Extensions of the Spending Constraint Model: Existence and Uniqueness of Equilibria

[Extended Abstract]

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1. INTRODUCTION

Although the study of market equilibria has occupied a central place within mathematical economics for over a century, polynomial time algorithms for such issues have started emerging only recently [6, 5, 8]. However, it is worth noting that whereas the traditional theory of market equilibria was built around tools from the fields of analysis and topology, we are attempting to build an algorithmic theory using discrete techniques, and traditional models and notions may not be amenable to such an extension.

Indeed, once the case of linear utility functions was solved [5], handling more complex utility functions required modifying the traditional model [8]. The modification is small, though fundamental, so that the resulting model also appears to be quite basic. Under this model, called the *spending constraint model*, utilities are specified not as a function of the amount of good obtained, but as a function of the amount of budget spent on that good. The model is a natural one, since people typically do have an estimate, either implicit or explicit, on how much they are willing to spend on each good.

[8] gives a polynomial time algorithm for the special case of decreasing step functions in this model. Two questions arise naturally: For more general utility functions in this model, can existence of equilibrium prices be established using traditional tools? If so, can they be computed or approximated efficiently by either extending or using the algorithm of [8]? In this paper, we provide affirmative answers to both questions for the case of continuous and decreasing functions. Additionally, we show that this model supports unique equilibrium prices, unlike the traditional model (a proof of uniqueness for decreasing step functions was given in [8]). Note that models yielding unique equilibrium prices have been considered desirable since this is another indication of stability in markets [2, 4].

The model proposed in [8] can be thought of as the spending constraint extension of Fisher's setting (see Section 2). In this paper, we define the spending constraint extension of the setting of Arrow and Debreu [1] and again consider the two questions stated above. We answer the first affirmatively and leave the second as an open problem (the extended model does not support unique equilibrium prices).

Thus the spending constraint model not only measures up to existential results, as in the traditional theory, but is also amenable to efficient algorithms, and exploring it further appears worthwhile. In this short note we only list our results; proofs will be provided in a forthcoming paper.

2. THE MODELS

Let A be a set of divisible goods, and B a set of buyers. Each buyer i has m_i amounts of money. The amount of good j available is b_j (which, w.l.o.g can be assumed to be unity). Utility functions are specified for each buyer, and given prices $\boldsymbol{p}$ for the goods, each buyer wants to spend her money on a bundle of goods that maximizes her utility. The problem is to find prices such that after each buyer is given an optimal bundle, the market clears, i.e., there is no surplus or deficiency of any good. Such prices are called **market clearing prices**.

Such a model was first defined by Irving Fisher in 1891, see [3]; Fisher's work was done contemporarily and independently of Walras' pioneering work [9] on modeling market equilibria. In his model, each buyer i had a utility function, U_{ij} for each good j; U_{ij} is a function of the amount, x_{ij}, of j that i receives. Fisher assumed that the U_{ij}'s were concave functions. [5] derive a polynomial time algorithm for the linear version of Fisher's problem.

In the model studied by Arrow and Debreu [1], there is no demarcation between buyers and sellers. Each agent comes to the market with an initial endowment of goods, and at the given prices, sells all her goods and wants to buy an optimal bundle, as per her utility function and the specified prices. Once again, the problem is to find prices at which market clears, i.e., after each agent is given her optimal bundle, there is no deficiency or surplus of any good.

The spending constraint model, introduced in [8], was defined in Fisher's setting, and is as follows. The *rate function* of buyer $i \in B$ for good $j \in A$ is specified via function $f_j^i : [0, e(i)] \to \mathbf{R}^+$, which gives the rate at which i derives utility per unit of j received, as a function of the amount of her budget spent on j. If the price of j is fixed at p_j per unit amount of j, then the function $g_j^i : [0, e(i)] \to \mathbf{R}^+$,

$$g_j^i(x) = \int_0^x \frac{f_j^i(y)}{p_j} dy.$$

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gives the utility derived by i on spending x dollars on good j. In this paper, we will assume that each f_j^i is continuous and monotonically decreasing. Let's call it *Model 1.* Under this model, g_j^i will be strictly concave and differentiable, and moreover, at any prices of goods, there is a unique allocation that maximizes i's utility.

We extend the spending constraint model to the Arrow-Debreu setting as follows. Each agent $i \in B$ comes to the market with a bundle of goods $\{e_{ij}\}_{j \in A}$. The rate function f_j^i now is a function from $[0,1]$ to $\mathbf{R}^+$ and specifies the rate at which i derives utility per unit of j received, as a function of the *fraction* of her budget spent on j. Observe that i's budget can only be determined once prices are fixed for all goods, and is given by $m_i = \sum_{j \in A} p_j e_{ij}$. The utility derived by i on spending x dollars on good j at prices $\boldsymbol{p}$ is given by

$$g_j^i(x, \boldsymbol{p}) = \int_0^{\frac{x}{m_i}} \frac{f_j^i(y)}{p_j} dy.$$

Once again, we will assume that each f_j^i is continuous and monotonically decreasing. Let's call it *Model 2.*

3. FISHER'S SETTING: MODEL 1

We use Brouwer's Fixed Point Theorem to prove:

THEOREM 1. *In Model 1, market clearing prices exist.*

Let $\xi(\mathbf{p})$ be the *total demand function*: its j^{th} component is the total demand for good j, in terms of money, at prices $\boldsymbol{p}$. We say that a market satisfies **weak gross substitutability** of goods if increasing the price of good j cannot decrease the demand for another good j', i.e.,

$$\frac{\partial \xi_j}{\partial p_{j'}} \geq 0, \forall\ j \neq j' \in A,$$

and that it satisfies **scale invariance** if

$$\xi(\mathbf{p}) = \xi(\lambda \mathbf{p}), \forall\ \lambda > 0.$$

It is easy to prove that Model 1 satisfies both these conditions.

THEOREM 2. *A market satisfying the two conditions stated above supports unique equilibrium prices.*

COROLLARY 3. *In Model 1, defined above, market clearing prices are unique.*

Note that in contrast, the Fisher's original model does not support unique equilibrium prices even for piece-wise linear utility functions (of course, it does for linear utilities). For example, consider a market with 2 buyers, having \$2 each, and 2 goods. Let $u_{11}(x) = u_{22}(x) = 10x$, if $0 \leq x \leq 1/2$ and 5 otherwise. Let $u_{12}(x) = u_{21}(x) = x$. Then both $(1,3)$ and $(3,1)$ are market clearing prices.

Next, we use the algorithm of [8] as a subroutine to derive an FPTAS for Model 1. The main idea is to approximate the f_j^i's by fine (depending on the error parameter) decreasing step functions. Suppose the step length is chosen to be ϵ and the subroutine returns price vector $\boldsymbol{p}$. Let U_i be the utility obtained by buyer i with this allocation w.r.t. the step utility functions, and U_i' be her utility, under the same allocation, but computed w.r.t. the original rate functions. Suppose that the derivative of f_j^i's is bounded by Δ and let $\mathbf{p}'$ be the unique equilibrium price under f_j^i's. Let j be such that $\frac{p_j}{p_j'}$ be minimized. The FPTAS follows from:

THEOREM 4. $\forall\ \epsilon > 0$,

1. $\forall\ i \in B, U_i' \geq U_i - \epsilon^2 \Delta$.

2. $p_j \geq p_j' - n\epsilon$.

4. ARROW-DEBREU SETTING: MODEL 2

We use Kakutani's Fixed Point Theorem to prove:

THEOREM 5. *Market clearing prices exist for Model 2.*

Model 2 does not support unique equilibrium prices. Consider two agents, each coming to the market with a unit amount of distinct goods. Suppose that the utility of each agent for her good far outweighs the utility for the other good. Then, for a whole continuum of prices we have market equilibria in which each agent buys only what she has. This example may also be pointing out the difficulty of obtaining a polynomial time algorithm for Model 2, even when restricted to linear utilities, the problem left open by [6]. The difficulty is: which equilibrium price should the algorithm shoot for? Note that even when a discrete algorithm is faced with multiple, though discrete, solutions, uniqueness is arbitrarily imposed – by breaking ties arbitrarily, and asking for the lexicographically first solution under the imposed ordering.

The next best recourse is to seek an FPTAS for this case, a result obtained by [7]. Their algorithm requires $O\left(\frac{n^2(n \log U + \log Mn^2)}{\epsilon}\right)$ max-flow computations (here U and M depend on the utilities and endowments of the buyers). We improve the running time as follows.

THEOREM 6. *For all $\epsilon > 0$ there is an algorithm that gives an ϵ-approximate market equilibrium and makes $O\left(\frac{mn^2}{\epsilon}\right)$ max-flow computations, where m is the number of buyers.*

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