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Model Implementation, Algorithms and Software Issues*

A General Equilibrium Model

of the Term Structure of Interest Rates

under Regime-switching Risk

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Introduction

- The aggregate economy is characterized by periodic shifts between distinct regimes of the business cycle

Hamilton (1989), Filardo (1994) and Diebold and Rudebusch (1996) among others.

- Markov regime-switching models of the short-term interest rate

Hamilton (1988), Garcia and Perron (1996), Gary (1996), Ang and Bekaert (1998).

- Dynamic term structure models under regime shifts:

Naik and Lee (1997), and Bansal and Zhou (2002).

- Dai and Singleton (2003): **the risk of regime shifts is not priced.**

- Dai, Singleton and Yang (2003) price regime-shift risk based on no-arbitrage condition.

Objectives

- Develop a dynamic term structure model under the systematic risk of regime shifts in a general equilibrium setting similar to that in CIR.
- How do regime shifts affect bond returns? How important is the regime-switching risk premium?

Main results

- We show that the regime-switching risk can be priced in a similar way as in the case of jump risk (e.g. Ahn and Thompson, 1988).
- We show that regime switching introduces a new source of time-variation in bond risk premiums, which is associated with the systematic risk of periodic shifts in bond prices due to regime changes.
- Closed-form approximate solution for the entire yield curve is obtained for an affine model.
- Empirical evidence suggests that the market price of the regime-shift risk is statistically significant, and mostly affect the long-end of the yield curve.

The Equilibrium Model

The Underlying Economy: Single good and a large number of infinitely lived and identical consumers similarly to that in CIR (1985a,b).

Modeling Regime Switching:

- *Hidden Markov Model:* Elliott & Mamon (2002), Elliott & Wilson (2002).
- *Conditional Markov Chain:* Bielecki & Rutkowski (2000), Yin & Zhang (1998)
- *Marked Point Process:* Landen (2000). Suppose N regimes.

Mark Space: $E = \{z = (i, j) : i \in \{1, \dots, N\}, j \in \{1, 2, \dots, N\}, i \neq j\}$.

Marked point process: for $A \in E$, $m(t, A)$ counts the cumulative number of regime shifts that belong to A during $(0, t]$ with *stochastic intensity kernel*:

$$\gamma_m(dt, dz) = h(z, x(t-))\mathbf{I}\{s(t-) = i\}\epsilon_z(dz)dt,$$

$h(z, x(t-))$: the regime-shift intensity at $z = (i, j)$,

$\epsilon_z(A)$: the Dirac measure at point z , ($\epsilon_z(A) = 1$ if $z \in A$ and 0, otherwise).

- $\gamma_m(dt, dz)$ is the conditional probability of a shift from regime i to regime j during $[t, t + dt)$ given $x(t-)$ and $s(t-) = i$.

Two State Variables: (s,x)

- The Regime: $s(t)$

$$ds(t) = \int_E \zeta(z)m(dt, dz)$$

where $\zeta(z) = \zeta((i, j)) = j - i$ with the compensator $\gamma_s(t)dt = \int_E \zeta(z)\gamma_m(dt, dz)$. For example, if there is a regime shift from i to j occurred at time t , the above equation implies $s_t = (j - i) + s_{t-} = j$.

- The other usual state variable: (μ_x and σ_x depend on $x(t), s(t)$)

$$dx(t) = \mu_x dt + \sigma_x dB_t$$

Investment Opportunities

- Production:

$$dy = y\mu_y dt + y\sigma_y dB_t + \int_E y\delta_y(z)m(dt, dz)$$

where $\delta_y(z)$ is the percentage change in y due to a regime shift,

- Competitive market for a default-free pure discount bond with the price:

$$dF = F\mu_F dt + F\sigma_F dB_t + \int_E F\delta_F(z)m(dt, dz)$$

- Competitive market for (local) risk-free lending and borrowing at $r(t)$.

1.3 The Consumer's Objective Function

- Maximize consumer's lifetime expected utility

$$E_0 \left[\int_0^{\infty} e^{-\rho t} U(c(t)) dt \right]$$

where $c(t)$ is the flow of consumption.

- Subject to the budget constraint

$$dw = w\mu_w dt + w\sigma_w dB_t + \int_E w\delta_w(z) m(dt, dz)$$

where

$$w\mu_w = w[\phi_1(\mu_y - r) + \phi_2(\mu_F - r) + r] - c$$

$$w\sigma_w = w[\phi_1\sigma_y + \phi_2\sigma_F]$$

$$w\delta_w(z) = w[\phi_1\delta_y(z) + \phi_2\delta_F(z)]$$

$w(t)$ is her wealth at time t ,

ϕ_1 is the proportion of her wealth invested in the physical production,

ϕ_2 is the proportion of her wealth invested in the discount bond.

The Equilibrium Short Rate

- Define: $J(w(t), s(t), x(t)) = \sup E_t \left[\int_t^\infty e^{-\rho(\tau-t)} U(c(\tau)) d\tau \right]$,
where the “sup” is over the admissible feedback control of (c, ϕ_1, ϕ_2) .
- CIR’s Notations: $Var(w^c) = (w\sigma_w)^2$, $Var(x) = \sigma_x^2$, $Cov(w^c, x) = (w\sigma_w)\sigma_x$.
And $\Delta_s f = f(s(t)) - f(s(t-))$ for any function $f(\cdot)$ that depends on $s(t)$.
- Parato Optimality implies no trading at equilibrium, so $\phi_1 = 1$ and $\phi_2 = 0$.

Proposition 1 The *equilibrium short-term interest rate* is given by

$$r = \mu_y^* - \left(-\frac{J_{ww}}{J_w} \right) \frac{Var(w^c)}{w} - \left(-\frac{J_{wx}}{J_x} \right) \frac{Cov(w^c, x)}{w} - \int_E \left(-\frac{\Delta_s J_w}{J_w} \right) \frac{\Delta_s w}{w} \gamma_m(dz)$$

where

$$\mu_y^* = \mu_y + \int_E \delta_y(z) \gamma_m(dz), \quad \text{and} \quad \gamma_m(dz) = h(z, x(t-)) \mathbf{I}\{s(t-) = i\} \epsilon_z(dz).$$

Remark Usually (for example, under log utility), $\left(-\frac{\Delta_s J_w}{J_w} \right) \frac{\Delta_s w}{w} > 0$. So, the impact of a systematic regime-switching risk is to lower the equilibrium short-term interest rate, as that of jump risk in Ahn and Thompson (1988).

The Equilibrium Bond Returns

Proposition 2 Let $\mu_F^* dt = E_{t-} \left(\frac{dF(t)}{F(t-)} \right)$ be the instantaneous expected rate of return of the discount bond. At equilibrium,

$$\begin{aligned} \mu_F^* - r &= \left[\left(-\frac{J_{ww}}{J_w} \right) Var(w) + \left(-\frac{J_{wx}}{J_w} \right) Cov(w, x) \right] \frac{F_w}{F} \\ &+ \left[\left(-\frac{J_{ww}}{J_w} \right) Cov(w, x) + \left(-\frac{J_{wx}}{J_w} \right) Var(x) \right] \frac{F_x}{F} \\ &+ \int_E \left(-\frac{\Delta_s J_w}{J_w} \right) \frac{\Delta_s F}{F} \gamma_m(dz) \end{aligned}$$

Remark 1 Rewrite the above equation as

$$\mu_F^* - r = -Cov \left(\frac{dJ_w^c}{J_w}, \frac{dF^c}{F} \right) - \int_E \frac{\Delta_s J_w}{J_w} \frac{\Delta_s F}{F} \gamma_m(dz)$$

where

$$Cov \left(\frac{dJ_w^c}{J_w}, \frac{dF^c}{F} \right) = \left(\frac{\sigma_{J_w}}{J_w} \right) \left(\frac{\sigma_{FF}}{F} \right) = \left(\frac{J_{ww} w \sigma_w + J_{wx} \sigma_x}{J_w} \right) \left(\frac{(w \sigma_w) F_w + \sigma_x F_x}{F} \right)$$

Regime-Switching Risk Premium

Remark 2 When $J_{wx} = 0$ and $F_w = 0$ (for example, under log utility),

$$\mu_F^* - r = \left(-\frac{J_{ww}}{J_w} \right) w\sigma_w \frac{\sigma_x F_x}{F} + \int_E \left(-\frac{\Delta_s J_w}{J_w} \right) \frac{\Delta_s F}{F} \gamma_m(dz)$$

- The first term is the instantaneous *diffusion risk premium*.
 $-\frac{\sigma_x F_x}{F}$ is the volatility of the bond return due to diffusions in $x(t)$.
 $-\left(-\frac{J_{ww}}{J_w} \right) w\sigma_w$ measures the extra rate of return per unit of such volatility and is referred as the market price of risk.

- The second term is analogously defined as the instantaneous *regime-switching risk premium*.
 $-\frac{\Delta_s F}{F}$ is the percentage change in bond price due to regime shifts
 $-\left(-\frac{\Delta_s J_w}{J_w} \right)$ measures the excess bond return per unit of such changes.

Simplification under Log-Utility for Default-free Bond

- Under Log-Utility, $J_{wx} = 0$, $J_w = \frac{1}{\rho w}$, and $J_{ww} = -\frac{1}{\rho w^2}$.
- “Default-free” implies $F_w = 0$, $F_{ww} = 0$ and $F_{wx} = 0$.

Proposition 3 Under $U(c) = \log(c)$ as CIR. The equilibrium short rate is

$$r = \mu_y - \sigma_y^2 + \int_E \lambda_s(z) \gamma_m(dz)$$

The price of a default-free pure discount bond $F(t, x(t), s(t), T)$ becomes,

$$F_t + (\mu_x - \sigma_y \sigma_x) F_x + \frac{1}{2} \sigma_x^2 F_{xx} + \int_E \Delta_s F (1 - \lambda_s(z)) \gamma_m(dz) = r F$$

for each $s \in \{1, 2, \dots, N\}$, with the boundary condition: $F(T, x, s, T) = 1$, and where $\Delta_s F = F(t, x(t), s(t-) + \zeta(z), T) - F(t-, x(t-), s(t-), T)$, and $\lambda_s(z) = \frac{\delta_y(z)}{1 + \delta_y(z)}$. Also, $F(t, x(t), s(t), T) = E^Q \left[\exp \left(- \int_t^T r(s) ds \right) \right]$.

A Term Structure with R-S Risk for Estimation

Further assume: $\mu_x = a_0(s) + a_1(s)x$, $\sigma_x = \sqrt{\sigma(s)x}$, $h(z, x(t-)) = e^{\eta_s(z)}$, $\sigma_y = \theta_x(s)\sqrt{\sigma(s)x}$, $\mu_y = x + \theta_x^2(s)\sigma(s)x - \int_E \lambda_s(z)\gamma_m(dz)$, and $\lambda_s(z) = 1 - e^{\theta_s(z)}$.

Then, $r = x$. That is,

$$dr(t) = (a_0(s) + a_1(s))r dt + \sqrt{\sigma(s)r(t)}dB_t.$$

Proposition 4 With the above assumptions, $F(t, \tau) = e^{A(\tau, s_t) + B(\tau, s_t)r_t}$: the approximate price at time t of a default-free pure discount bond with maturity τ ; $R(t, \tau) = -\frac{A(\tau, s_t)}{\tau} - \frac{B(\tau, s_t)r_t}{\tau}$: the τ -period interest rate, where $A(\tau, s)$ and $B(\tau, s)$ satisfy

$$\begin{aligned} -\frac{\partial B(\tau, s)}{\partial \tau} + [a_1(s) - \theta_x(s)\sigma(s)]B(\tau, s) + \frac{1}{2}\sigma(s)B^2(\tau, s) \\ + \int_E (e^{\Delta_s A} \Delta_s B) e^{\eta_s(z) + \theta_s(z)} \mathbf{1}(s = i) \epsilon_z(dz) = 1 \end{aligned} \quad (1)$$

and

$$-\frac{\partial A(\tau, s)}{\partial \tau} + a_0(s)B(\tau, s) + \int_E (e^{\Delta_s A} - 1) e^{\eta_s(z) + \theta_s(z)} \mathbf{1}(s = i) \epsilon_z(dz) = 0 \quad (2)$$

with boundary conditions $A(0, s) = 0$ and $B(0, s) = 0$, and $s = 1, 2$.

Empirical Results

Data Monthly interest rates from June 1964 to November 2000 obtained from CRSP. Eight series of interest rates with maturities 1m, 3m, 6m, 1-5 yrs. (6m, 5yr) are chosen to fit Models 1,2,3 for statistical analysis. All eight series are used to estimated the implied regimes.

Model 1: CIR has four parameters $(a_0, a_1, \sigma, \theta_x)$.

Model 2: CIR with two regimes but no R-S risk, has ten parameters $(a_0(1), a_1(1), \sigma(1), \theta_x(1), a_0(2), a_1(2), \sigma(2), \theta_x(2), \eta_s(1, 2), \eta_s(2, 1))$.

Model 3: CIR with two regimes and R-S risk, has twelve parameters $(a_0(1), a_1(1), \sigma(1), \theta_x(1), a_0(2), a_1(2), \sigma(2), \theta_x(2), \theta_s(1), \theta_s(2), \eta_s(1, 2), \eta_s(2, 1))$.

Econometric Methodology *Efficient Method of Moments* (EMM):

Bansal, Gallant and Tauchen (1995) and Gallant and Tauchen (1996, 2001).

Application to term structure under regime shifts: Bansal and Zhou (2002).

Efficient Method of Moments

Step 1: Projection Using quasi maximum likelihood to project the observed data (6m,5yr) to an auxiliary model close to the true data generating process.

- Gallant and Tauchen (2001) suggests a SNP model based on Hermite polynomial expansion as a convenient general purpose auxiliary model.
- The dimension of this auxiliary model is selected by minimizing BIC.
- The score function are used as moment conditions to compute a chi-square criterion function.

Step 2: Simulation Simulate (6m,5yr) according to the wanted stationary model to evaluate the expected value of the score and compute a chi-squared criterion function.

Step 3: Optimization A nonlinear optimizer is used to find the parameter setting that minimizes the criterion.

Advantage: If the auxiliary model is a close approximation of the true one, then EMM is asymptotic efficient, close to the efficiency of ML.

Conclusions and Future Works

- Affine regime-switching jump diffusion term structure model
 - Regime-switching risk on interest rate derivatives.
 - Optimal portfolio choice under regime-switching risk.
 - Monetary policy regimes.
 - Structural relation between business cycles and the yield curve.
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- Paper is available at <http://mendota.umkc.edu/paper-term.html>