What Does LIGO Measure?

The correspondence between mathematics and physics for gravitational wave observations

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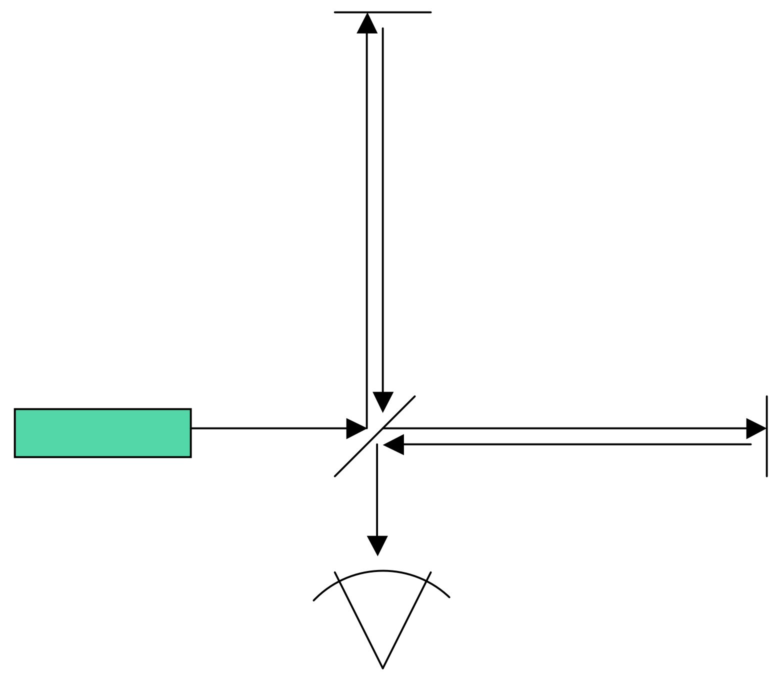
Goal, Caveat, Outline

- A mathematical description of the LIGO experiment
  - The physical observable and its relationship to the mathematical objects of GRT
- Caveat
  - Will work in language of perturbation theory
  - Implies (not yet discussed) distinction between background and waves

- Outline
  - Description of "LIGO"
  - Description of measurement
  - Relationship between observable and metric
  - Outline of calculation
  - Afterword: from "LIGO" to LIGO

Description of apparatus

- LIGO: Laser Interferometer Gravitational-wave Observatory
  - Idealize for discussion: one-bounce Michelson IFO
  - Laser
  - Beamsplitter
  - End-mirrors
- Physical extent of instrument is critical, but *measurements all take place at output port*

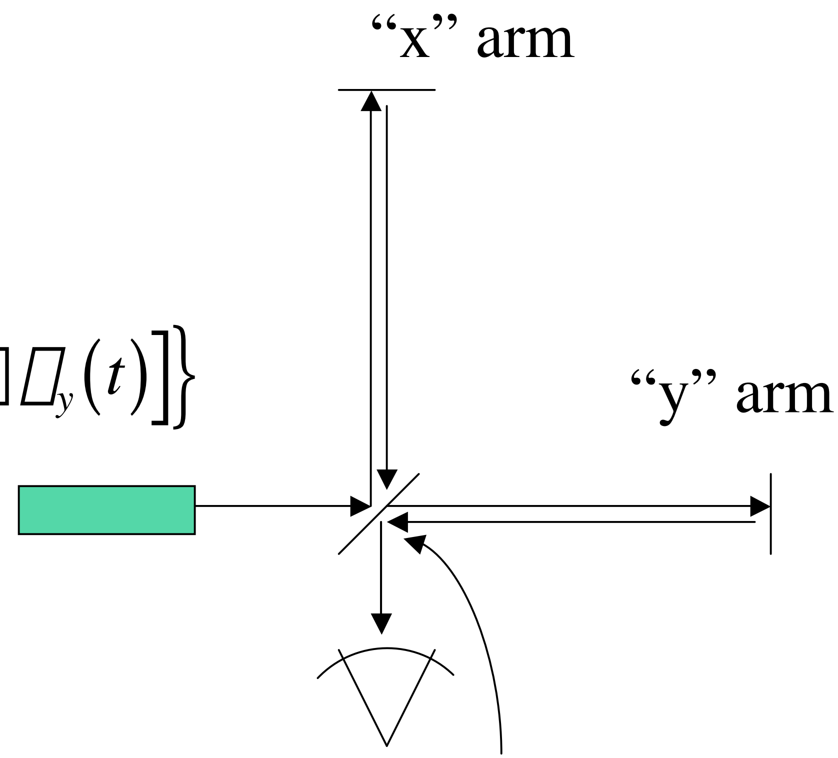


Description of measurement

- Measurement: field intensity at output port
- Intensity depends on phases $\phi_x(t)$, $\phi_y(t)$

$$\left|E_o(t)\right|^2 = 2\left|\mathcal{E}_o\right|^2\left\{1+\cos\left[\phi_x(t)-\phi_y(t)\right]\right\}$$

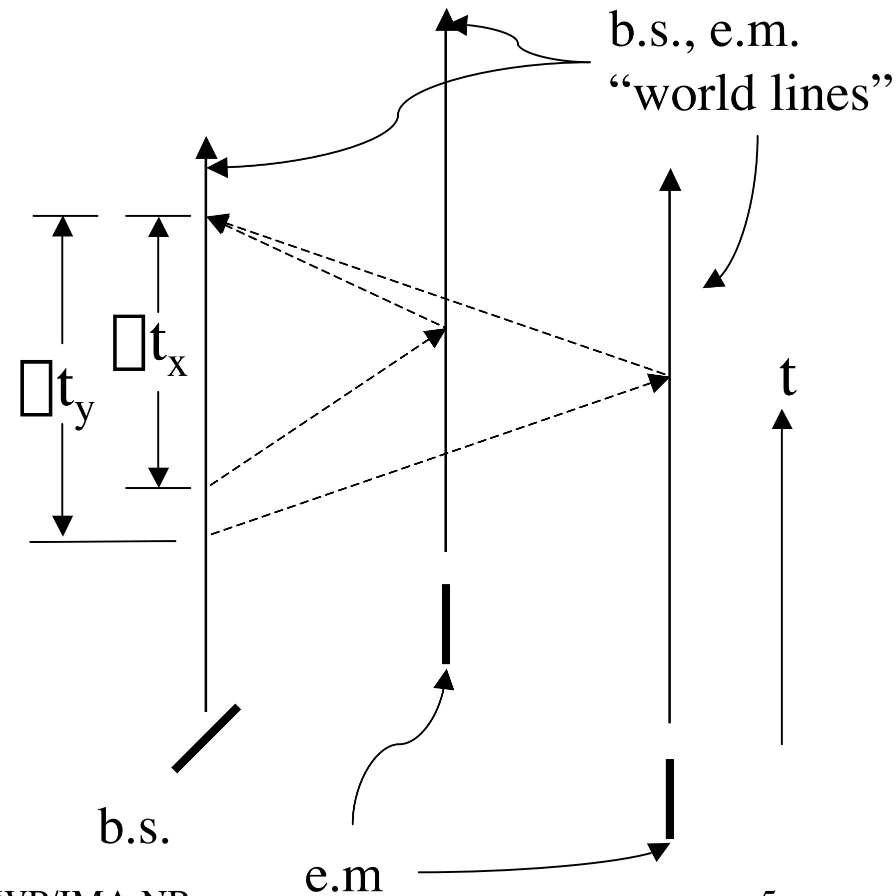
- What are phases $\phi_x(t)$, $\phi_y(t)$ of returning light?



$$E_o(t) = \mathcal{E}_o\left(e^{i\phi_x(t)} + e^{i\phi_y(t)}\right)$$

Relationship of measurement to metric

- What are phases $\phi_x(t)$, $\phi_y(t)$ of returning light?
- Phase is *constant* on *null geodesics*
  - Light returning now (t_o) emerged from laser at some time in past
  - $\phi_x(t_o) = \phi_l(t_o - \Delta t_x)$
  - $\phi_y(t_o) = \phi_l(t_o - \Delta t_y)$
  - Different paths, different elapsed times, different phases
- ***g** determines geodesics*



Outline of calculation (exercise for reader)

- Simplifying assumptions (all can be relaxed)
  - Grav.-wave perturbations on flat (Minkoskii) spacetime
  - Mirrors move on geodesics
  - Assume mirrors are, at some initial moment of time, at relative rest ($dx^i/dt = 0$, x^i spatial coordinate function, i=1..3)
- Work in TT gauge
  - Gauge choice simplifies *calculational* details
    - Though beware interpretation of non-invariant quantities

Outline of calculation: Preliminaries

- Recall TT gauge properties

$$g = \eta + h$$

$$\eta = -dt^2 + dx^2 + dy^2 + dz^2$$

$$0 = \partial_a\left(g^{ab} h_{bc}\right)$$

$$0 = U^a h_{ab}$$

$$0 = h_{ab} g^{ab}$$

- Show that
  - Mirrors initially at TT *coordinat*e rest remain at TT *coordinate* rest
    - I.e., spatial coordinate location of mirrors is time independent
- Show that
  - Time coordinate t is proper time for observer at rest in (t,x,y,z) coords.

Outline of calculation (exercise for reader)

- Evaluate when light *arriving at beamsplitter* along arm x (y) *now* (t_o) *left beamsplitter* along arm x (y)
  - Unperturbed arm length (constant t geodesic) L
  - $t = t_o - \Delta t$
  - $\Delta t = \Delta t_1 + \Delta t_2$
    - Δt_1 from end-mirror to beamsplitter
    - Δt_2 from beamsplitter to end-mirror
    - Terms gauge dependent, sum gauge *independent*
- For arm in direction x

$$\Delta t_x(t_0) = 2L\left[1 + H_x(t_0)\right]$$

Integral along light path

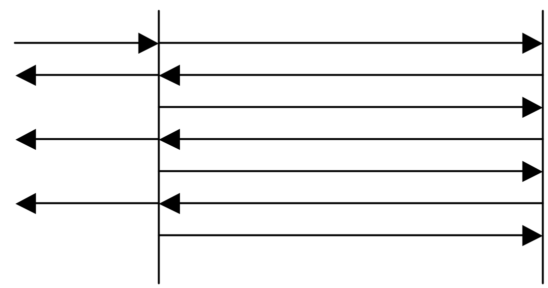
Projection of h along x arm

$$H_x(t_0) = \frac{1}{4L}\int_0^L \left[h_{xx}(t_0 - s, s, y, z) + h_{xx}(t_0 - L + s, L - s, y, z)\right] ds$$

Outline of calculation (exercise for reader)

- $\square_x(t_o)-\square_y(t_o)=2f_l L[H_x(t_o)-H_y(t_o)]$
- When $\square >> L$ $\quad H_x(t_0) = h_{xx}(t_0,0,0,0)/2$
  - $\square_x(t_o)-\square_y(t_o)=f_l L[h_{xx}(t_o,0,0,0)-h_{yy}(t_o,0,0,0)]$
  - "Small antenna" limit
- LIGO measures integral of metric perturbation along spacetime path light takes in moving from beamsplitter to end mirrors and back

Afterword: From "LIGO" to LIGO



- LIGO "more" than one-bounce Michelson
  - "resonant cavity" arms
- LIGO mirrors are accelerating
  - Not on geodesic trajectories
- Waves in curved background
  - Detector small, wavelength short, compared to curvature scale
    - Separate waves from background based on multiple length (& time) scales
  - Free mirrors don't remain at coordinate rest