The Conformal Approach to Numerical Relativity

or:
Putting GEO on the Grid (almost)

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To infinity and beyond . . .

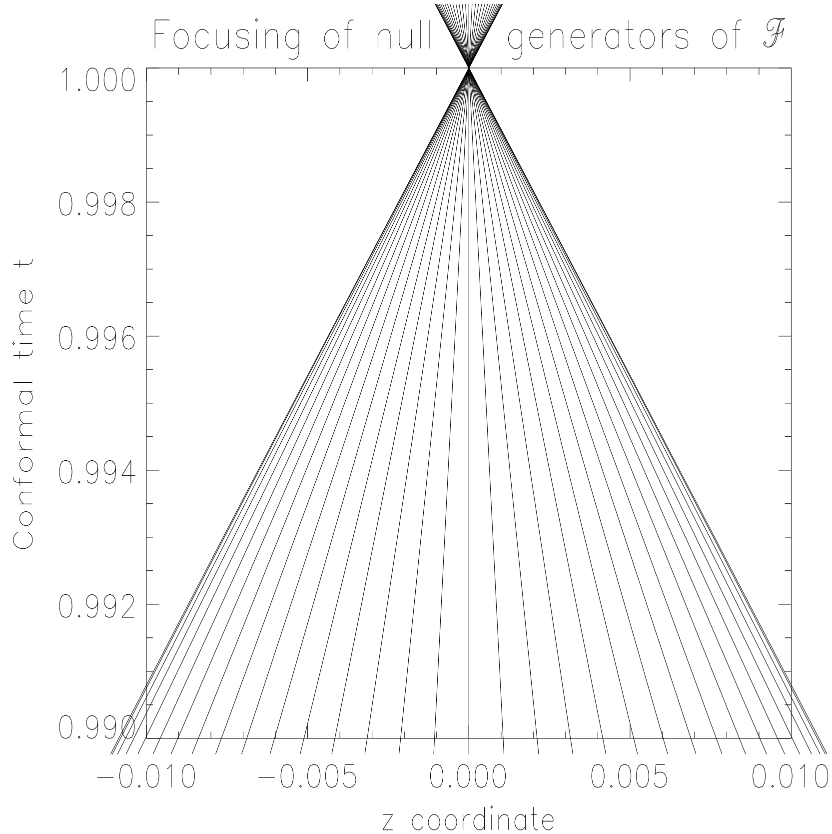

Figure 1: Generators of $\mathscr{I}^+$ focus at timelike infinity.

P. Hübner: Weak data evolve into regular i^+ – resolved as one grid cell!

what's it all about

basic idea: global methods work well for analytical problems –
could they also have advantages for numerical work?

hope: yes, and eventually we will be able to prove it

fact: global methods work fine for characteristic approach (caustics)
and Cauchy slices (but i^0 can't see radiation)

big Q: once the traditional (cutoff) and conformal approaches both work –
which one will be more efficient (e.g. computationally, conceptually, logistics)?

slightly more detail . . .

We want:
well defined & unambiguous numerical computation of generic isolated systems

Motivation for the conformal approach:
radiation, mass, angular momentum, boundary conditions etc. are only defined "at infinity" – approximations to infinity can be hard to control!

We use: Friedrich's conformal field equations:
physical metric: $\tilde{g}_{ab} = \Omega^{-2} g_{ab}$, physical spacetime $\tilde{\mathcal{M}} = \{p \in \mathcal{M} \,|\, \Omega(p) > 0\}$

hyperboloidal hypersurfaces: analogous to $t^2 - \vec{x}^2 = k^2$ in Minkowski space

3+1 split $\longrightarrow$ 57 Variables: h_{ab}, k_{ab}, $\gamma^a{}_{bc}$, ${}^{(0,1)}\hat{R}_a$, ${}^{(1,1)}\hat{R}_{ab}$, E_{ab}, B_{ab}, Ω, Ω_0, Ω_a, $\nabla^a \nabla_a \Omega$

functions of coordinates: N^a, $q = \log \frac{N}{\sqrt{\det h}}$, $R \Rightarrow$ regular symm. hyp. evolution system

(As Oscar and Mark have told us: this is not expected to work!)

Remark on compactifying Einstein Equations

Can obviously not be straightforward:

Einstein's vacuum equations in terms of Ω & g_{ab}:

$$\begin{aligned}\tilde{G}_{ab}[\Omega^{-2}g] \quad &= \quad G_{ab}[g] - \frac{2}{\Omega}\left(\nabla_a \nabla_b \Omega - g_{ab} \nabla_c \nabla^c \Omega\right) \\ &\quad - \frac{3}{\Omega^2} g_{ab} \left(\nabla_c \Omega\right) \nabla^c \Omega.\end{aligned}$$

singular for $\Omega = 0$, multiplication by Ω^2 also does not help here $\longrightarrow$ the principal part of PDEs encoded in G_{ab} would degenerate at $\Omega = 0$.

Multiply by Ω^2: for a vacuum spacetime $(\nabla_c \Omega) \nabla^c \Omega = 0$ @ $\mathscr{I}$ $\Rightarrow$ must consist of null surfaces!

numerical setup

pseudospectral solver for topologically trivial initial data

4th order method of lines: RK in time + centered in space + dissipation

a "transition layer" in the unphysical region is used to transform Einstein equations to trivial equations – stable with trivial copy at outermost gridpoint

Continuum problem is well posed, numerical results indicate numerical stability, theorems available to guarantee numerical stability for high enough dissipation (including boundary!), boundary conditions are physical.

Science fiction or the best possible method?

results I

Class. Quant. 18 (2001) p. 1871 & higher amplitudes

The initial conformal metric is chosen as

$$ds^2 = \left(1 + \frac{A}{3}\bar{\Omega}^2\left(x^2 + 2y^2\right)\right) dx^2 + dy^2 + dz^2$$

The boundary defining function is

$$\bar{\Omega} = \frac{1}{2}\left(1 - \left(x^2 + y^2 + z^2\right)\right) .$$

Gauge:

$$\begin{aligned} R &= 0, \\ N^a &= 0, \\ q &= 0 \quad \Rightarrow N = e^q\sqrt{\det h} = \sqrt{\det h} \end{aligned}$$

$A = 0 \longrightarrow g$ is Minkowski in standard coords.!

The complete future of (the physical part of) the initial slice can thus be reconstructed in a finite number of computational time steps!

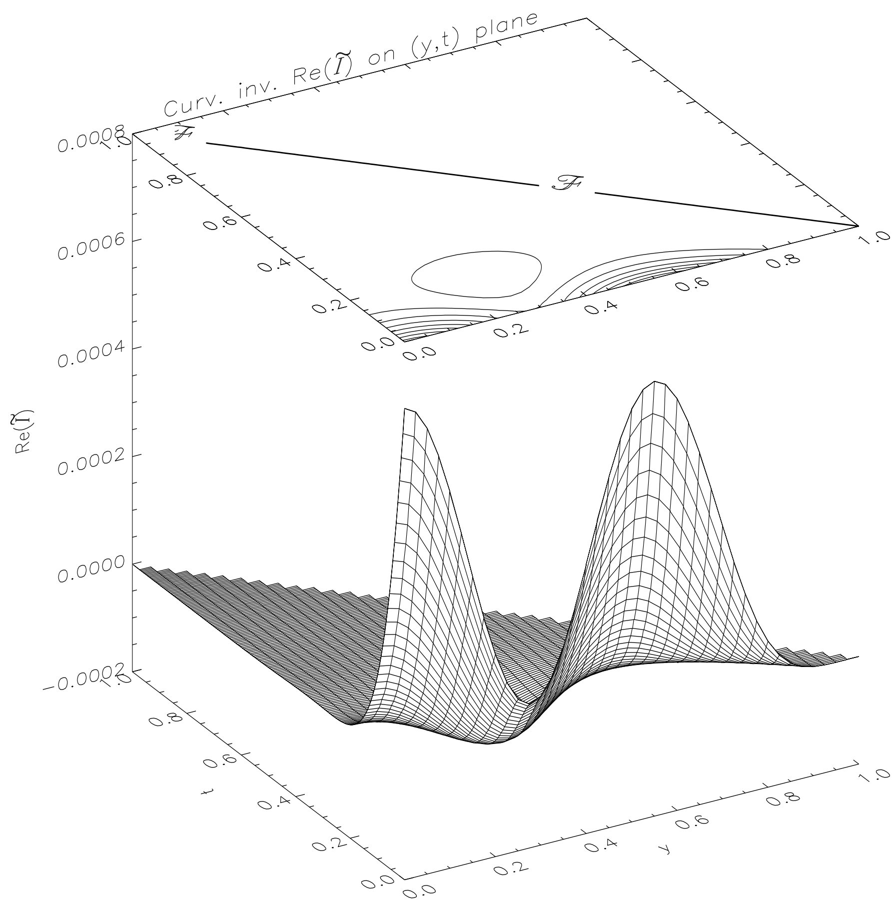

Figure 2: $\tilde{I} = \Omega^6 I$.

results II – Minkowski static gauge

(Marsha Weaver, see also V. Moncrief ITP UCSB online, Mini-program on black hole collisions)

$$ds^2 = -\Omega^2 dt^2 - 2r dr dt + dr^2 + r^2 d\Omega^2$$

$$\Omega = \frac{1-r^2}{2}.$$

$$R = 12\frac{(1-r^2)(3+r^2)}{(1+r^2)^3}$$

stable?

Missed out on Mexico . . . ?

Workshop on Formulations of Einstein Equations for Numerical Relativity, May 13-24 2002: two weeks on comparing codes and results

. . . there's still a chance to be part of a great experience!

http://www.aei.mpg.de/∼shawley/Mexico/ : the talks, the stories, the backstage info ...

cvs -d :pserver:cvs_anon@cvs.aei.mpg.de:/numrelcvs co CodeComparison

get your password now!

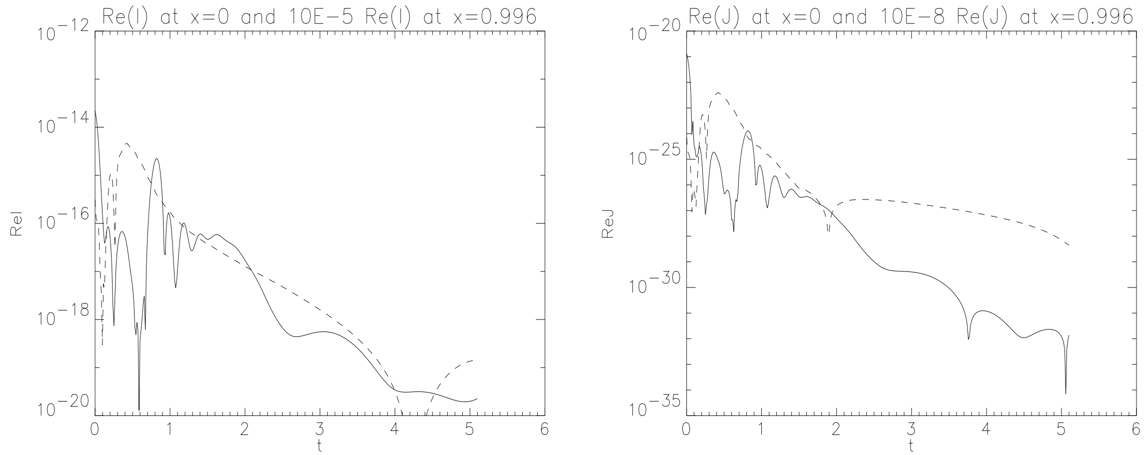

Figure 3: gridpoint at center (unbroken line), grid point at $x = 0.996$ (broken line).

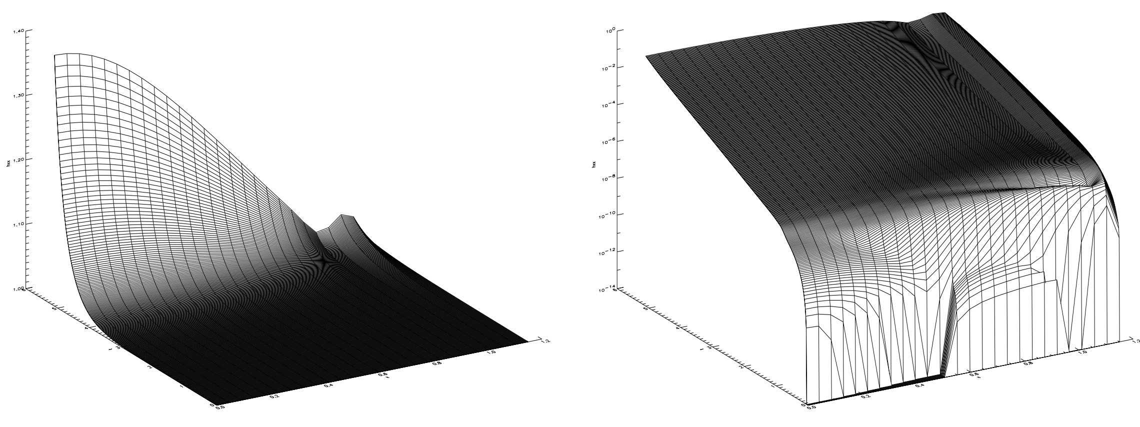

Figure 4: h_{xx} for $x \geq 0$ versus t with linear (left) and logarithmic (right) scaling.

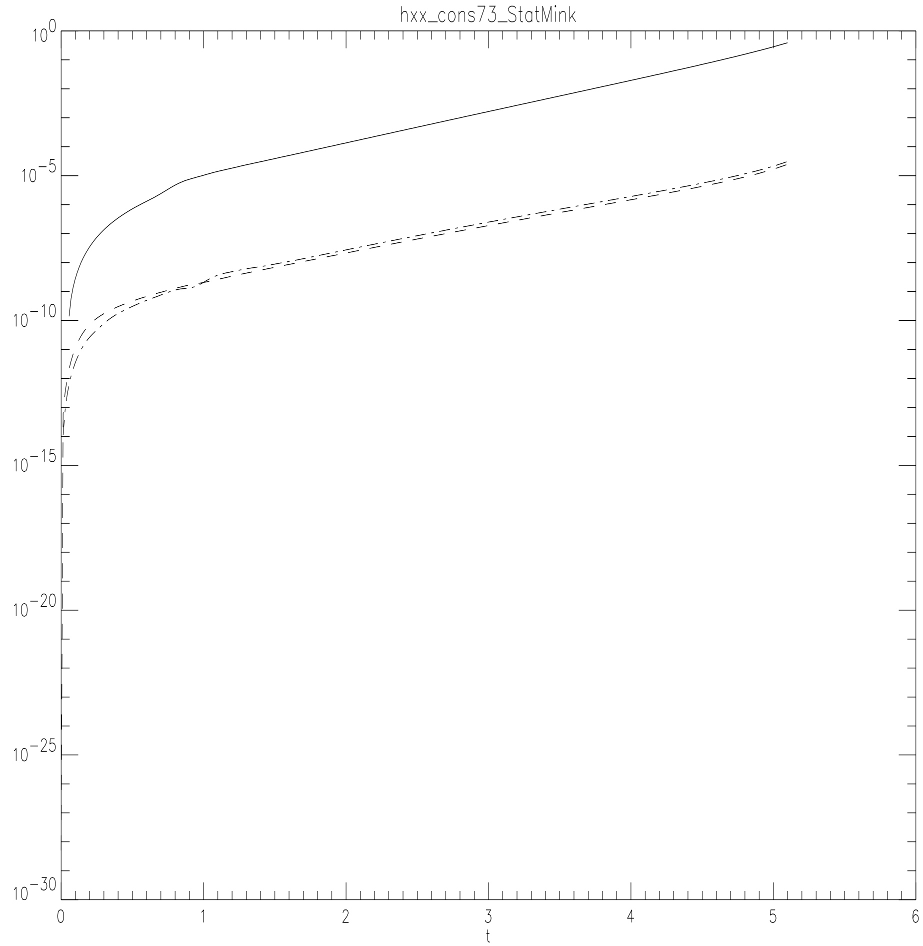

Figure 5: h_{xx} (unbroken) and constraints ${}^{(3)}\nabla_x h_{xx}$ & ${}^{(3)}\nabla_x \Omega = \Omega_x$.

results III – "Brill" ansatz

Axisymmetric Brill–wave type ansatz:

$$ds^2 = \omega^2 \left(e^{2Q}(d\rho^2 + dz^2) + \rho^2 d\varphi^2 \right) ,$$

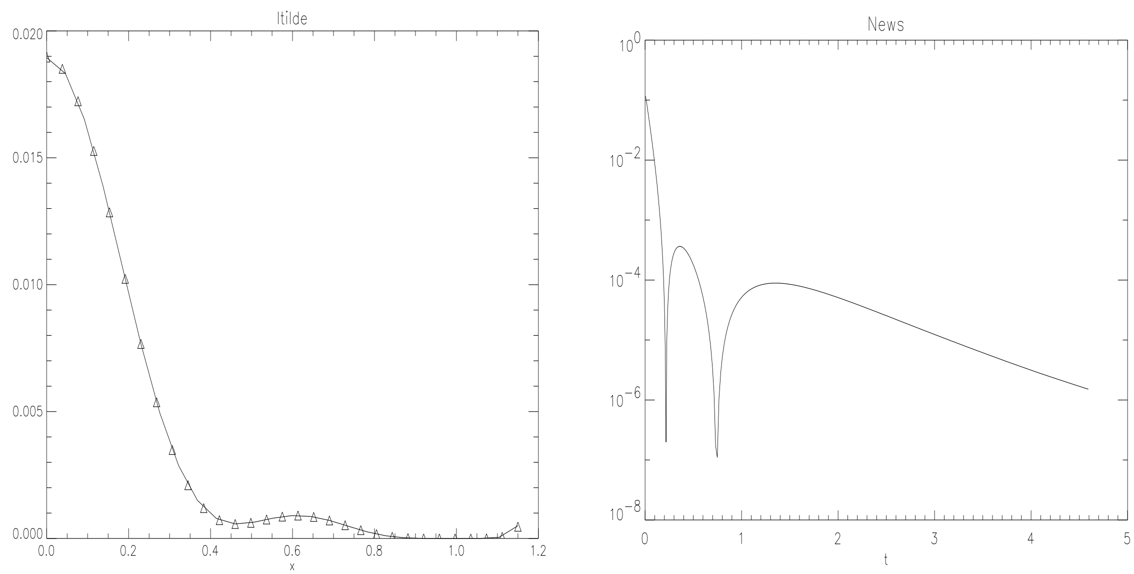

Figure 6: The left image shows the real part of the physical curvature invariant $\tilde{I} = \Omega^6 I$. The right image shows the corresponding mass loss function $\dot{M}_B$.

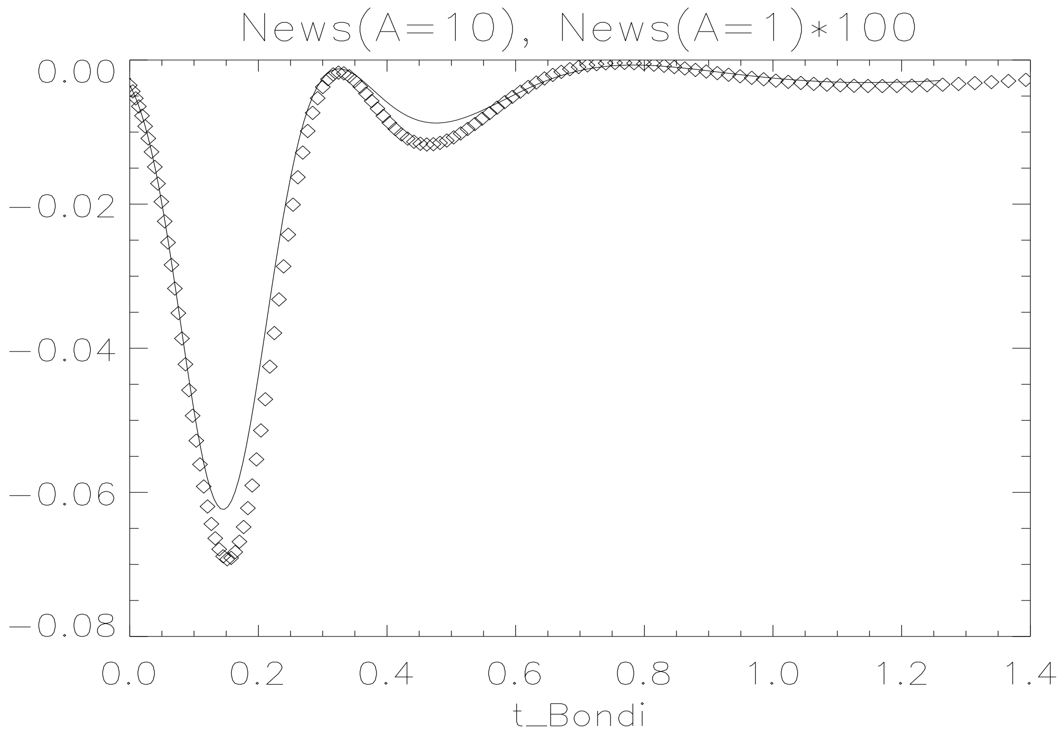

Figure 7: $A = 1$ vs. $A = 10$ “Brill” data.

bottom lines

- the conformal approach excels at radiation extraction
- evolution is numerically stable and quite robust
- instability appears on continuum level: interpretation as constraint violating modes – all Einstein equations want to be solved everywhere in the grid! – will require modification of equations – **Well-posedness is not enough!**
- a bad gauge can ruin the nicest geometry – the conformal approach wants a live gauge too . . .
- the cuts of $\mathscr{I}$ are spheres: isolated systems simulations want spherical boundaries
- initial data: a regular elliptic system is not yet available
- the standard 3+1 and conformal approach essentially seem to exhibit corresponding technical problems (but: ID and outer boundary condition / radiation extraction)

more bottom lines

- typical for compactified approaches: things happen faster, less room for cheating
- a consequence: we need good boundary conditions!
- coding: don't touch these equations with bare hands: use tools!