Perpetual Convertible Bonds

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1) **Description** The convertible bond is a financial instrument (issued by a firm) with the following provisions:

- pays a fixed amount at maturity (like a bond) and pays coupons
- can be converted by the bondholder for stock or can be called by the firm. This makes it a game between the firm and the bondholders.

2) **History**

- early work of BRENNAN & SCHWARTZ (1977, 1980): the value of the firm is the *primitive*; discrete coupons and dividends: *Linear* Black & Scholes equation
- INGERSOLL (1977): mostly no dividends, empirical discussion about continuous dividends that can cause early conversion. The equation is still *linear*.
- Mc CONNELL & SCHWARTZ (1986): equity is the primitive
- more recent papers concerned mostly with computational methods
3) Our Model and Solution of the Problem

The convertible bond is *perpetual*: it never matures.

\[ X(t) = S(t) + D(t) \]

- constant interest rate \( r \) for the money market
- \( \delta S_t dt \)-dividend payments and \( c dt \)-coupons in \( dt \) units of time (\( \delta, c > 0 \) are constant)
- At *conversion time*, \( D(t) = \gamma X(t) \) (\( 0 < \gamma < 1 \)-conversion factor)
- At *call time*, \( D(t) = \max\{K, \gamma X(t)\} \) (\( K > 0 \)-call value).
- constant volatility for \( X(t) \) (\( \sigma > 0 \))
- standing assumption
  \[ 0 < \delta < r \]
Objective: price $D(t)$ in terms of $X(t)$

$$D(t) = f(X(t))$$

(the value of the firm is the *primitive*)

There is *No-Arbitrage* (trading in the money market, stock and convertible bonds) if and only if

$$\mathcal{N} f(x) \triangleq r f(x) - (r x - c) f'(x) +$$

$$+ \delta(x - f(x)) f'(x) - \frac{1}{2} \sigma^2 x^2 f''(x) = c,$$

before bonds are called or converted.

*Risk neutral evolution* of the value of the firm:

$$dX(t) = r X(t) dt - \delta(X(t) - f(X(t))) dt -$$

$$- c dt + \sigma X(t) dW(t).$$

Consider *call* and *conversion* strategies:

- *call* as soon as $X(t) \geq C_a$
- *convert* as soon as $X(t) \geq C_o.$

for some $C_a$ and $C_o.$
Fix $C_a \geq K$ and $C_o > 0$. We need to solve equation (1) with the appropriate boundary conditions

$$f(C_a \wedge C_o) = \begin{cases} \gamma C_o & \text{if } C_o < C_a \\ \max\{K, \gamma C_o\} & \text{if } C_a \leq C_o \end{cases}$$

and

$$f(0) = 0.$$

**Theorem:** There exists a unique solution $f(\cdot, C_a, C_o)$ which satisfies the equation (1) and the boundary conditions.

**Remark:** $f(x, C_a, C_o)$ is the price of the bond corresponding to strategies $C_a, C_o$, as long as $x \leq C_a \wedge C_o$. Outside $(0, C_a \wedge C_o)$, the price $f(x, C_a, C_o)$ is either the call or the conversion value.

**Recall:** for fixed call and conversion levels ($C_a$ and $C_o$) we computed the corresponding price $f(x, C_a, C_o)$. 

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Game feature:

- need to compute the Min-Max with respect to call and conversion levels.

Theorem:
There exist optimal $C^*_a, C^*_o$ such that

$$f(x, C^*_a, C^*_o) = \min_{C_a \geq K} \max_{C_o > 0} f(x, C_a, C_o) = \max_{C_o > 0} \min_{C_a \geq K} f(x, C_a, C_o)$$

- $D(t) = f(X(t), C^*_a, C^*_o)$ is the price of the perpetual convertible bond

Qualitative description of $f(x, C^*_a, C^*_o)$:

- for fixed $r, c, \sigma, \delta < r, \gamma$, we have two thresholds

$$0 < K_1(r, c, \sigma, \delta, \gamma) \leq \frac{c}{r} < \frac{c}{\delta} \leq K_2(r, c, \sigma, \delta, \gamma)$$

which divide the behavior of the convertible bond in three cases, depending on the value of $K$:
Large call prices \((K > K_2)\).

Optimal conversion occurs before optimal call.

\[ 0 < C_o^* = \frac{K_2}{\gamma} < \frac{K}{\gamma} = C_a^* \]

\[
\min\{N f(x) - c, f(x) - \gamma x\} = 0 \text{ in } (0, \frac{K}{\gamma})
\]

\[ f(x) = \gamma x \text{ for } x \geq \frac{K}{\gamma} \]

Smooth fit at \(x = C_o^* = \frac{K_2}{\gamma}\).
Intermediate call prices \((K_1 \leq K \leq K_2)\).

Optimal call and optimal conversion occur at the same time.

\[
y = x
\]

\[
y = \gamma x
\]

\[
C^*_a = C^*_o = \frac{K}{\gamma}
\]

\[
\mathcal{N} f(x) - c = 0 \text{ on } (0, \frac{K}{\gamma})
\]

\[
f(x) = \gamma x \text{ for } x \geq \frac{K}{\gamma}
\]
Small call prices $(0 < K < K_1)$.

Optimal call occurs before optimal conversion.

Figure 3: Small call prices $K$

$$0 < C_{a^*} < C_{o^*} = \frac{K}{\gamma}$$

$$\max\{\mathcal{N}f(x) - c, f(x) - K\} = 0 \text{ on } (0, \frac{K}{\gamma})$$

$$f(x) = \gamma x \text{ for } x \geq \frac{K}{\gamma}$$

Smooth fit at $x = C_{a^*}$. 

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