Risk-tolerance wealth processes and sensitivity analysis of utility based prices

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Outline

In incomplete markets (**marginal**) prices of non traded derivative securities depend on

1. risk-preferences
2. total wealth x
3. the number q of the contingent claims in the portfolio:

$$p = p(x, q)$$

Our goal is to study the **sensitivity of prices with respect to quantities**:

$$\frac{\partial p}{\partial q}(x, 0) - ?$$

Model of a financial market

There are $d+1$ **traded** or **liquid** assets:

1. a **savings account**. We assume that the interest rate is 0:

$$B = \text{const}$$

2. d **stocks**. The price process S of the stocks is a RCLL process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, where T is a maturity.

Assumption *The model is arbitrage free.*

Fundamental Theorem of Asset Pricing: the model is arbitrage free iff

$$\mathcal{M} \neq \emptyset$$

where $\mathcal{M}$ is the family of equivalent local martingale measures for S.

Contingent claims

Consider a family of N **non-traded** or **illiquid** European contingent claims with

- maturity T
- payment functions $f = (f_i)_{1 \leq i \leq N}$

Valuation and pricing problems

Consider an investor which portfolio at time 0 consists of

1. initial capital x invested in the liquid securities (savings account and stocks)
2. quantities $q = (q_i)_{1 \leq i \leq N}$ of the non-traded contingent claims.

Question 1 What is the **cash value** $c(x, q)$ of the portfolio (x, q)?

Question 2 What is the **marginal** price $p(x, q)$ of the contingent claims f?

Main pricing methods

Arbitrage free pricing if f is replicable.

Utility based pricing if f is not replicable. The price depends on (x, q) and is **marginal**:

$$p(x, q) \neq \text{const.}$$

Utility function

Definition 1 A function $U : (0, \infty) \to \mathbf{R}$ is called a **utility function** if it satisfies the following conditions:

- U is strictly increasing
- U is strictly concave
- the derivative U' (marginal utility) satisfies:

$$U'(0) = \infty \quad U'(\infty) = 0$$

Example 1 (Power utility)

$$U(x, \alpha) = \frac{x^{1-\alpha} - 1}{1 - \alpha}, \quad (\alpha > 0).$$

For $\alpha = 1$ we have *Bernoulli utility*:

$$U(x; 0) = \log x.$$

Investment with random endowments

Consider an investor with the portfolio (x, q).

If the investor uses strategy $X \in \mathcal{X}(x)$, then total wealth at maturity equals

$$X_T + \langle q, f \rangle = X_T + \sum_{i=1}^{N} q_i f_i.$$

The goal of the investor is to maximize **the expected utility of terminal wealth**. The value function of the problem equals:

$$u(x, q) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T + \langle q, f \rangle)]$$

Utility based pricing

Marginal utility based price for the claims f given (x, q) is a vector $p(x, q)$ such that

$$u(x, q) \geq u(x', q')$$

for any pair (x', q') such that

$$x + \langle q, p(x, q) \rangle = x' + \langle q', p(x, q) \rangle.$$

If $u = u(x, q)$ is differentiable at (x, q) then

$$p(x, q) = \frac{u_q}{u_x}(x, q).$$

Interpretation: the investor **will not trade** claims at $p(x, q)$ even if he is allowed to do so.

Properties:

1. If f is replicable, then $p(x, q)$ equals the arbitrage free price for f.
2. The price $p(x, q)$ does not depend on q if and only if f is replicable.

Main difficulty: $p(x, q)$ is hard to compute.

Sensitivity analysis of utility based prices

We study **linear** approximation for $p(x, q)$:

$$p(x, q) \approx p(x) + D(x)q,$$

where

$$p^i(x) = p^i(x, 0), \quad i = 1, \dots, N$$

and

$$D^{ij}(x) = \frac{\partial p^i}{\partial q^j}(x, 0), \quad i, j = 1, \dots, N.$$

The matrix $D(x)$ measures the **sensitivity** of $p(x, q)$ with respect to q.

Question 3 (Quantitative) How to compute $p(x)$ and $D(x)$?

Closely related publications:

M. Davis (97) : duality formula for $p(x)$.

J. Kallsen (02) : formula for $D(x)$ for general semimartingale model but in a different framework of local utility maximization.

V. Henderson (02) : formula for $D(x)$ in the case of a Black-Scholes type model with basis risk and for power utility functions.

Question 4 (Qualitative) When the following desirable properties hold true?

1. The marginal utility based price $p(x) = p(x, 0)$ is **uniquely defined**.

2. The price $p(x, q)$ is locally **non constant** for non replicable claims:
   $D(x)q = 0 \quad \Leftrightarrow \langle q, f\rangle$ is replicable.

3. The sensitivity matrix $D(x)$ is **symmetric**, that is
   $$D^{ij}(x) = D^{ji}(x), \quad i, j = 1, \dots, N.$$

4. The sensitivity matrix $D(x)$ is **strictly negatively defined** for non-replicable claims:
   $\langle q, D(x)q\rangle < 0 \quad \Leftrightarrow \langle q, f\rangle$ is not replicable

Utility based valuation

Certainty equivalence value of portfolio (x, q) is the capital $c(x, q)$ such that

$$u(x, q) = u(c(x, q), 0).$$

The investor is **indifferent** between cash $c(x, q)$ and portfolio (x, q).

If $D(x)$ is symmetric then

$$c(x, q) \approx x + \langle q, p(x)\rangle + \frac{1}{2}\langle q, D(x)q\rangle.$$

Computation of $p(x) = p(x, 0)$

The idea belongs to Mark Davis. Define the conjugate function

$$V(y) = \max_{x>0} [U(x) - xy], \quad y > 0.$$

and consider the following **dual** optimization problem:

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}\left[V\left(y(\frac{d\mathbb{Q}}{d\mathbb{P}})\right)\right], \quad y > 0$$

We call $\mathbb{Q}(y)$ a **minimal martingale measure** for y if it attains the lower bound.

Mark Davis argued that if $x = -v'(y)$ then

$$p(x) = \mathbb{E}_{\mathbb{Q}(y)}[f].$$

The precise mathematical results are given in a paper with Julien Hugonnier and Walter Schachermayer.

Theorem (Hugonnier,K.,Schachermayer) *Let* $y > 0$ *and* $x = -v'(y)$. *The following conditions are equivalent:*

1. *for* **any** *bounded contingent claim* f *a utility based price* $p(x)$ *is uniquely defined.*
2. *the minimal martingale measure* $\mathbb{Q}(y)$ *exists.*

Moreover, in this case

$$p(x) = \mathbb{E}_{\mathbb{Q}(y)}[f].$$

Consider a maximal positive wealth process $\boldsymbol{X}$. We have that $\boldsymbol{X}$ is a uniformly integrable martingale for **some** (not all!) elements of $\boldsymbol{\mathcal{M}}$.

Theorem (Hugonnier,K.,Schachermayer) *Let* $y > 0$ *and* $x = -v'(y)$*. The following conditions are equivalent:*

1. *for* **any** *contingent claim* f *such that*
$$|f| \leq K(1 + X_T) \text{ for some } K > 0$$
a utility based price $p(x)$ *is uniquely defined.*

2. *the minimal martingale measure* $\mathbb{Q}(y)$ *exists and* X *is a uniformly integrable martingale under* $\mathbb{Q}(y)$.

Moreover, in this case

$$p(x) = \mathbb{E}_{\mathbb{Q}(y)}[f].$$

Risk-tolerance wealth process

Fix $x > 0$. Recall that $-U'(x)/U''(x)$ is called the **risk-tolerance** coefficient of U at x.

Denote by $\widehat{X}(x)$ the optimal solution of

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

Definition 2 (K., Sirbu) A maximal wealth process $R(x)$ is called the **risk-tolerance wealth process** if

$$R_T(x) = -\frac{U'(\widehat{X}_T(x))}{U''(\widehat{X}_T(x))}.$$

Theorem (K., Sirbu) *The following assertions are equivalent:*

1. *The risk-tolerance wealth process* $\boldsymbol{R(x)}$ *exists for all* $\boldsymbol{x > 0}$*.*

2. *minimal martingale measures* $\mathbb{Q}(\boldsymbol{y})$ *for different* $\boldsymbol{y > 0}$ *coincide:*

$$\mathbb{Q}(\boldsymbol{y}) = \widehat{\mathbb{Q}}, \quad \boldsymbol{y > 0}.$$

Typical examples:

1. Power utility functions.

2. Model is **complete** on the restriction to $\mathbf{F}^{\boldsymbol{S}}$ (models with **basis risk**).

Some properties of $R(x)$ (if it exists):

1. Initial value:

$$R_0(x) = -\frac{u'(x)}{u''(x)}.$$

2. Derivative of optimal wealth strategy:

$$\frac{R(x)}{R_0(x)} = \lim_{\Delta x \to 0} \frac{\widehat{X}(x+\Delta x) - \widehat{X}(x)}{\Delta x}.$$

Analysis of the sensitivity matrix $D(x)$

First, we answer "qualitative" questions.

Theorem (K., Sirbu) *The following assertions are equivalent:*

1. *The risk-tolerance wealth process* $R(x)$ *exists for all* $x > 0$.
2. *Any of the "qualitative" properties for* $D(x)$ *holds true.*
3. *We have*

$$p(x) = \textit{const}.$$

To compute $\boldsymbol{D(x)}$ we choose

$$\boldsymbol{R(x)/R_0(x) = \dot{X}(x)}$$

as a **numeraire** and denote

$\boldsymbol{f^R = fR_0(x)/R(x)}$: discounted contingent claims

$\boldsymbol{X^R = XR_0(x)/R(x)}$: discounted wealth processes

$\mathbb{Q}^{\boldsymbol{R}}$: the martingale measure for $\boldsymbol{R(x)/R_0(x)}$-discounted wealth processes:

$$\boldsymbol{\frac{d\mathbb{Q}^R}{d\widehat{\mathbb{Q}}} = \frac{R_T(x)}{R_0(x)}}$$

Recall the notation

$$u(x) = \max_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

Denote by

$$\alpha(x) = -\frac{xu''(x)}{u'(x)}$$

the relative risk-aversion coefficient of u

Theorem (K., Sirbu) *Assume that* $R(x)$ *exists. Then*

$$D(x) = -\frac{\alpha(x)}{x} \inf_{X} \mathbb{E}_{\mathbb{Q}^R} \left[(X_T^R - f^R)(X_T^R - f^R)'\right]$$

The solution of the mean-variance problem is formally given by the Kunita-Watanabe decomposition:

$$f^R = M + N, \quad N_0 = 0,$$

of $\mathbb{Q}^R$-martingale

$$f_t^R = \mathbb{E}_{\mathbb{Q}^R}\left[f^R | \mathcal{F}_t\right].$$

Here

1. M is $R(x/R_0(x)$-discounted wealth process. Interpretation: **hedging process**.

2. N is a martingale under $\mathbb{Q}^R$ which is orthogonal to all $R(x)/R_0(x)$-discounted wealth processes. Interpretation: **risk process**.

With these notations,

$$D(x) = - - \frac{\alpha(x)}{x}\mathbb{E}_{\mathbb{Q}^R}\left[N_T N_T'\right]$$

Question 5 How to compute $D(x)$ in *practice*?

Inputs:

1. $\mathbb{Q}$. *Already implemented!*
2. $R(x)/R_0(x)$. Recall that
$$\frac{R(x)}{R_0(x)} = \lim_{\Delta x \to 0} \frac{\widehat{X}(x+\Delta x) - \widehat{X}(x)}{\Delta x}.$$
Decide what to do with one penny!
3. Relative risk-aversion coefficient $\alpha(x)$. *Deduce from mean-variance preferences*. In any case, this is just a number!

Black and Scholes model with basis risk

Traded asset:

$$dS_t = S_t\,(\mu dt + \sigma dW_t)$$

Non traded asset:

$$d\widetilde{S} = (\widetilde{\mu} dt + \widetilde{\sigma} d\widetilde{W}_t),$$

Denote by

$$\rho = \frac{d\widetilde{W} dW}{dt}$$

the **correlation** coefficient between S and $\widetilde{S}$.
In practice, we want to chose S so that

$$\rho \approx 1.$$

Consider contingent claims

$$f = f(\widetilde{S})$$

whose payoffs are determined by $\widetilde{S}$.

To compute $D(x)$ assume (as an example) the following choices:

1. $\mathbb{Q}$ is a martingale measure for $\widetilde{S}$.
2. $R(x)/R_0(x) = 1$

Then

$$D_{ij}(x) = -\frac{\alpha(x)}{x}(1-\rho^2)\text{Covar}_{\mathbb{Q}}(f_i, f_j).$$

The existence of u''

Consider the problem of expected utility maximization:

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)], \quad x > 0$$

Question 6 When u'' exists and is strictly positive?

Theorem (K.,Sirbu) *Assume that U'' exists there are constants $c_1 > 0$ and $c_2 < \infty$ such that*

$$c_1 < -\frac{xU''(x)}{U'(x)} < c_2, \quad x > 0.$$

Then u'' exists and

$$c_1 < -\frac{xu''(x)}{u'(x)} < c_2, \quad x > 0.$$

Remark that, in general, both bounds c_1 and c_2 are needed for the existence of u''.

Summary

- Since utility based prices are difficult to compute, it is reasonable to approximate them.

- The following conditions are equivalent:
  - Approximate utility based prices have nice qualitative properties
  - Risk-tolerance wealth processes exist.
  - Minimal martingale measures do not depend on initial capital.

- To compute the first order correction we need to solve the mean-variance hedging problem, where the risk-tolerance wealth process plays the role of the numeraire.