Numerical relativity as an initial–boundary value problem: Well posedness, constraint preservation and numerical stability

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Numerical relativity 101

Some problems in numerical relativity

Recent progress

Initial value problem in NR

lot of work done

Initial-boundary value problem in NR

Little but promising work done

Numerical stability in NR

We should sit down and do some work...
Outline

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  - Numerical stability in NR → We should sit down and do some work...
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- Equivalent to convergence (Lax theorem)
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\[ \dot{u} = Au' \], \quad A \text{ has constant coefficients}
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Discretize them with the iterative Crank–Nicholson with p iterations (pICN) method:

$$u_{k}^{n+1} = \left[ 1 + 2 \sum_{j=1}^{p+1} (\lambda AD_0 4^{-1})^j \right] u_{k}^{n} , \quad \lambda = \frac{\Delta t}{\Delta x} ,$$

pICN is an explicit method for which $\rightarrow$ CN when $p \rightarrow \infty$. 
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pICN is an explicit method for which \(\rightarrow\) CN when \(p \rightarrow \infty\).
The scheme is stable if at fixed time \( t = n \Delta t \), \( |u^n| \leq c(t)||u^0|| \) with \( c \) independent of resolution and initial data.
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- **Necessary condition for stability:** $\rho(Q) \leq 1$ (von-Neumann)

- **Strongly hyperbolic case:** The scheme is numerically stable only if $p = 2, 3, 6, 7, 10, 11$ and is always second order accurate, large $p$ does not help (Teukolsky '00).
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- **Completely ill posed case:** The scheme is **unconditionally unstable** for all $p$. The numerical solution at fixed time grows as $u \propto \exp(t/\Delta x)$.

- **Weakly hyperbolic case:** The scheme is **unconditionally unstable** for all $p$. Two subcases:
  - The VN condition is violated: then $u \propto \exp(t/\Delta x)$.
  - The VN condition is satisfied: then $u \propto (t/\Delta x)^a$. → Careful!
Some remarks:

Many of the problems in numerical relativity can be traced down to a weakly hyperbolic formulation, or a formulation whose level of hyperbolicity is unknown. Example: the standard ADM equations.

Completely ill posed formulations are rarely used in NR. They are easily spotted out in simulations and discarded.
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Example: evolve

\[ ds^2 = e^A \sin(\omega t + \omega x)(-dt^2 + dx^2) + dz^2 + dy^2 \]

Choose \( A = 0.01 \rightarrow \). The only nontrivial component is \( g \approx 1 + A \sin(\omega t + \omega x) \).

Choose small frequencies in the initial data: \( \omega = 1 \)

The strongly hyperbolic case
The completely ill posed case

Instabilities become obvious in a very short time scale.
The weakly hyperbolic hyperbolic case

Errors in the metric
Densitizing the lapse (WH case)

Fourier components of the metric
Densitizing the lapse (WH case, 60 gridpoints)

All frequencies grow exponentially, some of them (in this example) are not excited initially and start at the order of truncation error. You don’t "see" the instability until everything goes to hell. Doing a convergence test with a low frequency initial data and couple of coarse resolutions can be misleading. Adding more dissipation might help, but exceeding the von-Neumann condition is even worse.
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- Doing a convergence test with a low frequency initial data and couple of coarse resolutions can be misleading.

- Adding more dissipation might help, but exceeding the von-Neumann condition \( \sigma \leq 1/(8\lambda) \) is even worse.
Some problems in NR

- The formulation is not well posed:

- Initial value problem: weakly hyperbolic formulations or formulations whose level of hyperbolicity is just unknown.

- Initial-boundary value problem: wrong boundary conditions (inconsistent with the evolution equations, with the constraints, or with both).

- The numerical scheme is unstable or its stability properties are unknown.

- More problems in NR

- Coordinate singularities: the gauge becomes ill defined after a finite time.

- The equations are well posed but there are frequency independent instabilities.

- The initial-boundary value problem is well posed but one does not control the incoming radiation.
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Some recent progress: the initial value problem

- Frequency independent stability properties studied by Lindblom and Scheel '02
- Approximate method to analytically compute energy norms growth around fixed backgrounds. Use to estimate the rate of growth of perturbations and stability of many-parameter formulations.
  - "Gauge conditions" relaxed by Sarbach and Tiglio '02
  - Formulations with algebraic lapse of the form or "live gauges of the form
  - There are gauge characteristic modes whose speed depend on the solution, except in the time harmonic or densitized lapse case.
  - One cannot assume a priori physical characteristic speeds
  - Careful with excision!
  - Does not seem possible to show well posedness with only strong hyperbolicity
  - Is the Bona-Masso formulation well posed?

- "Hyperbolicity" of BSSN by Rendall and Friedrich '01, Sarbach, Calabrese, Pullin and Tiglio '02, Nagy, Ortiz and Reula '02.
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For numerical evolution in bounded domains one needs:

- A well posed initial-boundary value problem at the continuum, such that:
  - the constraints are preserved as opposed to Maxwell, the constraints in Einstein’s equations in general do not propagate with zero speed.
  - an isolated system is represented (no incoming radiation)
  - a numerically stable (in the sense of Lax) scheme.

Why is it so difficult?: Consider a linear symmetric hyperbolic system:

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Basic energy estimate for well posedness
Recent progress: the initial-boundary value problem in NR

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$$u^{(+)} = Ru^{(-)} + g(t, x^A)$$

with $$R$$ small enough such that $$R^T R \leq 1$$, one can show that $$E(t) \leq E(0) \rightarrow$$ Basic energy estimate for well posedness
Recent progress: initial-boundary value problem in NR

- Doing so violates the constraints.
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Constraint-preserving well posed boundary conditions: a review
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Constraint-preserving well posed boundary conditions: a review

- **Stewart ’98 → 3D linear**
  - Used Laplace method.
  - Frittelli-Reula symmetric hyperbolic formulation
  - **No numerical** implementation up to now.
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- **Iriondo and Reula '01** → 1D nonlinear
  - Non linear spherical black holes + matter.
  - Formulation and gauge adapted to spherical symmetry.
  - Numerical “stability” of the semidiscrete problem through Strand’s ’94 symmetric operators + complete stability through 4th order Runge-Kutta (Kreiss and Wu ’93)
  - Nice and clean physics obtained in a very simple way.
Recent progress: initial-boundary value problem in NR

- Calabrese, Lehner and Tiglio ’01 → 1D nonlinear
- Non linear spherical black holes + matter
- Relaxed some of Iriondo–Reula’s conditions:
  - Einstein–Christoffel symmetric hyperbolic formulation.
  - Moved to lower (second) order scheme.
- Similar results → Remark for physicists: Good tail decay obtained with an observer in the last grid point
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Pulse falling into a black hole, Mass change of the black hole

Numerical relativity as an initial–boundary value problem: – p.13/15
Recent progress: initial-boundary value problem in NR

- **Szilagyi, Bishop, Gomez and Winicour ’02** → 3D linear
  - Uses harmonic gauge conditions \( \nabla^a \nabla_a x^\mu = 0 \)
  - Numerical implementation showing “robust stability” (no exponential growth in time)
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Szilgyi and Winicour '02 → 3D nonlinear
- Harmonic gauge.
- Robust stability
- At the linear level matched with Characteristic evolution (be patient, LL will talk about Cauchy-Characteristic matching in three days)
Recent progress: initial-boundary value problem in N

- Calabrese, Pullin, Reula, Sarbach and Tiglio ’02 → 3D linear
  - Generalized symmetric hyperbolic Einstein-Christoffel formulation
  - Cubic box, details of corners and edges taken into account
  - Stability of the semidiscrete problem through Strand's ’94 symmetric operators + Olsson’s ’95 generalization to 3D, complete stability through 3rd-4th order Runge-Kutta (Kreiss and Wu ’93)? Levy and Tadmor ’98, Tadmor ’01?
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- Well posed, constraint-preserving boundary conditions with no incoming radiation (control on the Weyl)
  - Friedrich and Nagy ’98 → seminal work, but not with NR in mind
  - Nagy and Reula, work in progress. Need to consider higher order systems