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Model Implementation, Algorithms and Software Issues

A General Equilibrium Model
of the Term Structure of Interest Rates
under Regime-switching Risk

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1. **Introduction**

2. **Objectives and Main results**

3. **The Equilibrium Model**
   
   (a) The Underlying Economy: (i) Two State Variables: (s,x)
   (ii) Investment Opportunities (iii) Consumer’s Objective Function.
   
   (b) The Equilibrium Short Rate
   
   (c) The Equilibrium Bond Returns
   
   (d) Special case under log Utility and Regime-Switching Risk Premium
   
   (e) Further special case under regime-switching affine model.

4. **Empirical Results**
   
   (a) Data; (b) Three Models; (c) Efficient Method of Moments (EMM)

5. **Conclusions and Future Works**
Introduction

- The aggregate economy is characterized by periodic shifts between distinct regimes of the business cycle
- Markov regime-switching models of the short-term interest rate
- Dynamic term structure models under regime shifts:
  Naik and Lee (1997), and Bansal and Zhou (2002).
- Dai and Singleton (2003): the risk of regime shifts is not priced.
Objectives
• Develop a dynamic term structure model under the systematic risk of regime shifts in a general equilibrium setting similar to that in CIR.
• How do regime shifts affect bond returns? How important is the regime-switching risk premium?

Main results
• We show that the regime-switching risk can be priced in a similar way as in the case of jump risk (e.g. Ahn and Thompson, 1988).
• We show that regime switching introduces a new source of time-variation in bond risk premiums, which is associated with the systematic risk of periodic shifts in bond prices due to regime changes.
• Closed-form approximate solution for the entire yield curve is obtained for an affine model.
• Empirical evidence suggests that the market price of the regime-shift risk is statistically significant, and mostly affect the long-end of the yield curve.
The Equilibrium Model

The Underlying Economy: Single good and a large number of infinitely lived and identical consumers similarly to that in CIR (1985a,b).

Modeling Regime Switching:


Mark Space: $E = \{z = (i, j) : i \in \{1, \ldots, N\}, j \in \{1, 2, \ldots, N\}, i \neq j\}$.

Marked point process: for $A \in E$, $m(t, A)$ counts the cumulative number of regime shifts that belong to $A$ during $(0, t]$ with stochastic intensity kernel:

$$\gamma_m(dt, dz) = h(z, x(t-)) \mathbf{I}\{s(t-) = i\} \epsilon_z(dz) dt,$$

$h(z, x(t-))$: the regime-shift intensity at $z = (i, j)$,

$\epsilon_z(A)$: the Dirac measure at point $z$, ($\epsilon_z(A) = 1$ if $z \in A$ and 0, otherwise).

• $\gamma_m(dt, dz)$ is the conditional probability of a shift from regime $i$ to regime $j$ during $[t, t + dt)$ given $x(t-)$ and $s(t-) = i$. 
Two State Variables: (s,x)

- The Regime: \( s(t) \)

\[
ds(t) = \int_E \zeta(z)m(dt, dz)
\]

where \( \zeta(z) = \zeta((i,j)) = j - i \) with the compensator \( \gamma_s(t)dt = \int_E \zeta(z)\gamma_m(dt, dz) \).

For example, if there is a regime shift from \( i \) to \( j \) occurred at time \( t \), the above equation implies \( s_t = (j - i) + s_{t-} = j \).

- The other usual state variable: \((\mu_x \text{ and } \sigma_x \text{ depend on } x(t), s(t))\)

\[
dx(t) = \mu_x dt + \sigma_x dB_t
\]

Investment Opportunities

- Production:

\[
dy = y\mu_y dt + y\sigma_y dB_t + \int_E y\delta_y(z)m(dt, dz)
\]

where \( \delta_y(z) \) is the percentage change in \( y \) due to a regime shift,

- Competitive market for a default-free pure discount bond with the price:

\[
dF = F\mu_F dt + F\sigma_F dB_t + \int_E F\delta_F(z)m(dt, dz)
\]

- Competitive market for (local) risk-free lending and borrowing at \( r(t) \).
1.3 The Consumer’s Objective Function

- Maximize consumer’s lifetime expected utility

\[
E_0 \left[ \int_0^\infty e^{\rho t} U(c(t)) dt \right]
\]

where \( c(t) \) is the flow of consumption.

- Subject to the budget constraint

\[
dw = w\mu_{w}dt + w\sigma_{w}dB_{t} + \int_{E} w\delta_{w}(z)m(dt,dz)
\]

where

\[
w\mu_{w} = w[\phi_{1}(\mu_{y} - r) + \phi_{2}(\mu_{F} - r) + r] - c
\]
\[
w\sigma_{w} = w[\phi_{1}\sigma_{y} + \phi_{2}\sigma_{F}]
\]
\[
w\delta_{w}(z) = w[\phi_{1}\delta_{y}(z) + \phi_{2}\delta_{F}(z)]
\]

\( w(t) \) is the her wealth at time \( t \),
\( \phi_{1} \) is the proportion of her wealth invested in the physical production,
\( \phi_{2} \) is the proportion of her wealth invested in the discount bond.
The Equilibrium Short Rate

- Define: \( J(w(t), s(t), x(t)) = \sup E_t \left[ \int_t^\infty e^{-\rho(t-\tau)} U(c(\tau)) d\tau \right] \), where the “sup” is over the admissible feedback control of \((c, \phi_1, \phi_2)\).
- CIR’s Notations: \( Var(w_c) = (w\sigma_w)^2 \), \( Var(x) = \sigma_x^2 \), \( Cov(w_c, x) = (w\sigma_w)\sigma_x \). And \( \Delta_s f = f(s(t)) - f(s(t-)) \) for any function \( f(\cdot) \) that depends on \( s(t) \).
- Parato Optimality implies no trading at equilibrium, so \( \phi_1 = 1 \) and \( \phi_2 = 0 \).

**Proposition 1** The equilibrium short-term interest rate is given by

\[
r = \mu^*_y - \left( \frac{-J_{ww}}{J_w} \right) \frac{Var(w_c)}{w} - \left( \frac{-J_{wx}}{J_x} \right) \frac{Cov(w_c, x)}{w} - \int_E \left( \frac{-\Delta_s J_w}{J_w} \right) \frac{\Delta_s w}{w} \gamma_m(dz)
\]

where

\[
\mu_y^* = \mu_y + \int_E \delta_y(z) \gamma_m(dz), \quad \text{and} \quad \gamma_m(dz) = h(z, x(t-)) I\{s(t-) = i\} \varepsilon_z(dz).
\]

**Remark** Usually (for example, under log utility), \( \left( \frac{-\Delta_s J_w}{J_w} \right) \frac{\Delta_s w}{w} > 0 \). So, the impact of a systematic regime-switching risk is to lower the equilibrium short-term interest rate, as that of jump risk in Ahn and Thompson (1988).
Proposition 2  Let $\mu^*_F dt = E_{t-} \left( \frac{dF(t)}{F(t-)} \right)$ be the instantaneous expected rate of return of the discount bond. At equilibrium,

$$\mu^*_F - r = \left[ \left( -\frac{J_{ww}}{J_w} \right) Var(w) + \left( -\frac{J_{wx}}{J_w} \right) Cov(w, x) \right] \frac{F_w}{F}$$

$$+ \left[ \left( -\frac{J_{ww}}{J_w} \right) Cov(w, x) + \left( -\frac{J_{wx}}{J_w} \right) Var(x) \right] \frac{F_x}{F}$$

$$+ \int_E \left( -\frac{\Delta_s J_w}{J_w} \right) \frac{\Delta_s F}{F} \gamma_m(dz)$$

Remark 1  Rewrite the above equation as

$$\mu^*_F - r = -Cov \left( \frac{dJ^c_w}{J_w}, \frac{dF^c}{F} \right) - \int_E \frac{\Delta_s J_w}{J_w} \frac{\Delta_s F}{F} \gamma_m(dz)$$

where

$$Cov \left( \frac{dJ^c_w}{J_w}, \frac{dF^c}{F} \right) = \left( \sigma J_w \right) \left( \frac{\sigma F}{F} \right) = \left( \frac{J_{ww} w \sigma_w + J_{wx} \sigma_x}{J_w} \right) \left( \frac{(w \sigma_w) F_w + \sigma_x F_x}{F} \right)$$
Regime-Switching Risk Premium

Remark 2 When $J_{wx} = 0$ and $F_w = 0$ (for example, under log utility),

$$\mu_F^* - r = \left( -\frac{J_{ww}}{J_w} \right) w_\sigma^* F_x^x + \int_E \left( -\frac{\Delta_s J_w}{J_w} \right) \frac{\Delta_s F}{F} \gamma_m(dz)$$

- The first term is the instantaneous diffusion risk premium. $\frac{\sigma^* F_x^x}{F}$ is the volatility of the bond return due to diffusions in $x(t)$. $\left( -\frac{J_{ww}}{J_w} \right) w_\sigma^*$ measures the extra rate of return per unit of such volatility and is referred as the market price of risk.

- The second term is analogously defined as the instantaneous regime-switching risk premium. $\frac{\Delta_s F}{F}$ is the percentage change in bond price due to regime shifts. $\left( -\frac{\Delta_s J_w}{J_w} \right)$ measures the excess bond return per unit of such changes.
Simplification under Log-Utility for Default-free Bond

- Under Log-Utility, $J_{wx} = 0$, $J_w = \frac{1}{\rho_w}$, and $J_{ww} = -\frac{1}{\rho_w}$.
- “Default-free” implies $F_w = 0$, $F_{ww} = 0$ and $F_{wx} = 0$.

**Proposition 3** Under $U(c) = \log(c)$ as CIR. The equilibrium short rate is

$$r = \mu_y - \sigma_y^2 + \int_E \lambda_s(z) \gamma_m(dz)$$

The price of a default-free pure discount bond $F(t, x(t), s(t), T)$ becomes,

$$F_t + (\mu_x - \sigma_y \sigma_x) F_x + \frac{1}{2} \sigma_x^2 F_{xx} + \int_E \Delta_s F(1 - \lambda_s(z)) \gamma_m(dz) = r F$$

for each $s \in \{1, 2, \cdots, N\}$, with the boundary condition: $F(T, x, s, T) = 1$, and where $\Delta_s F = F(t, x(t), s(t-)) + \zeta(z, T) - F(t-, x(t-), s(t-), T)$, and

$$\lambda_s(z) = \frac{\delta_y(z)}{1+\delta_y(z)}.$$  Also, $F(t, x(t), s(t), T) = E^Q\left[ \exp \left( - \int_t^T r(s) ds \right) \right]$. 
A Term Structure with R-S Risk for Estimation

Further assume: $\mu_x = a_0(s) + a_1(s)x$, $\sigma_x = \sqrt{\sigma(s)x}$, $h(z, x(t-)) = e^{\eta_s(z)}$, $\sigma_y = \theta_x(s)\sqrt{\sigma(s)x}$, $\mu_y = x + \theta_x^2(s)\sigma(s)x - \int_E \lambda_s(z)\gamma_m(dz)$, and $\lambda_s(z) = 1 - e^{\theta_s(z)}$.

Then, $r = x$. That is,

$$dr(t) = (a_0(s) + a_1(s))rdt + \sqrt{\sigma(s)r(t)}dB_t.$$  

**Proposition 4**  With the above assumptions, $F(t, \tau) = e^{A(\tau, s_t) + B(\tau, s_t)r_t}$: the approximate price at time $t$ of a default-free pure discount bond with maturity $\tau$;

$R(t, \tau) = -\frac{A(\tau, s_t)}{\tau} - \frac{B(\tau, s_t)r_t}{\tau}$: the $\tau$-period interest rate, where $A(\tau, s)$ and $B(\tau, s)$ satisfy

$$-\frac{\partial B(\tau, s)}{\partial \tau} + [a_1(s) - \theta_x(s)\sigma(s)] B(\tau, s) + \frac{1}{2} \sigma(s) B^2(\tau, s)$$

$$+ \int_E (e^{\Delta_s A} \Delta_s B) e^{\eta_s(z) + \theta_s(z)} 1(s = i)\epsilon_z(dz) = 1$$  \hspace{1cm} \text{(1)}$$

and

$$-\frac{\partial A(\tau, s)}{\partial \tau} + a_0(s) B(\tau, s) + \int_E (e^{\Delta_s A} - 1) e^{\eta_s(z) + \theta_s(z)} 1(s = i)\epsilon_z(dz) = 0$$  \hspace{1cm} \text{(2)}$$

with boundary conditions $A(0, s) = 0$ and $B(0, s) = 0$, and $s = 1, 2$.  

12


**Empirical Results**

**Data**  Monthly interest rates from June 1964 to November 2000 obtained from CRSP. Eight series of interest rates with maturities 1m, 3m, 6m, 1-5 yrs. (6m, 5yr) are chosen to fit Models 1,2,3 for statistical analysis. All eight series are used to estimated the implied regimes.

**Model 1:** CIR has four parameters \( (a_0, a_1, \sigma, \theta_x) \).

**Model 2:** CIR with two regimes but no R-S risk, has ten parameters \( (a_0(1), a_1(1), \sigma(1), \theta_x(1), a_0(2), a_1(2), \sigma(2), \theta_x(2), \eta_s(1,2), \eta_s(2,1)) \).

**Model 3:** CIR with two regimes and R-S risk, has twelve parameters \( (a_0(1), a_1(1), \sigma(1), \theta_x(1), a_0(2), a_1(2), \sigma(2), \theta_x(2), \theta_s(1), \theta_s(2), \eta_s(1,2), \eta_s(2,1)) \).

Efficient Method of Moments

Step 1: Projection Using quasi maximum likelihood to project the observed data (6m,5yr) to an auxiliary model close to the true data generating process.
• Gallant and Tauchen (2001) suggests a SNP model based on Hermite polynomial expansion as a convenient general purpose auxiliary model.
• The dimension of this auxiliary model is selected by minimizing BIC.
• The score function are used as moment conditions to compute a chi-square criterion function.

Step 2: Simulation Simulate (6m,5yr) according to the wanted stationary model to evaluate the expected value of the score and compute a chi-squared criterion function.

Step 3: Optimization A nonlinear optimizer is used to find the parameter setting that minimizes the criterion.

Advantage: If the auxiliary model is a close approximation of the true one, then EMM is asymptotic efficient, close to the efficiency of ML.
Conclusions and Future Works

- Affine regime-switching jump diffusion term structure model
- Regime-switching risk on interest rate derivatives.
- Optimal portfolio choice under regime-switching risk.
- Monetary policy regimes.
- Structural relation between business cycles and the yield curve.

- Paper is available at http://mendota.umkc.edu/paper-term.html