Black Hole Excision Using Pseudospectral Collocation Methods

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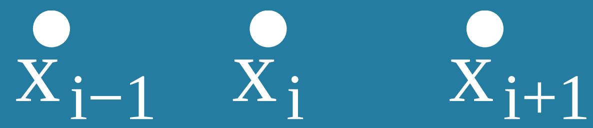
Outline:

- A Pseudospectral Evolution Algorithm
- Boundary Conditions
- Excision
- Our Primary Difficulty

Introduction: Spectral methods

- Finite difference approach:



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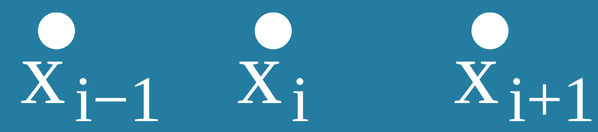

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  - ⋆ Write *solution* as sum of N basis functions $\phi_k(x)$.

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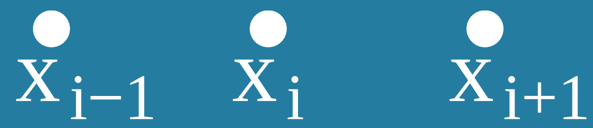

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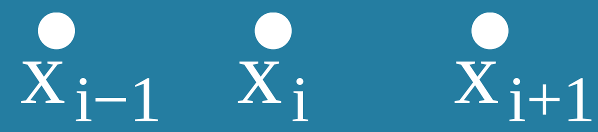

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Why Spectral Methods for Black Holes?

- Exponential convergence for smooth solutions. (error $\sim e^{-N}$ vs N^{-2} for FD)

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Common side effects include. . .

- More restrictive CFL condition
  (for Chebyshev basis, CPU $\sim N^{D+2}$ vs N^{D+1} for FD)
- Adding/removing "grid points" changes entire grid.
- . . .

Pseudospectral Collocation Evolution Algorithm

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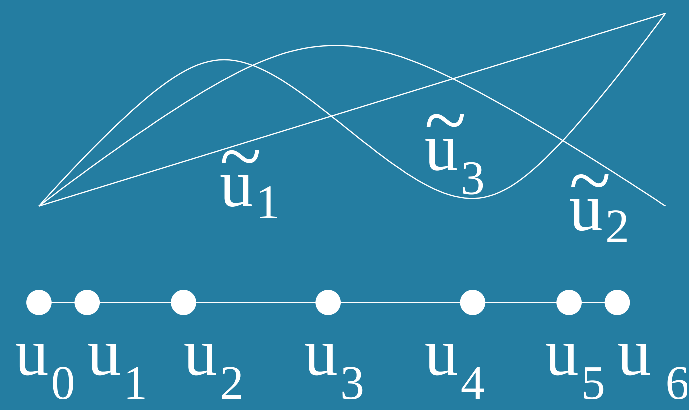

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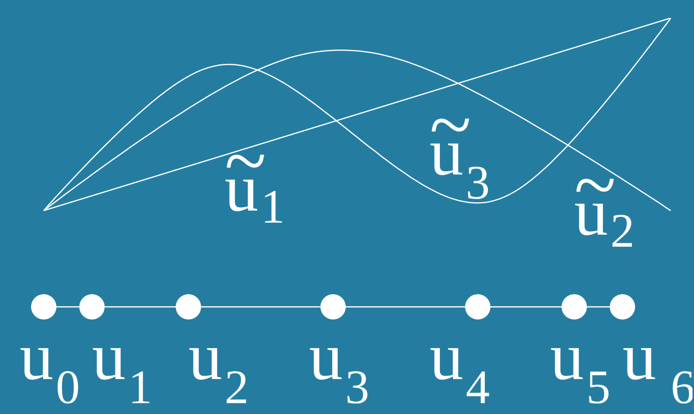

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⋆ Write (1) as ODEs for each $u_n(t)$: $\partial_t u_n(t) = F_n(\{u_n\}, t)$

- ∗ Evaluate spatial derivatives analytically.
- ∗ Evaluate nonlinear terms by multiplying $u_n(t)$.

Treatment of Boundaries

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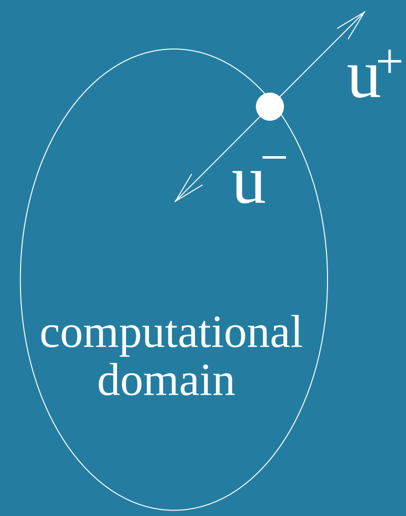

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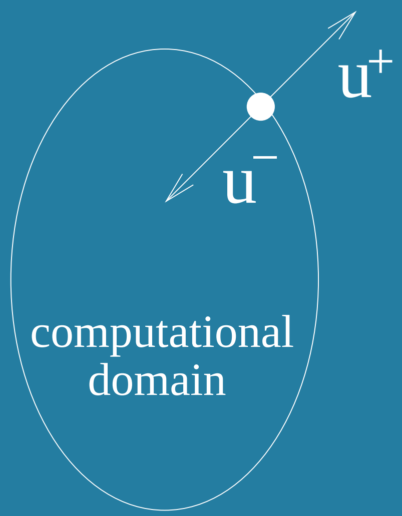

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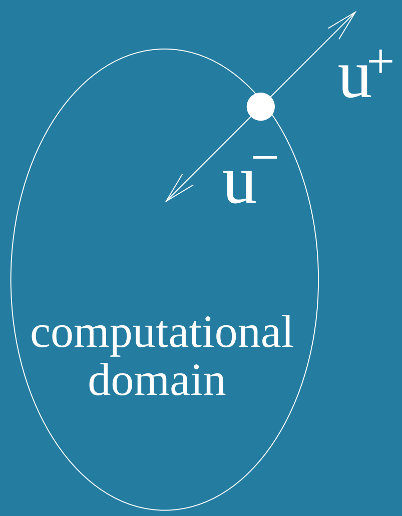

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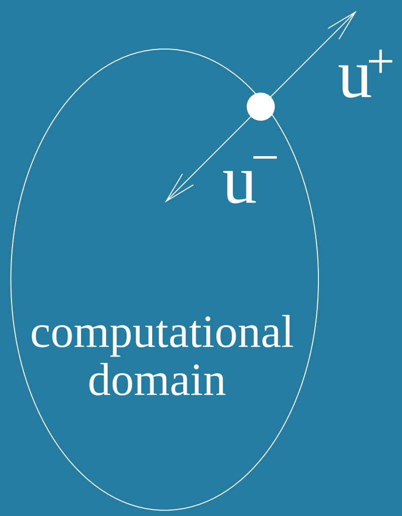

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Specify ingoing fields

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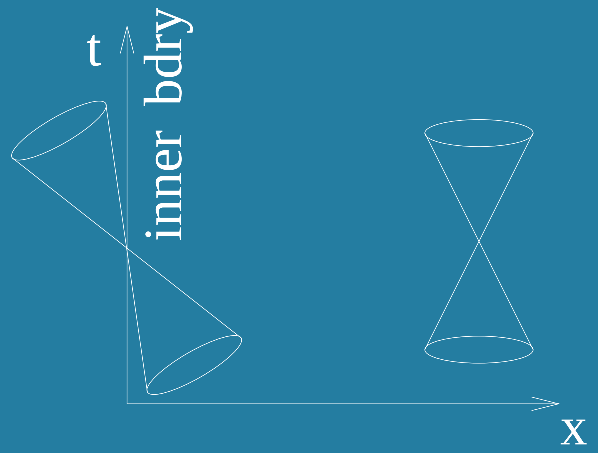

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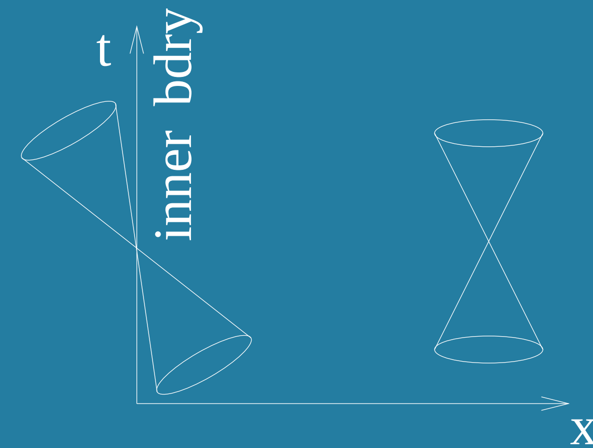

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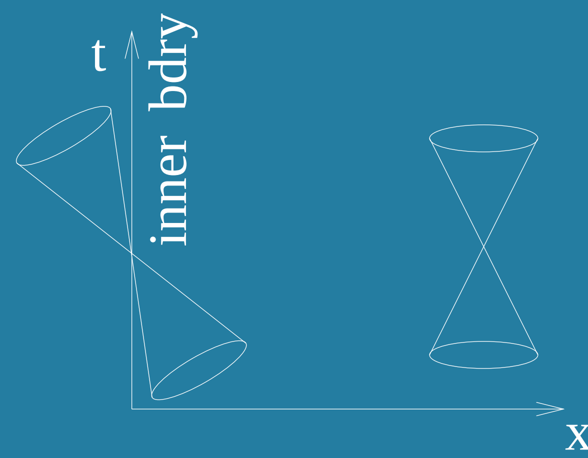

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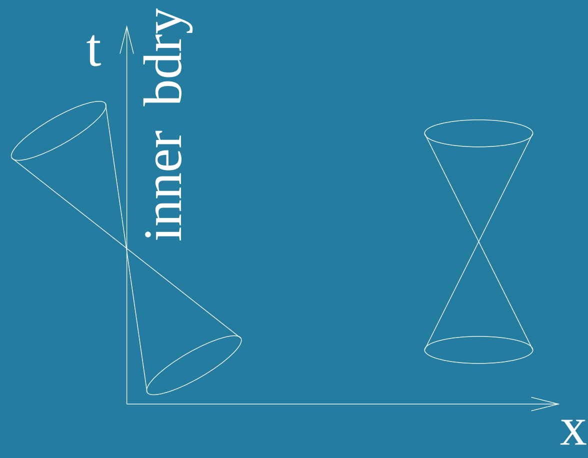

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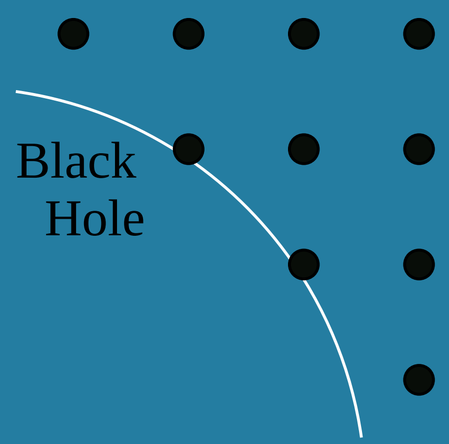

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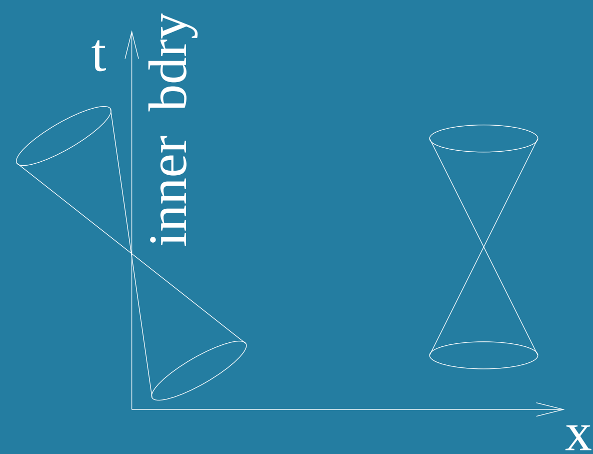

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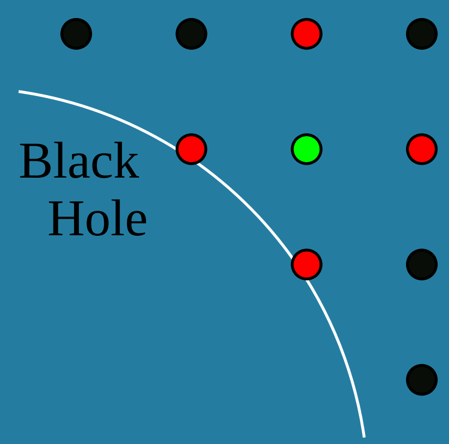

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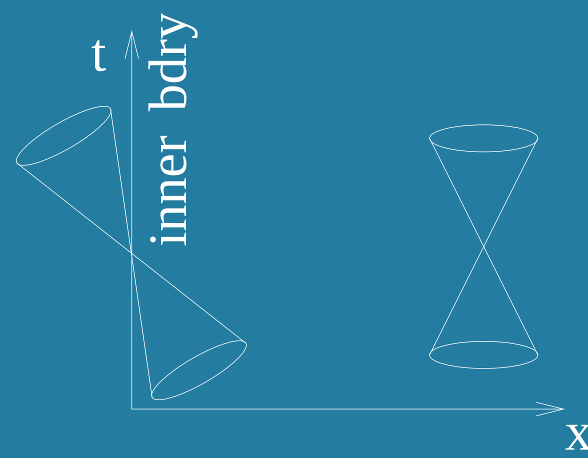

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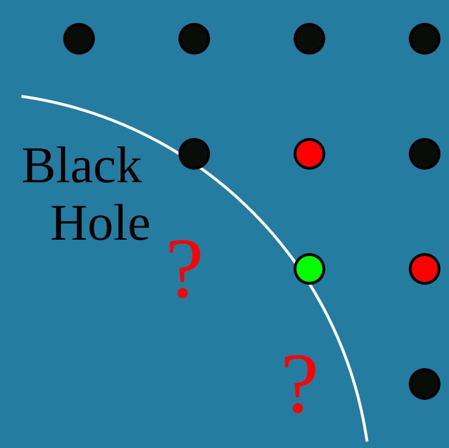

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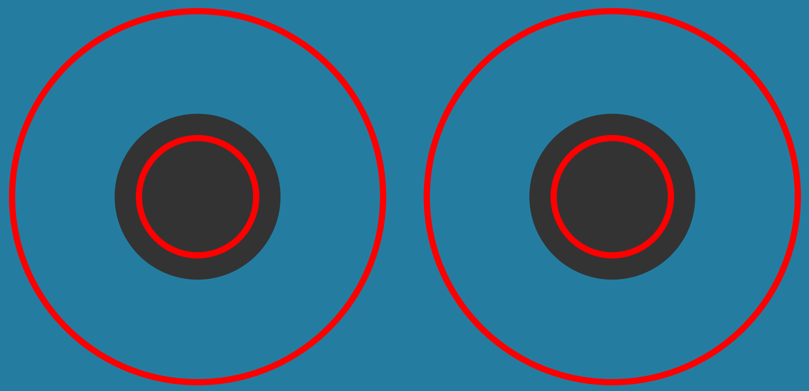

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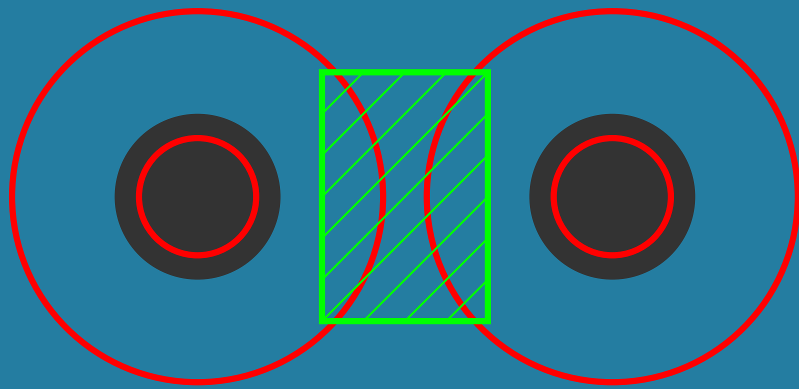

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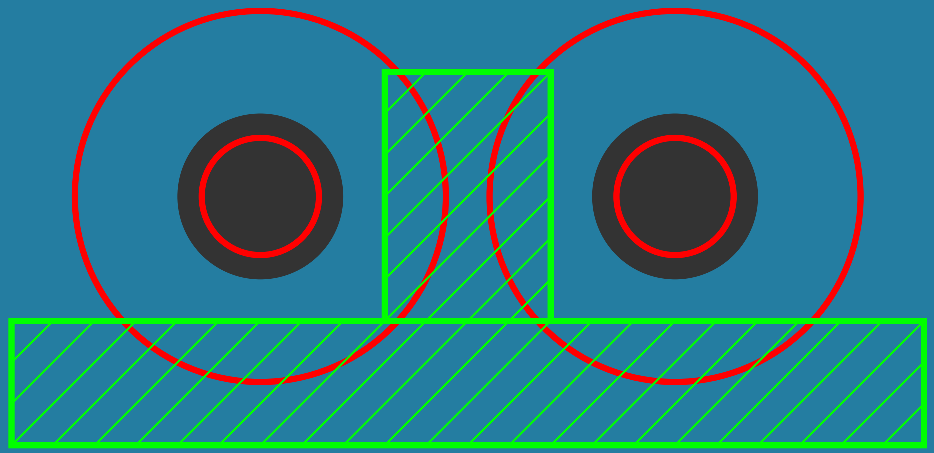

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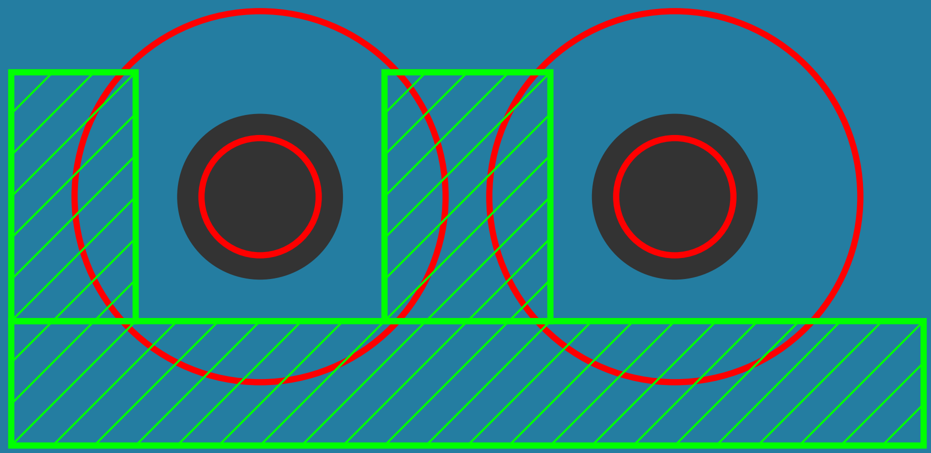

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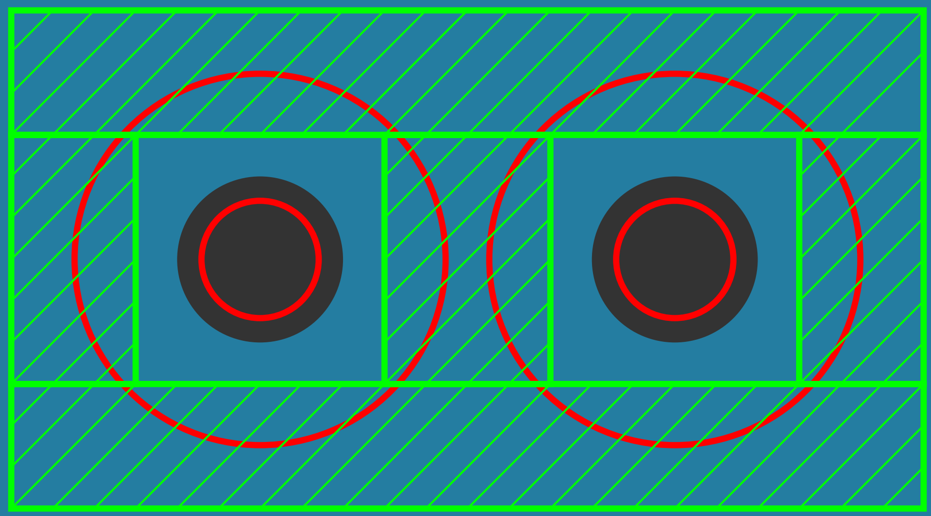

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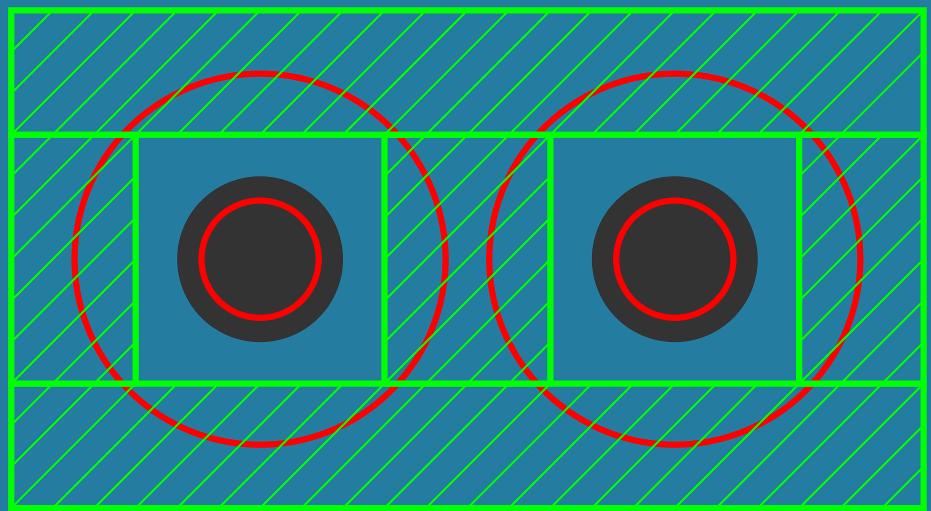

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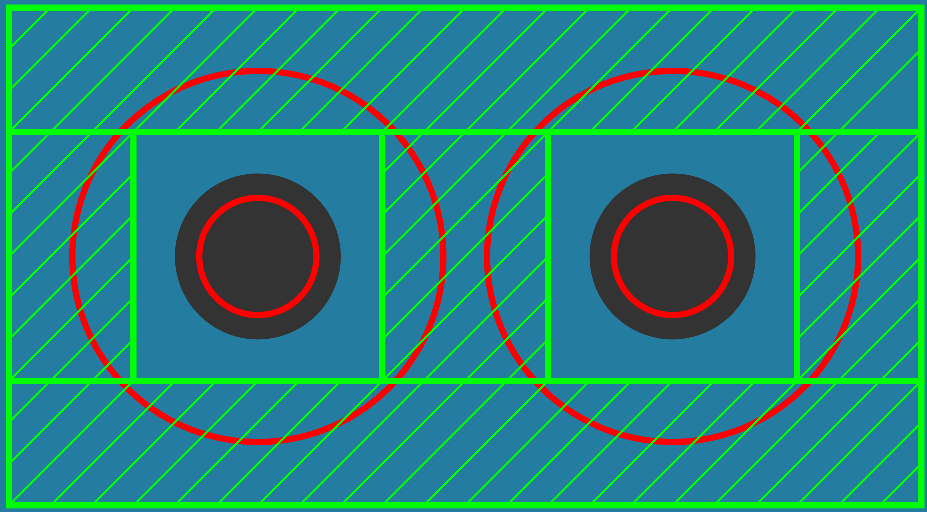

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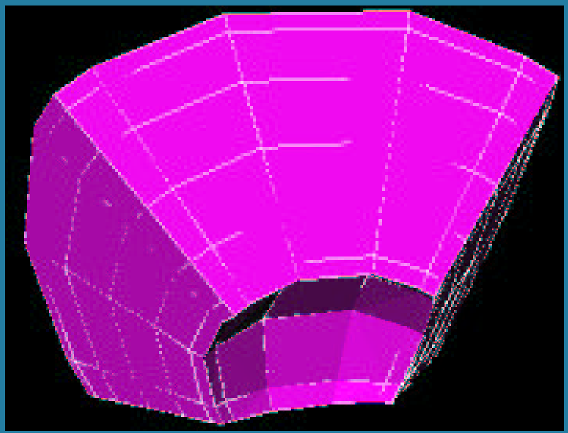
- Can use more complicated domains



The "constraint-violation problem"

(one of *many* problems that must be dealt with. . .)

- Einstein's equations ⇒ constraints + evolution.
  (EM analogue: $\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \qquad \partial_t \vec{E} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) - 4\pi\vec{J}$)
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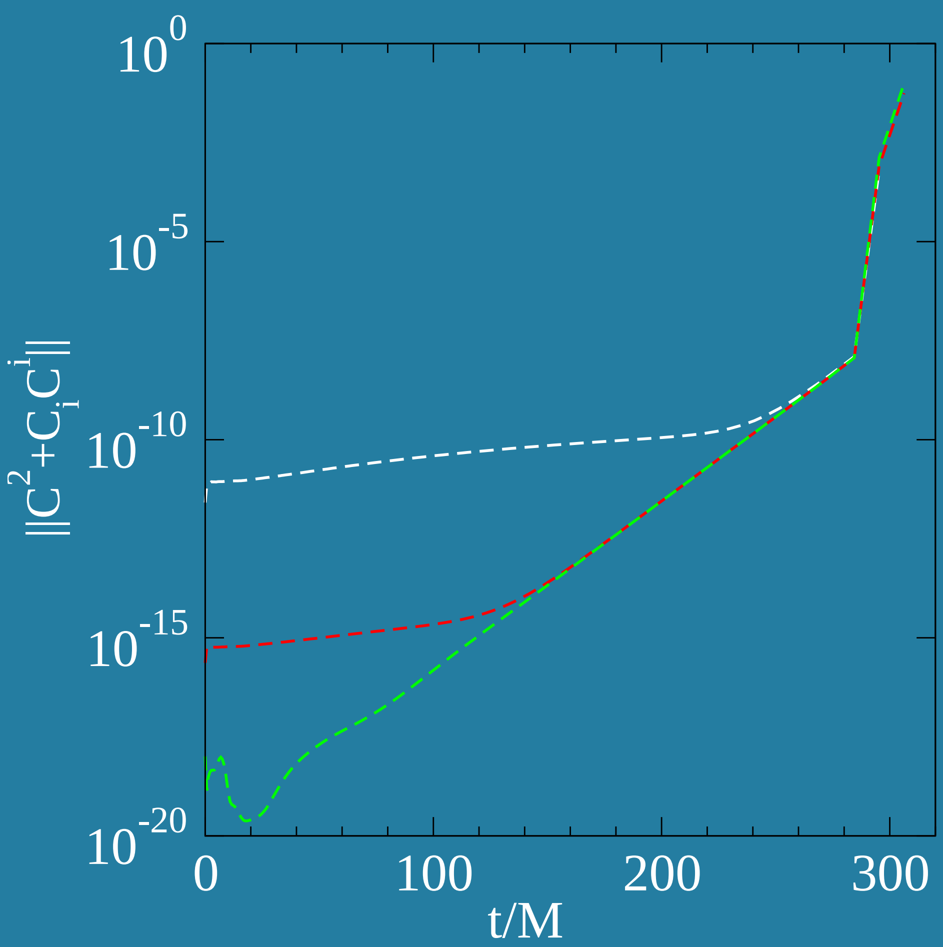

Parameterized evolution equations

Kidder,Scheel,Teukolsky Phys.Rev. D64 (2001) 064017

- Take “1st order ADM” version of Einstein’s equations

$$\begin{aligned}\partial_t g_{ij} &= \ldots\\ \partial_t K_{ij} &= \ldots\\ \partial_t d_{kij} &= \ldots\end{aligned}$$

$$\begin{aligned}\mathcal{C} &= \bar{R} + K^2 - K_{ij}K^{ij} = 0\\ \mathcal{C}_i &= D^j K_{ij} - D_i K = 0\\ \mathcal{C}_{lkij} &= \partial_{[l} d_{k]ij} = 0 \qquad (d_{kij} \equiv \partial_k g_{ij})\end{aligned}$$

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- Add constraints to the evolution equations
- Use densitized lapse: $N = g^{\sigma}Q$
- Change variables, e.g. $P_{ij} = K_{ij} + \hat{z} g_{ij} K$

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\begin{aligned}
\partial_t g_{ij} &= \ldots \\
\partial_t K_{ij} &= \ldots + \gamma g_{ij}\mathcal{C} \quad + \zeta g^{kl}\mathcal{C}_{k(ij)l} \\
\partial_t d_{kij} &= \ldots + \eta g_{k(i}\mathcal{C}_{j)} + \chi g_{ij}\mathcal{C}_k
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C} &= \bar{R} + K^2 - K_{ij}K^{ij} = 0 \\
\mathcal{C}_i &= D^j K_{ij} - D_i K = 0 \\
\mathcal{C}_{lkij} &= \partial_{[l} d_{k]ij} = 0 \qquad\qquad (d_{kij} \equiv \partial_k g_{ij})
\end{aligned}
$$

- Add constraints to the evolution equations
- Use densitized lapse: $N = g^{\sigma} Q$
- Change variables, e.g. $P_{ij} = K_{ij} + \hat{z} g_{ij} K$

$\Rightarrow$ Total of 12 parameters

(9 for symmetric hyperbolic system with physical characteristic speeds)

$\Rightarrow$ Fix 10 params for simplicity. Vary params γ and $\hat{z}$.

Results of varying parameters

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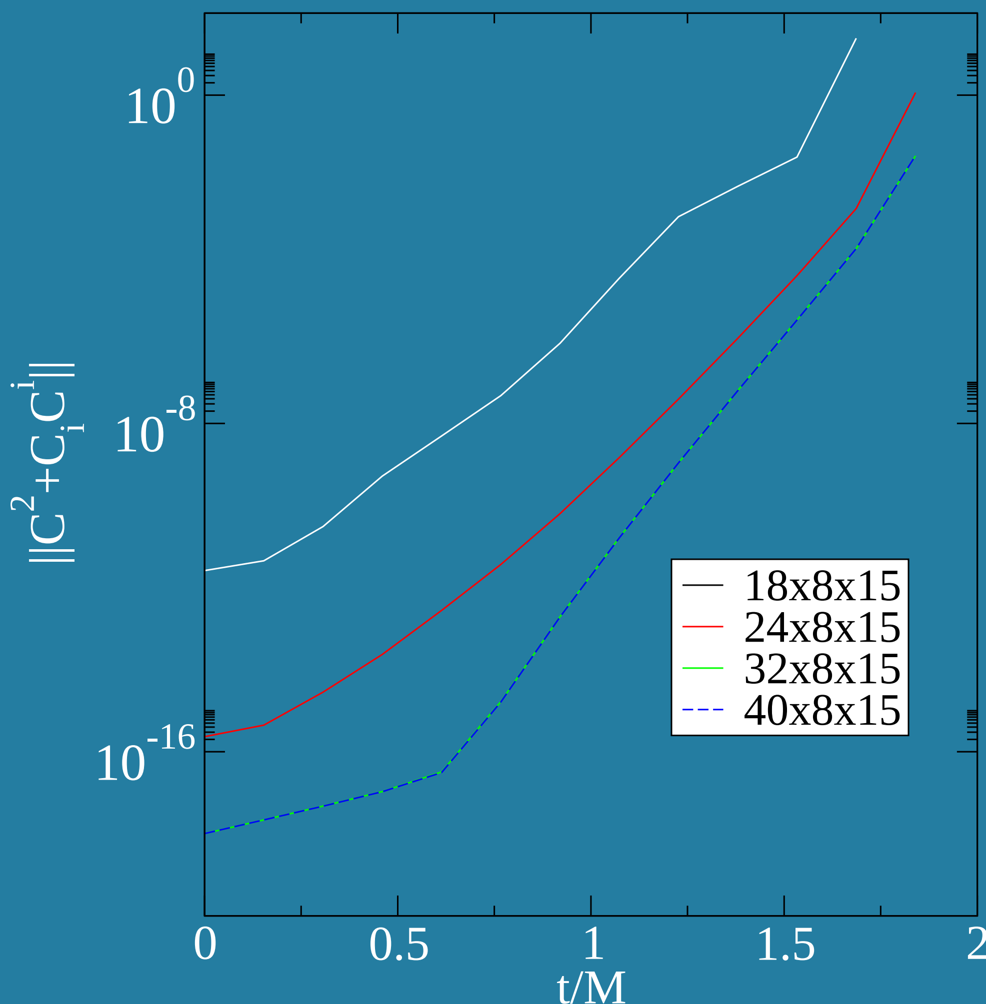
- 3D numerical evolution of Schwarzschild BH
  - ⋆ Pseudospectral method
  - ⋆ Horizon excision
  - ⋆ 3 different resolutions
  - ⋆ 3D spherical shell domain

$\gamma = 10,\ \hat{z} = -1/4$



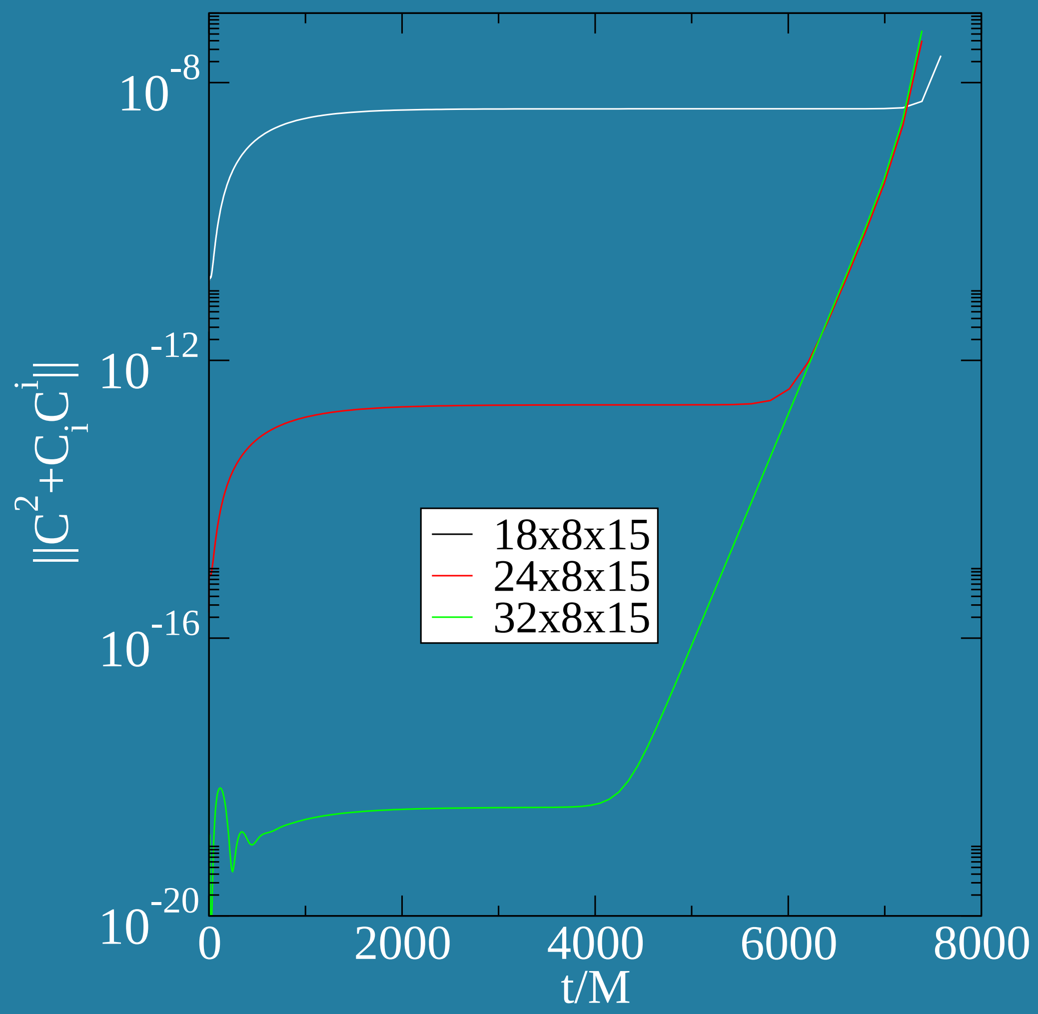
Results of varying parameters

- 3D numerical evolution of Schwarzschild BH
  - ⋆ Pseudospectral method
  - ⋆ Horizon excision
  - ⋆ 3 different resolutions
  - ⋆ 3D spherical shell domain

- Change only γ and $\hat{z}$
  Choice guided by analytical estimate of growth rate of energy norm, (Lindbom and Scheel, gr-qc/0206035)

- Do not change:
  - ⋆ Numerical method
  - ⋆ Initial data
  - ⋆ Boundary conditions
  - ⋆ Any terms with derivatives

$$\gamma = -16,\ \hat{z} = -0.42$$



Basic problem is not numerical

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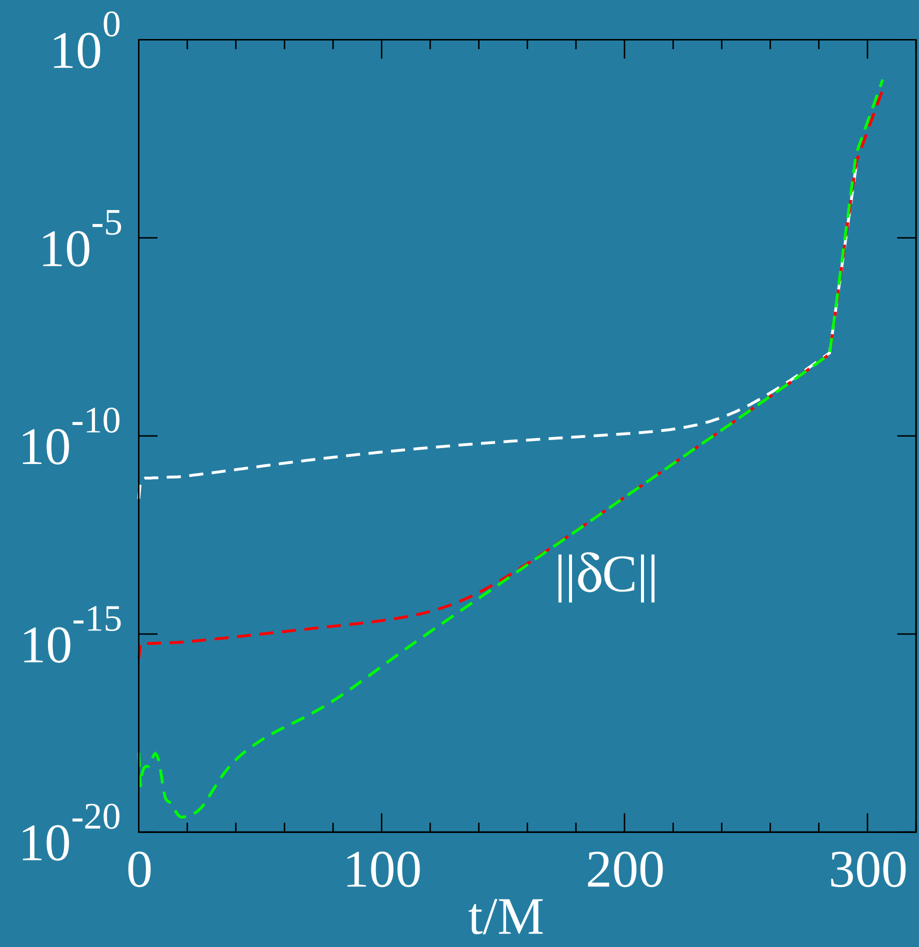
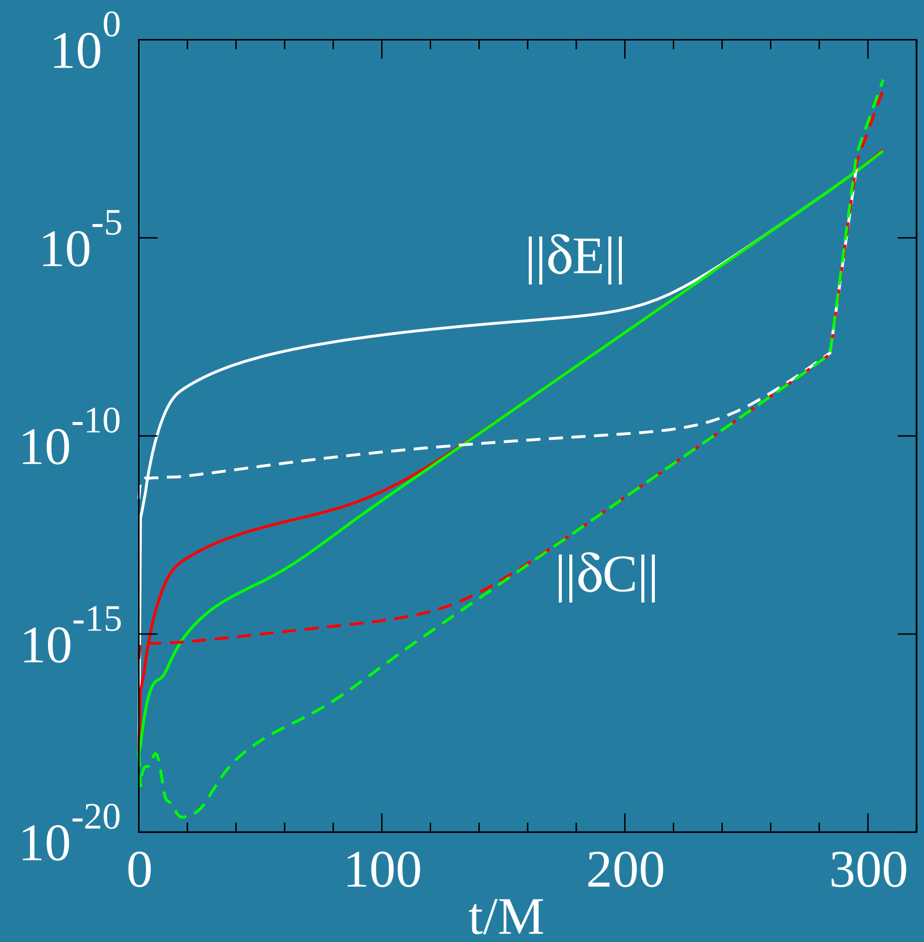
- Constraints $\sim e^{\lambda t}$, $\lambda = 0.0489$ (nonlinear)



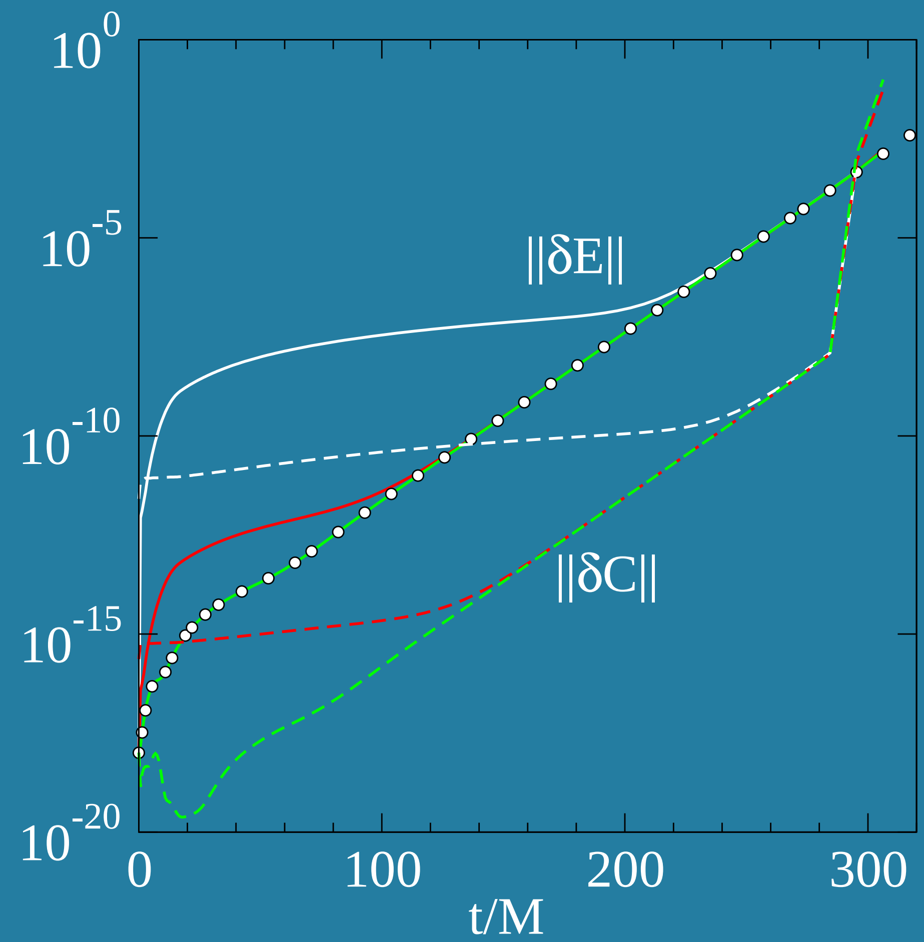
Basic problem is not numerical

- Constraints $\sim e^{\lambda t}$, $\lambda = 0.0489$ (nonlinear)
- "Energy" $\sim e^{\lambda t}$, $\lambda = 0.0490$ (nonlinear)



Basic problem is not numerical

- Constraints $\sim e^{\lambda t}$, $\lambda = 0.0489$ (nonlinear)
- "Energy" $\sim e^{\lambda t}$, $\lambda = 0.0490$ (nonlinear)
- "Energy" $\sim e^{\lambda t}$, $\lambda = 0.0489$ (linear)



Basic problem is not numerical

- Constraints $\sim e^{\lambda t}$, $\lambda = 0.0489$ (nonlinear)
- "Energy" $\sim e^{\lambda t}$, $\lambda = 0.0490$ (nonlinear)
- "Energy" $\sim e^{\lambda t}$, $\lambda = 0.0489$ (linear)
- Exact expression for any δu solving evolution equations

$$1/\tau = \frac{\int \left(C_{\alpha\beta}\, \delta u^{\alpha}\, \delta u^{\beta} - \nabla_i\, \delta E^i \right) \sqrt{g}\, d^3x}{2 \int \delta E \sqrt{g}\, d^3x}$$

$\lambda = 0.0489$