

Perpetual Convertible Bonds

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1) Description The convertible bond is a financial instrument (issued by a firm) with the following provisions:

- pays a fixed amount at maturity (like a bond) and pays coupons
- can be converted by the bondholder for stock or can be called by the firm. This makes it a game between the firm and the bondholders.

2) History

- early work of BRENNAN & SCHWARTZ (1977, 1980): the value of the firm is the *primitive* ; discrete coupons and dividends : *Linear* Black & Scholes equation
- INGERSOLL (1977): mostly no dividends, empirical discussion about continuous dividends that can cause early conversion. The equation is still *linear*.
- Mc CONNELL & SCHWARTZ (1986): equity is the primitive
- more recent papers concerned mostly with computational methods

3) Our Model and Solution of the Problem

The convertible bond is *perpetual*: it never matures.

$X(t)$ -total value of the firm

$S(t)$ - value of the equity

$D(t)$ -value of the convertible bonds.

$$X(t) = S(t) + D(t)$$

- constant interest rate r for the money market
- $\delta S_t dt$ -dividend payments and $c dt$ -coupons in dt units of time ($\delta, c > 0$ are constant)
- At *conversion time* , $D(t) = \gamma X(t)$ ($0 < \gamma < 1$ -*conversion factor*)
- At *call time*, $D(t) = \max\{K, \gamma X(t)\}$ ($K > 0$ -*call value*).
- constant volatility for $X(t)$ ($\sigma > 0$)
- standing assumption

$$0 < \delta < r$$

Objective: price $D(t)$ in terms of $X(t)$

$$D(t) = f(X(t))$$

(the value of the firm is the *primitive*)

There is *No-Arbitrage* (trading in the money market, stock and convertible bonds) if and only if

$$\begin{aligned} \mathcal{N}f(x) \stackrel{\Delta}{=} & rf(x) - (rx - c)f'(x) + \\ & + \delta(x - f(x))f'(x) - \frac{1}{2}\sigma^2x^2f''(x) = c, \end{aligned} \quad (1)$$

before bonds are called or converted.

Risk neutral evolution of the value of the firm:

$$\begin{aligned} dX(t) = & rX(t)dt - \delta(X(t) - f(X(t)))dt - \\ & - cdt + \sigma X(t)dW(t). \end{aligned}$$

Consider *call* and *conversion* strategies :

- *call* as soon as $X(t) \geq C_a$
- *convert* as soon as $X(t) \geq C_o$.

for some C_a and C_o .

Fix $C_a \geq K$ and $C_o > 0$. We need to solve equation (1) with the appropriate boundary conditions

$$f(C_a \wedge C_o) = \begin{cases} \gamma C_o & \text{if } C_o < C_a \\ \max\{K, \gamma C_o\} & \text{if } C_a \leq C_o \end{cases}$$

and

$$f(0) = 0.$$

Theorem: There exists a unique solution $f(\cdot, C_a, C_o)$ which satisfies the equation (1) and the boundary conditions.

Remark: $f(x, C_a, C_o)$ is the price of the bond corresponding to strategies C_a, C_o , as long as $x \leq C_a \wedge C_o$. Outside $(0, C_a \wedge C_o)$, the price $f(x, C_a, C_o)$ is either the call or the conversion value.

Recall: for fixed *call* and *conversion* levels (C_a and C_o) we computed the corresponding price $f(x, C_a, C_o)$.

Game feature:

- need to compute the *Min-Max* with respect to *call* and *conversion* levels.

Theorem:

There exist optimal C_a^*, C_o^* such that

$$\begin{aligned} f(x, C_a^*, C_o^*) &= \min_{C_a \geq K} \max_{C_o > 0} f(x, C_a, C_o) = \\ &= \max_{C_o > 0} \min_{C_a \geq K} f(x, C_a, C_o) \end{aligned}$$

- $D(t) = f(X(t), C_a^*, C_o^*)$ is the price of the perpetual convertible bond

Qualitative description of $f(x, C_a^*, C_o^*)$:

- for fixed $r, c, \sigma, \delta < r, \gamma$, we have two thresholds

$$0 < K_1(r, c, \sigma, \delta, \gamma) \leq \frac{c}{r} < \frac{c}{\delta} \leq K_2(r, c, \sigma, \delta, \gamma)$$

which divide the behavior of the convertible bond in three cases, depending on the value of K :

Large call prices ($K > K_2$).

Optimal conversion occurs before optimal call.

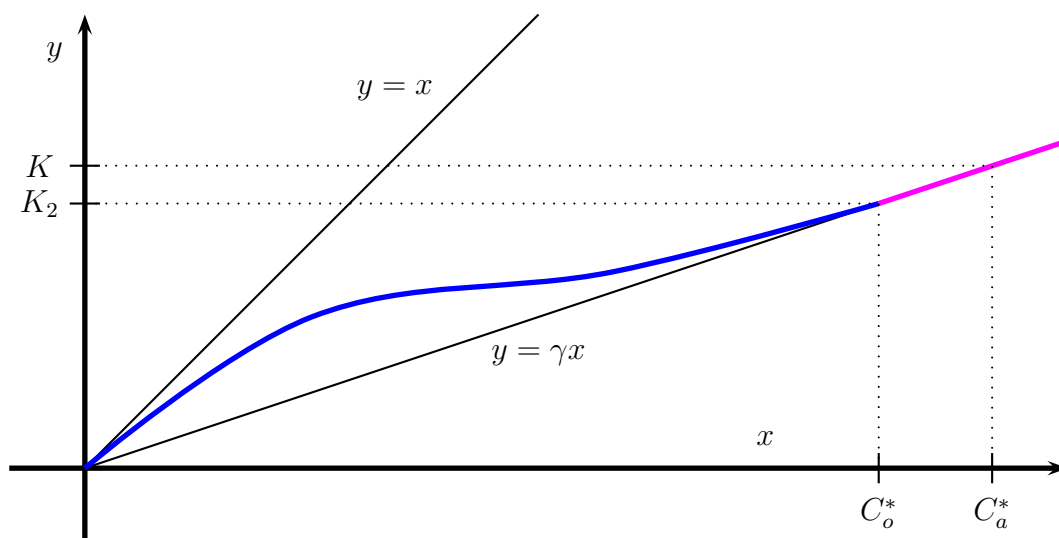


Figure 1: Large call prices K

$$0 < C_o^* = \frac{K_2}{\gamma} < \frac{K}{\gamma} = C_a^*$$

$$\min\{\mathcal{N}f(x) - c, f(x) - \gamma x\} = 0 \text{ in } (0, \frac{K}{\gamma})$$

$$f(x) = \gamma x \text{ for } x \geq \frac{K}{\gamma}$$

Smooth fit at $x = C_o^* = \frac{K_2}{\gamma}$.

Intermediate call prices ($K_1 \leq K \leq K_2$).

Optimal call and *optimal conversion* occur at the same time.

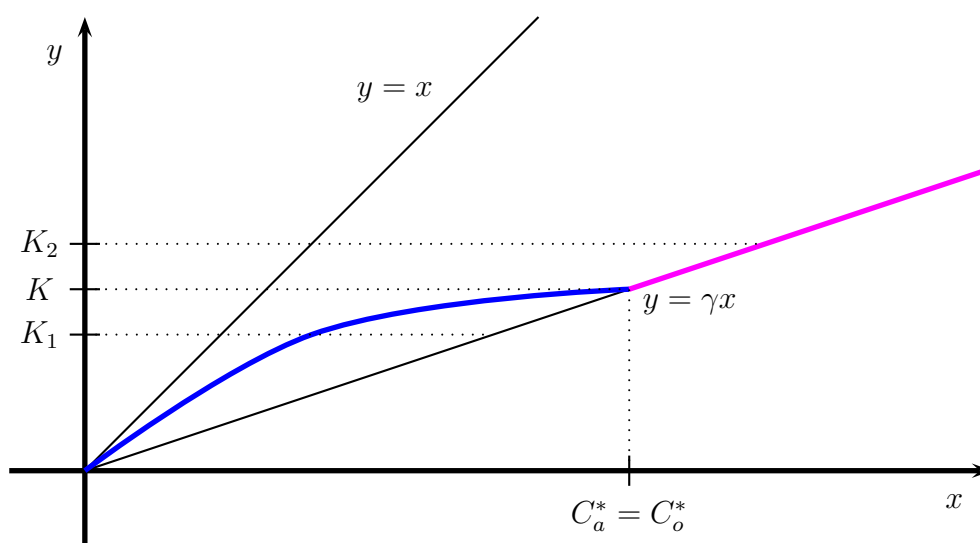


Figure 2: Intermediate call prices K

$$C_o^* = C_a^* = \frac{K}{\gamma}$$

$$\mathcal{N}f(x) - c = 0 \text{ on } \left(0, \frac{K}{\gamma}\right)$$

$$f(x) = \gamma x \text{ for } x \geq \frac{K}{\gamma}$$

Small call prices ($0 < K < K_1$).

Optimal call occurs before optimal conversion.

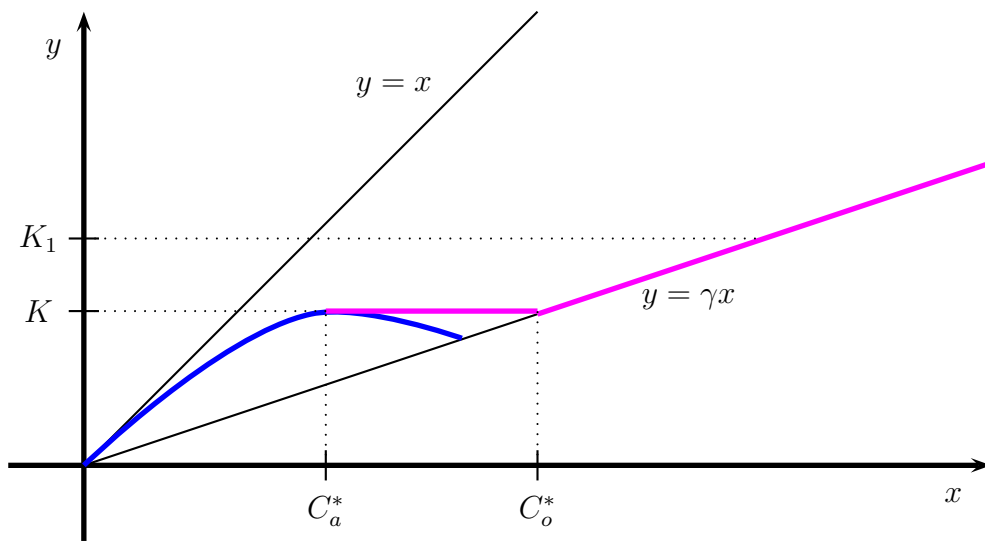


Figure 3: Small call prices K

$$0 < C_a^* < C_o^* = \frac{K}{\gamma}$$

$$\max\{\mathcal{N}f(x) - c, f(x) - K\} = 0 \text{ on } (0, \frac{K}{\gamma})$$

$$f(x) = \gamma x \text{ for } x \geq \frac{K}{\gamma}$$

Smooth fit at $x = C_a^*$.