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Inverse Problems and Optimal Design Problems
A Level Set Approach for
Acknowledgments
minimize data fit. We can pose a least-squares problem to differentiate equation. The relation between unknown and data is given by a partial differential equation.

![Diagram of incident wave and scatterer]

**Example:** Inverse scattering, nondestructive testing.

Geometry from observed data.

Many inverse problem involve the determination of an unknown context.
pose an optimization problem associated with the objective.

The physical phenomenon is modeled by a partial differential equation; the objective is found from the solution of the PDE.

Examples: Structures, diffraction optics.

Many optimal design problems involve finding a geometry that

maximizes an objective.
\[ \{ 0 > (\hat{f}, x) \phi : (\hat{f}, x) \} = S \]

In level set representation, we consider a function \( \phi \) so that

\( (\hat{f}, x) \phi \) and assume the points are connected by some spline.

\[ \forall \theta \in (\hat{f}, x) \]

If \( \theta \) is more general, we can put marker points on the boundary

\( \partial \theta \)

If \( S \) is a region in 2D, \( \hat{f} \) is star-shaped, we can represent it in polar coordinates.

What is level set?
\[ 0 = (A \cdot x) \phi \]

\[ (A \cdot x) \]

\[ (\theta) R = 1 \]

Set.

Parameterizations of a curve in polar, marker points, and level.
Level set method easily accommodates topology of the unknown.

Why level set?
\[
\min_{\phi} \| \beta - (\phi)\mathcal{A} \|_2
\]

We choose a least-squares formulation

\[
\beta = (\phi)\mathcal{A}
\]

Inverse problems is to find in the equation unknown to data. We are given measured data \( \beta \), therefore the inverse problem, we denote by \( (\phi)\mathcal{A} \) the mapping from the inverse problem \( \{0 > (x)\phi : x \} = \mathcal{S} \)

We parameterize the unknown domain by the level set function \( (x)\phi \)

Formulation of Inverse Problems
min \quad (\phi)_{\mathcal{D}}^{\phi} \quad \text{subject to} \quad (\phi)_{\mathcal{A}}^{\phi} = 0.

We wish to solve

\begin{align*}
0 &= (\phi)_{\mathcal{D}} \quad \text{and} \quad (\phi)_{\mathcal{A}}
\end{align*}

subject to many constraints.

Associated with this geometry, we have an objective, and a

\begin{align*}
\{0 > (x)\phi : x\} &= S
\end{align*}

the level set formulation as

In a design problem, the unknown is a geometry \( S \), represented in

Formulation of optimal design problems
\[ \{ 0 < (x) \phi : x \} \quad \text{for} \quad \exists d \quad \{ 0 > (x) \phi : x \} \quad \text{for} \quad \forall d \right\} = (x)d \]

In level set formulation

\[ \left. \begin{array}{c}
S \ni x \quad \text{for} \quad \exists d \\
S \notin x \quad \text{for} \quad \forall d \\
\end{array} \right\} = (x)d \]

Consider a function whose value is defined through the domain \( S \) with respect to geometry parameters.

With problems involving geometry because we need to differentiate with gradients involving functions involved. This is especially tricky.

In any gradient-based descent algorithm for optimization, we will

Gradient calculation
requires a little work. respect to a function for a fixed geometry. The second term The first term is "easy", we know how to take variation with
\[ \phi D \quad \mathcal{H}^{\phi} D = \mathcal{H}^{\phi} \]

We use chain rule

\[ \cdot (\phi d)^{\mathcal{H}} \]

Let the function be
\[ \phi \Delta \Delta = (x)u \]

\[ xp (x)f (x)d\phi = \int \Psi (x)d\phi = (x)f \text{ against a test function} \]

The variation integrated against \( d\phi \).
\[ \exists x \forall \varphi \left| \frac{\phi(\varphi \Delta)}{\varphi \Delta} (1d - zd) \right| = d \varphi \]

Identity

\[ \exists x \forall \varphi \left( x \varphi \cdot \int (x)u \cdot x \varphi (1d - zd) \right) = \langle f, d \varphi \rangle \]

Therefore

\[ 0 < (x)u \cdot x \varphi (1d - zd) \]
\[ 0 > (x)u \cdot x \varphi (1d - zd) \]

Also
When you make a variation on \( x \), the variation in the function \( x \) is

\[
\delta \frac{|\phi \Delta|}{\phi \delta} (1d - \varepsilon d) = \phi \delta \cdot d \phi \Delta = d \phi
\]

The previous formula now becomes

\[
x \delta \cdot \phi \Delta + \phi \delta = 0.
\]

From this, we can take a variation
\[
(x)sp \left( \frac{\phi \Delta}{\phi \theta} \right) X \int_{s \theta \in x} = \left| \frac{\phi \Delta}{\phi \theta} \right| \cdot H \cdot D
\]

Because \( H \cdot D \) is a linear operator, it must have a representation
\[
\left| \frac{\phi \Delta}{\phi \theta} \right| \cdot H \cdot D \cdot (I_d - z_d) = \phi \theta \cdot H^\phi D
\]

Putting it together, we have
\[
\phi \theta \cdot H \cdot D = \phi \theta \cdot H^\phi D
\]

So, for \( ((\phi)d)H \), we can now calculate the directional derivative.
\[ |\phi\Delta| (x) T a + |\phi\Delta| (x) Y (1d - zd) - = \phi q \]

determined

We modify the gradient with Lagrange multiplier to be

\[ 0 = (\phi) C \]

In the case where there is a constraint

\[ x \quad \text{for all} \quad |(x)\phi\Delta| (x) Y (1d - zd) - = (x)\phi q \]

for all of \( x \), we can extend the definition to all of \( x \) so that for \( x \) is defined as above, since \( x \) is defined by any \( \phi q \)

This states that the steepest descent direction is specified by

\[ S \subseteq S \quad \text{for} \quad |(x)\phi\Delta| (x) Y (1d - zd) - = (x)\phi q \]

Hence, the steepest descent direction associated with \( \phi \) is
\[ \zeta \| \delta - (\phi)\mathcal{V} \| = (\phi)\mathcal{J} \]

Since we can view

We can perform similar calculation in the case of inverse problems

\[ 0 \approx (\phi\mathcal{g} + \phi)\mathcal{G} \]

The multiplier can be chosen so that

\[ (x)sp \frac{|\phi\Delta|}{\phi\mathcal{G}} (x)T \int_{\mathcal{G}} = \phi\mathcal{g} \cdot (\phi)\mathcal{G}\phi\mathcal{J} \]

where
more general procedure with well-developed theory

• Differentiation of PDEs with respect to geometry (Zolesio);

• Variational level set method (Chain, Merriman, Osher and Zhao);

Other approaches:
where $v$ is the velocity normal to the curve at $x$.

\[ 0 = |\phi \Delta (t, x) v + \frac{\mathcal{L}}{\phi C} \]

We view level set function to be evolving, so curve evolution can be described by evolutes, the zero level set moves. Its movement evolves \( \{0 = (t, x) \phi : x\} \) as \( (t, x)\phi \) moves. The zero level set evolves, so we view level set function to be evolving, so curve evolution
the form of $\Lambda$ does not restrict motion to the normal to the curve.

\[(\dot{t},x)\phi \Delta \cdot (\dot{t},x)\Lambda - = (\dot{t},x)\phi \theta\]

Note recent work by Burger where evolution is given by Sethian.

\[|(x)\phi \Delta| \text{ is computed in computing and}\]

\[(x)Y (\mathbb{1}d - zd) - = (\dot{t},x)\nu\]

We view this as a curve evolution rule with velocity

\[\forall x \quad |(x)\phi \Delta| (x)Y (\mathbb{1}d - zd) - = (x)\phi \theta\]

Recall that we have evolution of all level curves.

Respects the constraints, extending $v(\mathbb{R})$ to the domain allows us to find $v(x, \dot{t})$ that optimizes the objective and also...
\[(x)\phi \text{ update} \bullet \]
\[(x)v \text{ get velocity} \bullet \]
Newton’s method
approximation (when needed, solve for via
solve for Lagrange multiplier \(v \text{ linear} \bullet \)
\((\phi)D^\phi D \text{ and } (\phi)H^\phi H \bullet \)
do while not optimal
\[(x)\phi \text{ initial guess for} \]
subject to \((\phi)D \text{ optimize}\)
\((\phi)H \text{ optimize}\)
Algorithm
on $\varnothing \quad f = u \cdot n \Delta$

Assume that we measure when current $\varnothing \mid n$ is applied.

In $\varnothing \quad 0 = n \Delta(x) \partial \cdot \Delta$

Let $n(x)$ be the electrical potential.

with Bruno Lunea (circa 1997)

Electrical Impedance Tomography
The inverse problem is to find $\mathcal{S}$ from the measurements.

Corresponding to the boundary, denoted by $(\Omega, \gamma_2)$, we do this experiment for two $f$'s, $f_1$ and $f_2$, and measure the

\[
\begin{align*}
\mathcal{S} & \subseteq x \quad \text{for } \forall d \\
\mathcal{S} & \not\subseteq x \quad \text{for } \forall d
\end{align*}
\]

which also know that
See also result by Ito, Kunisch, and Li.

Convergence is very slow.

- Gradient calculation by adjoint state method
- Forward problem solved by Immersed Interface Method

Computational details:

\[ \min \| [b_1, b_2] - \phi(\mathbf{A}) \|_2 \]

Here the mapping \( \mathbf{A} \) takes \( f \) to \( (f_1, f_2) \). We solve the least squares problem.
Data fit: Initial vs. Final
receiver points.

We solve a point source problem and record $n$ at all the 10

\[ 0 = n(x) \partial_x \psi + n \nabla \]

Let $n^2$ represent the amplitude in Helmholtz's equation

with Ambiez Litman and Dominique Lesselier

Reconstruction of sound speed anomaly
Also recent results by Dorn, Miller and Rapaport.

\[ h_p \left( \frac{h}{n} \right) \left( \frac{h}{x} \right) \mathcal{D} \int \left( \frac{1}{\mathcal{E}} \phi - \frac{2}{\mathcal{E}} \phi \right) \psi - (x)_{\text{inc}} n = (x)n \]

We solve the forward problem using the Liipmann-Schwinger equation.

The inverse problem is to find \( \mathcal{S} \) from the measured data.

\[
\begin{cases}
\mathcal{S} \ni x & \text{for } \mathcal{E} d \\
\mathcal{S} \not\ni x & \text{for } \mathcal{H} d
\end{cases}
= (x)d
\]

Again, we assume
Suppose that \( \forall i > i_1 > ... > i_4 \# \) and all \( S \). \( S \ni x \) for all \( \forall d \)
\[
\{ S \ni x \text{ for all } i_d \} = (x)d
\]

We assume that \( (x)d \) is of the form
\[
\forall \emptyset \text{ on } 0 = n
\]
\[
\forall \emptyset \text{ in } n(x)d\chi = n \nabla
\]

Let \( U \) be a domain in \( \mathbb{R}^2 \). Let \( u \) satisfy \( (x)n \) with Stanley Osber

An eigenvale optimization problem
\[ T = \chi^1 - \chi^2 \text{ subject to } \min \|S\| \text{ or minimization.} \]

\[ Y = \|S\| \text{ subject to } \max \chi^1 \text{ or } \max \chi^1 - \chi^2 \]

Problem 2.

Problem 1.
using level set, they did not use gradients to obtain velocity.

Sethian and Weigmann considered classical structural optimization
Arc Method with reduced gradient.
Curve evolution equation can be interpreted as the classical
over the curve must be zero.
Interpretation. It states that to preserve area, the total velocity
Linearized Lagrange multiplier update in Problem 1 has a nice
Example 2: Minimize $x_1$
Example 3: Maximize $\lambda_2 - \lambda_1$
Example 4: Fix $\lambda_2 - \lambda_1$ and minimize area
take a Newton step towards constraint
\[ \log(c - \alpha^w) + \log \alpha^{w+1} \]
\( \alpha \)
\( \alpha^w \)
\( \alpha \) and \( c + \alpha^w \) and \( c \)

Solve eigenvalue problem to determine \( m \)
do what not optimal
initital guess for \( (x) \phi \) for

Ad hoc algorithm:
is maximized.

Pick a number \( c \), and \( S \) such that the gap around \( c \), namely

Example 7: Maximize spectral gap around a value
Optimal design in heat conduction

Joint work with T.-T. Chene (UCSD) and S. Other (UCI)
That is, find \( S \) such that the flux over \( I \) is maximized.

\[
\text{maximize } \int_I n \cdot u \, ds = \| S \|
\]

Optimal design problem is

\[ X = \| S \| \]

Moreover, we constrain the size of \( S \)

\[ S \ni x \quad \text{for} \quad \exists d \quad \left\{ \begin{array}{l}
S \ni x \quad \text{for} \quad 1 \leq d \\
S \notin x \quad \text{for} \quad 1 < d
\end{array} \right\} = (x)_d
\]

Again, we give bottom, flux-free on the sides.

Let \( (x)_n \) be temperature.

We specify mixed boundary conditions - Dirichlet on the top and

\[
0 = n \Delta (x)_d \cdot \Delta
\]
Gradient calculation by adjoint state method

Forward problem solved by 'Ghost Fluid' Method

Computational details:

Our approach looks for a non-homogenized solution by penalizing the perimeter. The relaxed solution to the problem is a composite. Our approach optimizes the problem:

We can also modify the problem to consider structural rigidity. We can also modify the problem to consider optimal design of bars with the maximum torsional rigidity. A small change in the partial differential equation allows us
implemented.

size of domain, distance between sets, are being

Other geometrical constraints, such as lower bound on

Lots of application in design and inverse problems.

Lots of theoretical questions need attention.

Many improvements and refinement needed.

Approach.

Geometry

Problem involving Geometry

Approach for solving inverse and design

Discussion