What Does LIGO Measure?

The correspondence between mathematics and physics for gravitational wave observations

Lee Samuel Finn
Center for Gravitational Wave Physics
Goal, Caveat, Outline

- A mathematical description of the LIGO experiment
  - The physical observable and its relationship to the mathematical objects of GRT
- Caveat
  - Will work in language of perturbation theory
  - Implies (not yet discussed) distinction between background and waves

- Outline
  - Description of “LIGO”
  - Description of measurement
  - Relationship between observable and metric
  - Outline of calculation
  - Afterword: from “LIGO” to LIGO
Description of apparatus

• LIGO: Laser Interferometer Gravitational-wave Observatory
  – Idealize for discussion: one-bounce Michelson IFO
  – Laser
  – Beamsplitter
  – End-mirrors

• Physical extent of instrument is critical, but measurements all take place at output port
Description of measurement

- Measurement: field intensity at output port
- Intensity depends on phases \( \varphi_x(t) \) and \( \varphi_y(t) \)

\[
|E_o(t)|^2 = 2|\mathcal{E}_o|^2 \left\{ 1 + \cos \left[ \varphi_x(t) \right] \right\}
\]

- What are phases \( \varphi_x(t) \), \( \varphi_y(t) \) of returning light?

\[
E_o(t) = \mathcal{E}_o \left( e^{i\varphi_x(t)} + e^{i\varphi_y(t)} \right)
\]
Relationship of measurement to metric

- What are phases $\phi_x(t)$, $\phi_y(t)$ of returning light?

- Phase is *constant* on *null* geodesics
  - Light returning now ($t_o$) emerged from laser at some time in past
  - $\phi_x(t_o) = \phi_x(t_o - t_x)$
  - $\phi_y(t_o) = \phi_y(t_o - t_y)$
  - Different paths, different elapsed times, different phases

- $g$ determines geodesics
Outline of calculation (exercise for reader)

- Simplifying assumptions (all can be relaxed)
  - Grav.-wave perturbations on flat (Minkoskii) spacetime
  - Mirrors move on geodesics
  - Assume mirrors are, at some initial moment of time, at relative rest \( (dx^i/dt = 0, x^i \text{ spatial coordinate function, } i=1..3) \)

- Work in TT gauge
  - Gauge choice simplifies *calculational* details
    - Though beware interpretation of non-invariant quantities
Outline of calculation:
Preliminaries

- Recall TT gauge properties

\[ g = \Box + h \]
\[ \Box = \Box dt^2 + dx^2 + dy^2 + dz^2 \]
\[ 0 = \Box_a (g^{ab} h_{bc}) \]
\[ 0 = U^a h_{ab} \]
\[ 0 = h_{ab} g^{ab} \]

- Show that
  - Mirrors initially at TT coordinate rest remain at TT coordinate rest
  - I.e., spatial coordinate location of mirrors is time independent

- Show that
  - Time coordinate t is proper time for observer at rest in (t,x,y,z) coords.
Outline of calculation (exercise for reader)

- Evaluate when light arriving at beamsplitter along arm x (y) now \( t_o \) left beamsplitter along arm x (y)
  - Unperturbed arm length (constant t geodesic) L
  - \( t = t_o - \bar{t} \)
  - \( \bar{t} = \bar{t}_1 + \bar{t}_2 \)
    - \( \bar{t}_1 \) from end-mirror to beamsplitter
    - \( \bar{t}_2 \) from beamsplitter to end-mirror
    - Terms gauge dependent, sum gauge independent

- For arm in direction x
  \[
  \bar{t}_x(t_0) = 2L \left[ 1 + H_x(t_0) \right]
  \]
  Integral along light path

  \[
  H_x(t_0) = \frac{1}{4L} \int_0^L h_{xx}(t_0 \bar{s}, s, y, z) + h_{xx}(t_0 \bar{L} + s, L \bar{s}, y, z) \, ds
  \]
  Projection of \( h \) along x arm

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Outline of calculation
(exercise for reader)

• $\Box_x(t_o)-\Box_y(t_o)=2f_l L[H_x(t_o)-H_y(t_o)]$

• When $\Box >> L$, $H_x(t_0) = h_{xx}(t_0,0,0,0)/2$
  - $\Box_x(t_o)-\Box_y(t_o)=f_l L[h_{xx}(t_o,0,0,0)-h_{yy}(t_o,0,0,0)]$
  - “Small antenna” limit

• LIGO measures integral of metric perturbation along spacetime path light takes in moving from beamsplitter to end mirrors and back
Afterword: From “LIGO” to LIGO

- LIGO “more” than one-bounce Michelson
  - “resonant cavity” arms
- LIGO mirrors are accelerating
  - Not on geodesic trajectories
- Waves in curved background
  - Detector small, wavelength short, compared to curvature scale
    - Separate waves from background based on multiple length (& time) scales
  - Free mirrors don’t remain at coordinate rest