Monte Carlo Methods for American Options: Overview and New Results

Paul Glasserman
Columbia University

Pricing American Options

- Pricing American option = solving optimal stopping problem
- Deterministic numerical methods (binomial lattices, PDE methods) in dimensions 1-3
- Lots of problems in dimension 5-80
- Monte Carlo potentially attractive --- how to combine simulation and dynamic programming?
- We give
  - overview of a class of related methods and ideas
  - a new result on a regression-based method

Monte Carlo for American Options: A Circle of Ideas

- Weighted backward induction on independent paths
  - Weights from likelihood ratios
  - Weights from calibration
  - Weights from regression
- High bias from peeking ahead
- Low bias from suboptimality
- Duality

Formulation

Exercise opportunities at $0=t_0<t_1<\ldots<t_m$

Underlying Markov process $X(t)$, $X_i=X(t_i)$

Exercise at t_i in state x pays $h_i(x) \geq 0$

Value at date i in state x

$$V_i(x) = \sup_{\tau \in \mathrm{T}_i} E[h_\tau(X_\tau) \mid X_i = x]$$

$\mathrm{T}_i = \{i, i+1, \ldots, m\}$ – valued stopping times

Find $V_0(X_0)$

Dynamic Programming

$$V_m = h_m$$

$$V_i(x) = \max\left\{h_i(x), E[V_{i+1}(X_{i+1}) \mid X_i = x]\right\}, \quad i = 0,1,\ldots,m-1$$

Continuation values

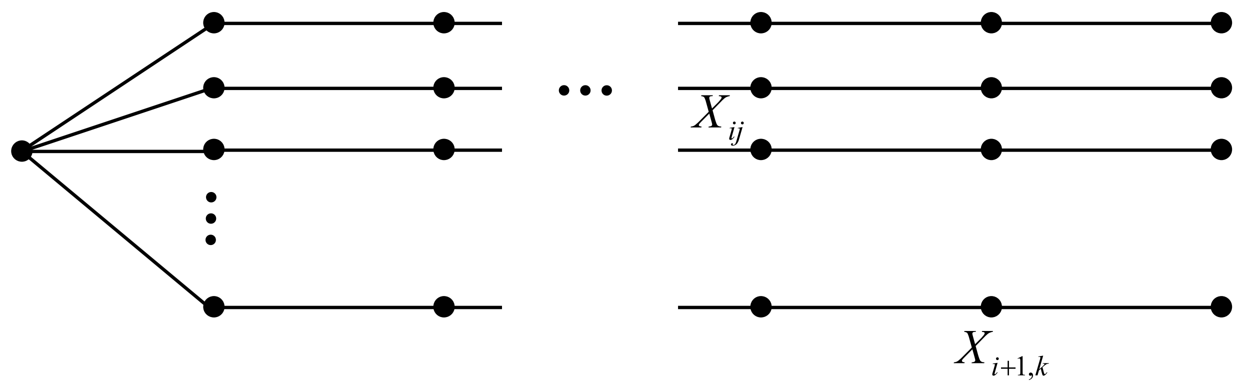
$$C_i(x) = E[V_{i+1}(X_{i+1}) \mid X_i = x]$$

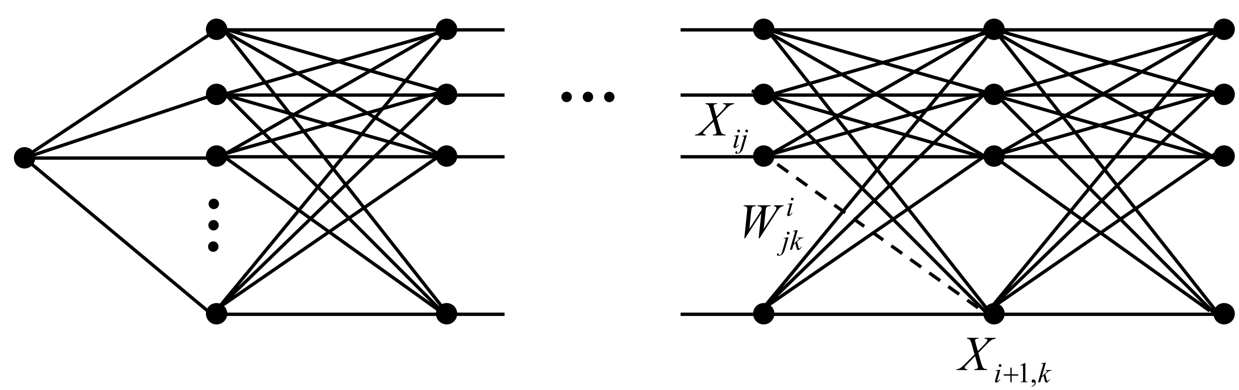
$$V_i = \max(h_i, C_i)$$

$$C_m = 0$$

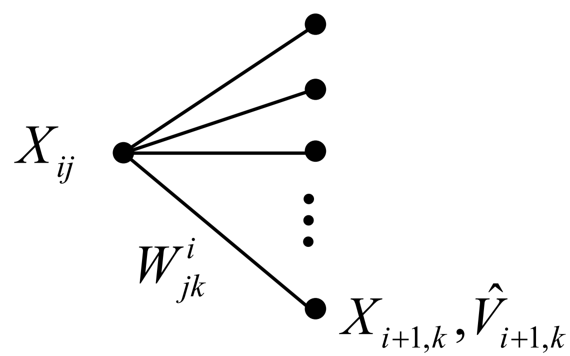
$$C_i(x) = E[\max\{h_{i+1}(X_{i+1}), C_{i+1}(X_{i+1})\} \mid X_i = x]$$

Stochastic Mesh





Weighted Backward Induction



Estimate $C_i(X_{ij}) = E[V_{i+1}(X_{i+1}) \mid X_i = X_{ij}]$

using $\hat{C}_{ij} = \frac{1}{b}\sum_{k=1}^{b} W^i_{jk} \hat{V}_{i+1,k}$

Weights correct for the fact that downstream nodes have the "wrong" distribution : $X_{i+1,k}$ does not have the law of $(X(t_{i+1}) \mid X(t_i) = X_{ij})$

High Bias

Basic algorithm

$$\hat{V}_M = h_M$$

$$\hat{V}_{ij} = \max\left\{ h_i(X_{ij}), \frac{1}{b}\sum_{k=1}^{b} W^i_{jk} \hat{V}_{i+1,k} \right\}$$

Suppose weights "work" for true value

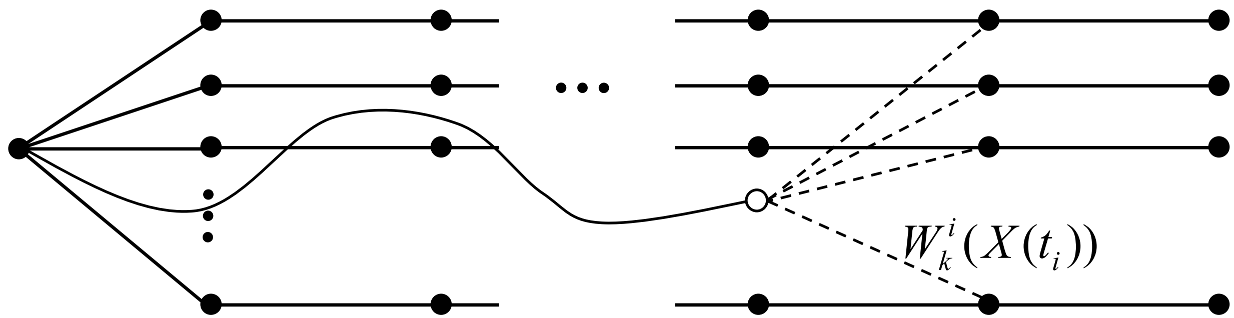
$$E[W^i_{jk} V_{i+1}(X_{i+1,k}) \mid X_i = X_{ij}] = C_i(X_i)$$

Then induction shows

$$E[\hat{V}_{ij} \mid X_{ij}] \geq V_i(X_{ij})$$

by Jensen's inequality and convexity of max

Low Bias: Mesh-Defined Stopping Rule



Simulate independent path

At each exercise date t_i estimate continuation value

$$\hat{C}_i(X(t_i)) = \frac{1}{b}\sum_{k=1}^{b} W_k^i(X(t_i))\hat{V}_{i+1,k}$$

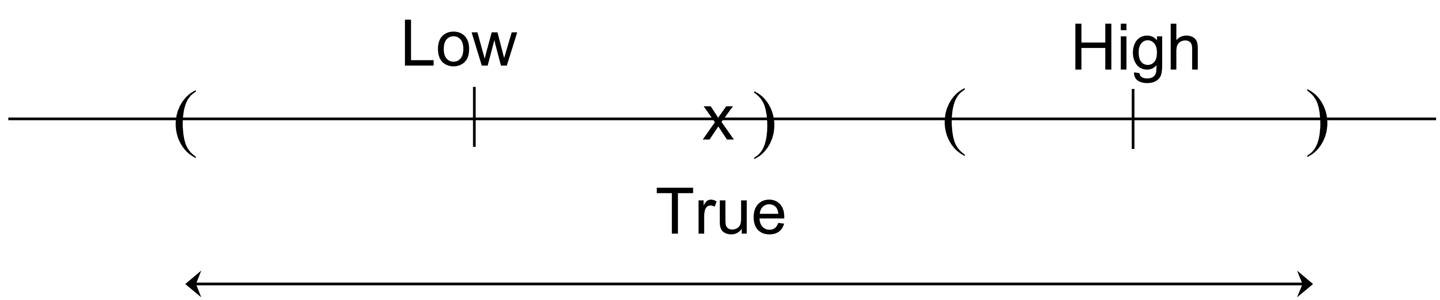
Stopping rule $\quad \hat{\tau} = \min\left\{ i : h_i(X(t_i)) \geq \hat{C}_i(X(t_i)) \right\}$

Estimate $h_{\hat{\tau}}(X(t_{\hat{\tau}}))$ (separates decision from payoff)

Stopping rule is suboptimal so

$$E[h_{\hat{\tau}}(X(t_{\hat{\tau}}))] \leq V_0(X_0)$$

Interval Estimate



Choice of Weights

Unweighted average

$$\frac{1}{b}\sum_{k=1}^{b} V_{i+1}(X_{i+1,k}) \to E[V_{i+1}(X_{i+1})] \neq E[V_{i+1}(X_{i+1}) \mid X_i = X_{ij}]$$

Suggests weights should be likelihood ratios relating conditional and unconditional distributions

By Radon-Nikodym Theorem, these are the only weights that work, if we consider a sufficiently rich class of V_{i+1}

Simplest case

$$W_{jk}^{i} = \frac{f_i(X_{ij}, X_{i+1,k})}{g_{i+1}(X_{i+1,k})} = \frac{\text{transition density}}{\text{marginal density}}$$

Convergence results (Broadie-Glasserman 1997, Avramidis-Matzinger 2002) for alternative constructions and LR weights

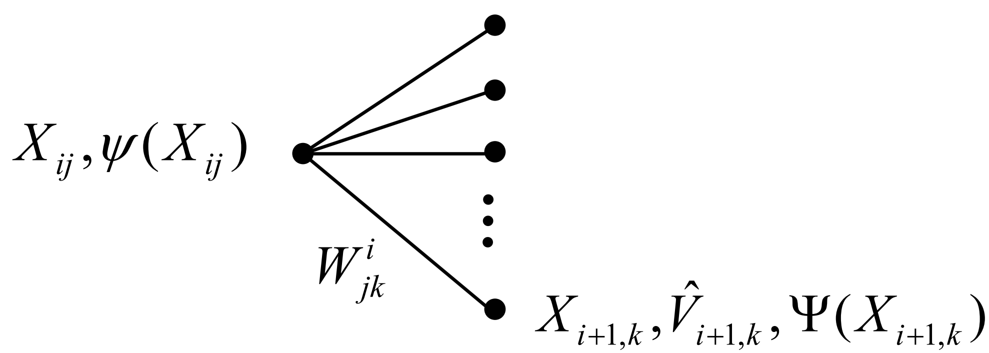
Central limit theorem? Choice of conditioning?

Weights Through Calibration

Transition density

- may be unknown
- may fail to exist (dim(state)>dim(Brownian motion))

Weights through calibration (B-G-Ha 2001)



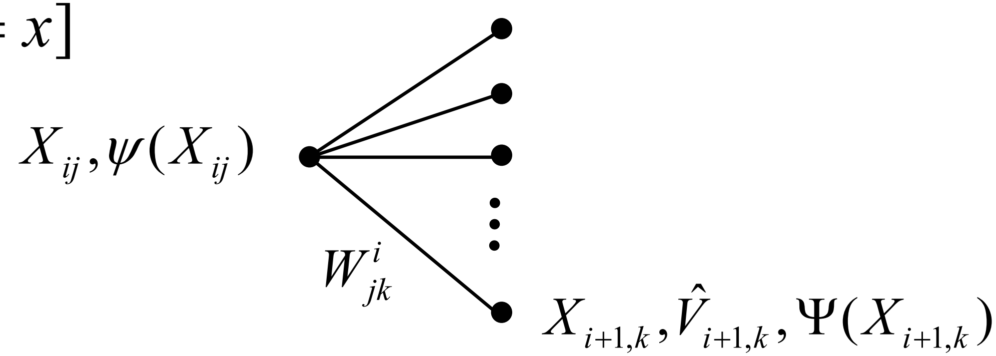
Given known functions

$$\psi(x) = E[\Psi(X_{i+1}) \mid X_i = x]$$

choose weights that “price” Ψ correctly

Weights Through Calibration

$$\psi(x) = E[\Psi(X_{i+1}) \mid X_i = x]$$



Choose weights so that

$$\frac{1}{b}\sum_{k=1}^{b} W^i_{jk} \Psi(X_{i+1,k}) = \psi(X_{ij})$$

From all feasible solutions, choose one minimizing

$$\sum_{k=1}^{b} H(W^i_{jk}),$$ some convex H (e.g., quadratic, entropy)

These are maximally uniform weights (in majorization ordering)

Weights Through Calibration

- Multiperiod version of Weighted Monte Carlo (in the sense of Avellaneda et al.)
- Analysis of WMC in Glasserman-Yu (2003) shows that over a single period
  – In limit as b increases, equivalent to a (signed) change of measure determined by $G = (H')^{-1}$

$$\frac{1}{b}\sum_{k=1}^{b} W_{jk}^{i} V(X_{i+1,k}) \Rightarrow E[V(X_{i+1})G(\mu + \lambda'\Psi(X_{i+1})) \mid X_i]$$

  – Correctly prices V in span of Ψ
  – For quadratic H, equivalent to regression on Ψ rather than ψ
  – Regression on Ψ rather than ψ produces less-dispersed estimates

Multiple periods? Choice of objectives? CLT?

Regression-Based Methods

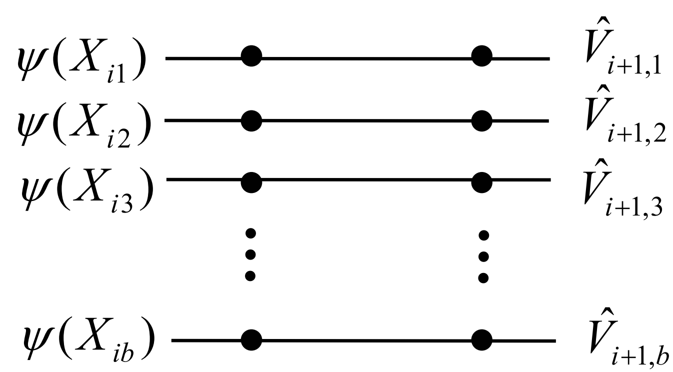
Carrière (1996), Longstaff-Schwartz (2001), Tsitsiklis-Van Roy (1999,2001)

Posit approximation of the form

$$C_i(x) \approx \beta' \psi(x)$$

$$V_{i+1}(X_{i+1,j}) = \beta' \psi(X_{ij}) + \varepsilon$$

Estimate coefficients through regression

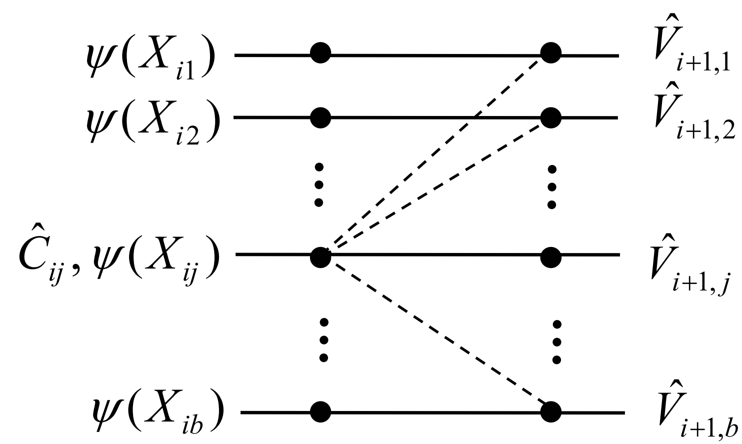


Regression-Implied Mesh Weights

$$\hat{\beta} = B_{\psi}^{-1} B_{\psi V}, \qquad B_{\psi V} = \frac{1}{b}\sum_{k=1}^{b} \psi(X_{ik})\hat{V}_{i+1,k}$$

so

$$\begin{aligned}\hat{C}_{ij} &= \psi(X_{ij})'\hat{\beta} \\ &= \psi(X_{ij})' B_{\psi}^{-1} B_{\psi V} \\ &= \frac{1}{b}\sum_{k=1}^{b} \underbrace{\psi(X_{ij})' B_{\psi}^{-1}\psi(X_{ik})}_{W_{jk}^{i}} \hat{V}_{i+1,k}\end{aligned}$$



Backward Induction

Tsitsiklis-Van Roy (1999,2001):

$$\hat{V}_{ij} = \max(h_i(X_{ij}), \hat{C}_i(X_{ij}))$$

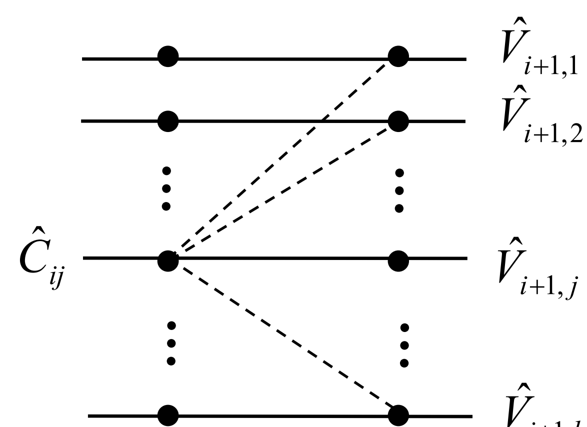
"High" mesh estimator with regression weights

Longstaff-Schwartz (2001):

$$\hat{V}_{ij} = \begin{cases} h_i(X_{ij}), & \text{if } h_i(X_{ij}) \geq \hat{C}_i(X_{ij}) \\ \hat{V}_{i+1,j}, & \text{if } h_i(X_{ij}) < \hat{C}_i(X_{ij}) \end{cases}$$

Interleaves "high" and "low"

Numerically, T-V with second pass similar to L-S



More on Weights

Special form of regression weights allows fast calc.

$$\hat{C}_{ij} = \frac{1}{b} \sum_{k=1}^{b} \underbrace{\psi(X_{ij})' B_{\psi}^{-1} \psi(X_{ik})}_{W_{jk}^{i}} \hat{V}_{i+1,k}, \quad O(b^2)$$

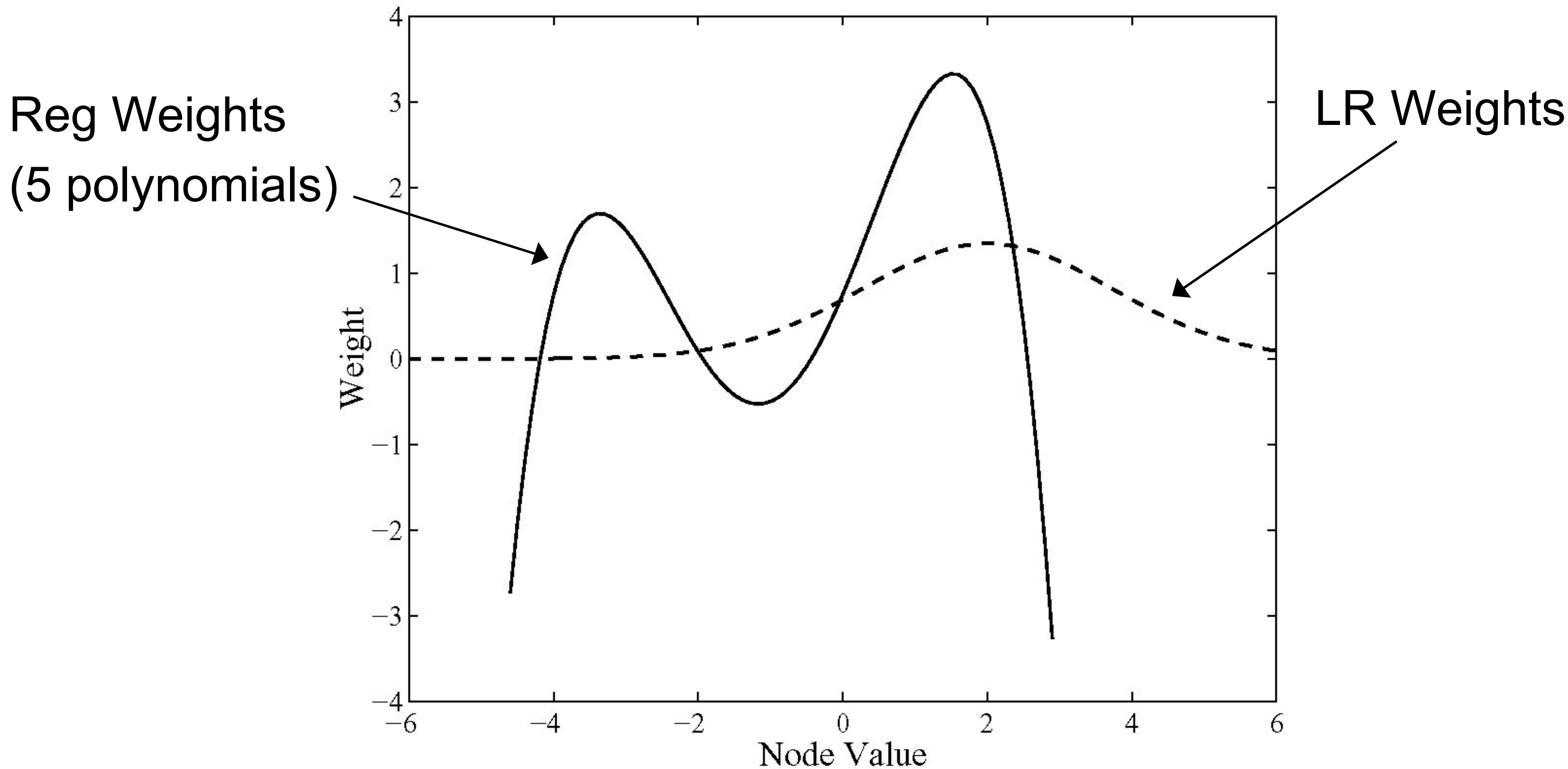
$$\hat{C}_{ij} = \beta' \psi(X_{ij}), \qquad O(bK),\ K = \text{dim}\, \psi << b$$

big advantage

Requires little to get *some* price

More on Weights

- Regression weights ignore spacing t_{i+1}-t_i
- Weights can be surprising



Convergence

- Convergence of regression-based methods requires increase in K as well as b
- Clément, Lamberton, Protter (2002) prove convergence of L-S method as first b then K increase
- What about rate as both increase together?
- We consider (joint work with Bin Yu)
  - Brownian motion, geometric Brownian motion
  - polynomial ψ
  - worst-case convergence
  - single-period and multiple periods

Formulation: Single-Period Brownian Case

- Dates $t_1 < t_2$
- Hermite polynomials $\psi_k(X(t_1)), k = 0,1,...,K$ normalized so $E[\psi_k^2(X(t_1))] = 1,$ makes $B_\psi = I$
- Downstream value spanned:

$$Y = \sum_{k=0}^{K} a_k \psi_k(X(t_2))$$

$$E[Y \mid X(t_1)] = \sum_{k=0}^{K} \beta_k \psi_k(X(t_1))$$

- Criterion:

$$\text{MSE}(\hat{\beta}) = \sum_k \text{Var}[\hat{\beta}_k] = \int [\hat{\beta}'\psi(x) - \beta'\psi(x)]^2 \phi(x)\, dx$$

Theorem

Let $c = 2\log\left(2 + \sqrt{t_2 / t_1}\right)$; $\qquad N = \#$paths

If $K_N = (1-\delta)\log N / c$, for some $\delta > 0$

$$\lim_{N\to\infty} \sup_{\|\beta\|=1} \text{MSE}(\hat{\beta}) = 0$$

If $K_N = (1+\delta)\log N / c$, for some $\delta > 0$

$$\lim_{N\to\infty} \sup_{\|\beta\|=1} \text{MSE}(\hat{\beta}) = \infty$$

Worst case attained at $\beta = (0,\ldots,0,1)'$

Estimated MSE

N	500	1000	2000	4000	8000	16000	32000	64000	128000
K=1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
2	0.08	0.04	0.02	0.01	0.01	0.01	0.01	0.01	0.01
3	1.67	0.31	0.17	0.08	0.04	0.02	0.01	0.01	0.01
4	5.6	3.0	1.6	1.73	0.36	0.18	0.09	0.05	0.02
5	52.7	23.4	13.5	6.0	3.1	1.5	0.8	1.4	0.2
6	427.2	155.7	93.3	38.4	24.0	10.8	6.2	3.1	1.5
7	2403	1202	600.8	300.4	150.2	75.1	37.5	18.8	9.4
8	11447	5723	2862	1431	715.4	357.7	178.9	89.4	44.7
9			9856	4928	2464	1232	616	308	154
10					6109	3054	1527	764	381
11							2810	1405	702
12									1023
Bound	2.5	2.8	3.1	3.4	3.7	3.9	4.2	4.5	4.8

Theorem: Lognormal Case

$$\psi_k(X(t),t) = X(t)^k e^{-k^2 t/2}$$

If $K_N = \sqrt{\frac{(1-\delta)\log N}{5t_1 + t_2}}$, for some $\delta > 0$

$$\lim_{N\to\infty} \sup_{\|\beta\|=1} \text{MSE}(\hat{\beta}) = 0$$

If $K_N = \sqrt{\frac{(1+\delta)\log N}{3t_1 + t_2}}$, for some $\delta > 0$

$$\lim_{N\to\infty} \sup_{\|\beta\|=1} \text{MSE}(\hat{\beta}) = \infty$$

i.e., need $N = O(\exp(K^2))$

Multiperiod Results

- Independent paths at each step (for tractability)
- "Spanning" assumption not preserved by backward induction
- We assume

$$E[h_n^4(X_n)] \le (t_n / t_{n-1})^{2K} E[\psi_{nK}(X_n)^4]$$

- Get general bound on error $E\left[\left\|C_n - \hat{C}_n\right\|^2\right]$

- Single-period rates sufficient for convergence of multiperiod problem

Comments

- No free lunch
- Just polynomials? Negative rate determined by "co-kurtosis"

$$\frac{E[\psi_K(X(t_2),t_2)^2\psi_K(X(t_1),t_1)^2]}{(E[\psi_K(X(t_2),t_2)\psi_K(X(t_1),t_1)])^2}$$

Numerically find exponential growth for some bounded families

- How relevant is worst case?
- Egloff (2003) gets more positive conclusion with bounds based on VC-dimension (for bounded payoffs)

Concluding Remarks

- Several techniques turn out to be closely related
- Differ in
  - choice of weights
  - combinations of high- and low-bias elements
- Other strategies for choosing weights?
- Dual formulation (Haugh-Kogan 2001,Rogers 2001)
  - minimizing over martingales instead of maximizing over stopping times
  - combine with any suboptimal policy to get bounds
  - when regression is valid, residuals give optimal martingale
  - "High" estimate has same form as dual
- Lots of questions about convergence remain open

Dual Formulation

Haugh & Kogan (2001), Rogers (2001)

For any martingale $0 = M_0, M_1, \ldots, M_m$

$$V_0(X_0) \le E\left[\max_{0 \le k \le m}\{h_k(X_k) - M_k\}\right]$$

Equality if M is the martingale part of V (in the sense of Doob-Meyer decomposition)

Andersen-Broadie (2001): Given any stopping rule, use martingale part of (suboptimal) value; this gives upper bound to accompany low estimate

Is this upper bound related to other methods?

More on Duality

- If the regression is valid in the sense that

$$V_{i+1}(X_{i+1}) = \beta'\psi(X_i) + \varepsilon_{i+1}, \qquad E[\varepsilon_{i+1} \mid X_i] = 0$$

then residuals define the optimal martingale

$$M_i = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i$$

Dual is the high estimator

- Every high estimator can be written as

$$\max_{0 \le k \le m} \left\{ h_k(X_k) - \sum_{i=1}^{k} \varepsilon_i \right\}, \qquad \varepsilon_{i+1} = \hat{V}_{i+1}(X_{i+1}) - \hat{C}_i(X_i)$$

these may not be martingale differences