbb{E}^\star \left\{ e^{-r(T-t)} \left(\exp\left(\frac{L_T}{T}\right) - K_1 S_T - K \right)^+ \mid V_t, L_t \right\}$$

where $V_t = (S_t, Y_t, Z_t)$, and the running sum process (L_t) is given by

$$dL_t \quad = \quad \ln S_t \, dt$$

Approximate GAOs Prices (fixed strike case $K_1 = 0$)

$$\mathbf{P_G(t,x,y,z,L)} \approx \widetilde{\mathbf{P_G}}(\mathbf{t,x,z,L})$$

where

$$\begin{aligned}\widetilde{P_G} &= P_0^{fix} - V_0^{\delta}(T-t)\sqrt{2}\frac{\partial P_0^{fix}}{\partial \sigma} + V_1^{\delta}(T-t)x\frac{\partial^2 P_0^{fix}}{\partial x \partial \sigma} \\ &\quad - \frac{V_2^{\varepsilon}(T-t)^2}{2}\frac{\partial P_0^{fix}}{\partial x} + \frac{(V_2^{\varepsilon} - V_3^{\varepsilon})(T-t)^3}{3}\frac{\partial^2 P_0^{fix}}{\partial x^2} + \frac{V_3^{\varepsilon}(T-t)^4}{4}\frac{\partial^3 P_0^{fix}}{\partial x^3}\end{aligned}$$

The leading order term is the homogenized "Black-Scholes " price:

$$P_0^{fix}(t,x,L;\bar{\sigma}) = \exp\left(\frac{L - t\ln x}{T} + \ln x + R\right)\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2)$$

$$\frac{\partial P_0^{fix}}{\partial \sigma} = \frac{T-t}{3}\bar{\sigma}x^2\frac{\partial^2 P_0^{fix}}{\partial x^2} - \frac{T-t}{6}\bar{\sigma}x\frac{\partial P_0^{fix}}{\partial x}$$

with formulas for $R(t,T,z)$, $d_1(x,z,L)$ and $d_2(x,z,L)$.

Computation of $\mathbf{h_G}$

Approximate the "perfect" h_G

$$h_G(t,x,y,z,L) = -\frac{1}{P_G} a' \nabla_{(v,L)} P_G$$

by approximating $P_G(t,x,y,z,L)$ by $\widetilde{P_G}(t,x,z,L) \quad \Longrightarrow$

$$\widetilde{\mathbf{h_G}}(t,x,z,L) =$$

$$\frac{-\frac{\partial \widetilde{\mathbf{P_G}}}{\partial x}}{\widetilde{\mathbf{P_G}}(t,x,z,L)} \begin{pmatrix} f(y,z)x \\ 0 \\ 0 \\ 0 \end{pmatrix} - \nu_s \sqrt{\mathbf{2}\delta} \frac{\bar{\sigma}(z) \frac{\partial P_0^{fix}}{\partial \sigma}}{P_0^{fix}(t,x,z,L)} \begin{pmatrix} \rho_2 \\ \rho_{12} \\ \sqrt{1-\rho_2^2-\rho_{12}^2} \\ 0 \end{pmatrix}$$

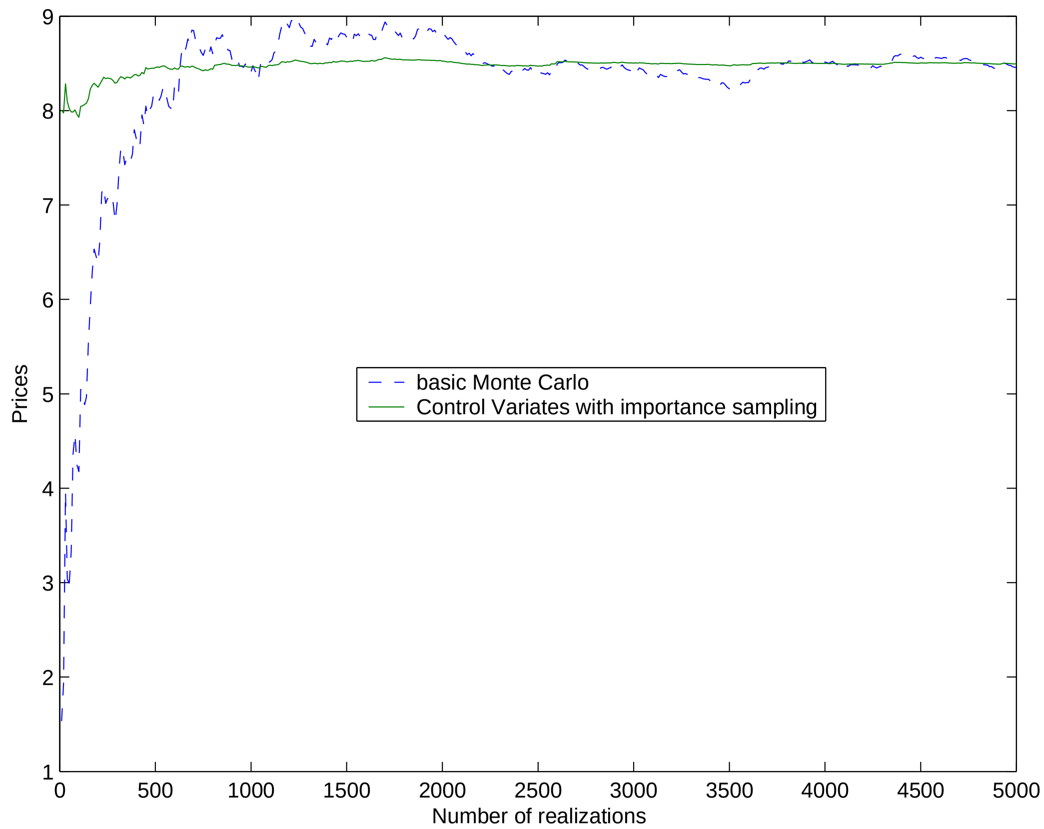
Importance Sampling GAOs Monte Carlo

r	m_f	m_s	ν_f	ν_s	ρ_1	ρ_2	ρ_{12}	Λ_f	Λ_s	$f(y,z)$
10%	-0.8	-0.6	0.7	1	-0.2	-0.2	0	0	0	$\exp(y+z)$

$\$S_0$	Y_0	Z_0	L_0	$\$K$	T years
100	-1	-0.5	0	110	1

α	δ	P_G^{MC}	$P_G^{IS}(\widetilde{P_G})$
100	0.05	0.048341 (7.97)	0.006334 (7.76)
75	0.1	0.043363 (7.57)	0.007707 (7.46)
50	0.5	0.051290 (7.45)	0.009676 (7.17)
25	1	0.058433 (7.31)	0.014814 (6.96)

AAOs Monte Carlo by C.V. GAOs and I.S.



$\mathbf{P_A^{CV}} = \mathbf{P_A^{MC}} - \mathbf{1} \left(\mathbf{P_G^{MC}} - \mathbf{P_G^{IS}}(\widetilde{\mathbf{P_G}}) \right)$ with $\alpha = 75$ and $\delta = 0.1$

Variance reduced from $(\mathbf{1.5411})\mathbf{10^{-4}}$ to $(\mathbf{1.6201})\mathbf{10^{-6}}$ with respective sample means 8.4604 and 8.4965 ($\Delta t = 0.005, N = 5000$).