

Discontinuous Markov Processes and Pseudo Differential Operators

(Boundary Potential Theory)

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Symmetric α -Stable Process

is a Lévy process with density function $p(t, x, y) = p(t, x - y)$:

$$\int_{\mathbf{R}^n} e^{ix \cdot \xi} p(t, x) dx = e^{-t|\xi|^\alpha}.$$

Here $\alpha \in (0, 2]$.

$\alpha = 2$: Brownian Motion with generator Δ

$\alpha < 2$: Discontinuous process with generator

$$\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$$

Dirichlet form: $(\mathcal{E}, W^{\alpha/2, 2}(\mathbf{R}^n))$

$$\begin{aligned} \mathcal{E}(u, v) &= (-\Delta^{\alpha/2} u, v)_{L^2(\mathbf{R}^n)} \\ &= c \int_{\mathbf{R}^n \times \mathbf{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\alpha}} dx dy. \end{aligned}$$

Killed Symmetric α -Stable Process

Let X be symmetric α -stable in \mathbf{R}^n . For an open set $D \subset \mathbf{R}^n$, define

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$

Let

$$X_t^D(\omega) = \begin{cases} X_t(\omega), & \text{if } t < \tau_D(\omega), \\ \partial, & \text{if } t \geq \tau_D(\omega), \end{cases}$$

where ∂ is a coffin state added to D .

Process X^D is called the symmetric α -stable process in D . Its infinitesimal generator L is the “Dirichlet” $\Delta^{\alpha/2}$ in D .

Dirichlet form $(\mathcal{E}, W_0^{\alpha/2, 2}(D))$:

$$\mathcal{E}(u, u) = (-Lu, u)_{L^2(D)} = c \int_{D \times D} \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} dx dy + \int_D u(x)^2 \kappa_D(x) dx,$$

where

$$\kappa_D(x) = \mathcal{A}(n, -\alpha) \int_{D^c} \frac{1}{|x - y|^{n+\alpha}} dy.$$

If D is Lipschitz, then $\kappa_D(x) \approx \delta_D(x)^{-\alpha}$. Here $\delta_D(x) = \text{dist}(x, D^c)$.

Potential Theory for X^D , 1997–

Two-sided Green function estimates, Poisson kernel estimates, Martin boundary and Martin kernel estimates, boundary Harnack inequality, conditional guage theorem, intrinsic ultracontractivity, ...

Chen, Song, Wu, ...

Bogdan, Kulczycki, Byczkowski, ...

When D is Lipschitz, $X_{\tau_D-} \in D$ and so X^D has no non-zero bounded harmonic function.

Censored (or Resurrected) α -Stable Process

A censored α -stable processes in $D \subset \mathbf{R}^n$ is obtained from X^D through resurrection:

Start with a symmetric α -stable process X in \mathbf{R}^n from a point in a domain D ; at the place where X jumps out of D , start a new independent symmetric α -stable process; when the latter jumps out of D , at the place it jumps out we start yet another independent symmetric α -stable process; repeat this procedure countably many times. This gives us a new process Y , which we call the censored α -stable process.

The infinitesimal generator of Y is

$$\mathcal{L} = \Delta^{\alpha/2} + \kappa_D(x)$$

with Dirichlet boundary condition and its Dirichlet form is:

$$\mathcal{E}(u, u) = c \int_{D \times D} \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} dx dy.$$

Theorem 1 (Bogdan-Burdzy-Chen 01/03)

Suppose $D \subset \mathbf{R}^n$ is bounded Lipschitz. Then Y is transient if and only if $\alpha > 1$. In this case, for $x \in D$,

$$\mathbf{P}_x(\zeta < \infty \text{ and } \lim_{t \uparrow \zeta} Y_t \in \partial D) = 1.$$

Green function $G(x, y)$:

$$\int_D G(x, y) \varphi(y) dy = \mathbf{E}^x \left[\int_0^\zeta \varphi(Y_s) ds \right].$$

Theorem 2 (Chen-Kim 02)

Let $D \subset \mathbf{R}^n$ is bounded $C^{1,1}$ and $\alpha > 1$. Then

$$G(x, y) \approx \min \left\{ \frac{1}{|x - y|^{n-\alpha}}, \frac{\delta_D(x)^{\alpha-1} \delta_D(y)^{\alpha-1}}{|x - y|^{n-2+\alpha}} \right\}.$$

Here $f \approx g$ means $c_1 f \leq g \leq c_2 g$.

The proof uses Boundary Harnack inequality and Hardy's inequality.

Harmonicity

Definition 3 Let O be an open subset of D .
 $f : D \rightarrow \mathbb{R}$ is

(1) **harmonic** in O with respect to Y if

$$\mathbf{E}_x[|f(Y_{\tau_B})|; \tau_B < \zeta] < \infty \quad \text{and}$$

$$f(x) = \mathbf{E}_x[f(Y_{\tau_B}); \tau_B < \zeta], \quad x \in B,$$

for every open set B with $\bar{B} \subset D$;

(2) **superharmonic** in O with respect to Y if
 f is lower semicontinuous in O and

$$\mathbf{E}_x[f^-(Y_{\tau_B}); \tau_B < \zeta] < \infty \quad \text{and}$$

$$f(x) \geq \mathbf{E}_x[f(Y_{\tau_B}); \tau_B < \zeta], \quad x \in B,$$

for every open set B with $\bar{B} \subset D$.

Facts:

- f is harmonic in O iff $(\Delta^{\alpha/2} + \kappa_D)f = 0$ in O .
- f is superharmonic in U iff $(\Delta^{\alpha/2} + \kappa_D)f \leq 0$ in O .

Theorem 4 (*Boundary Harnack principle*)
(*Bogdan-Burdzy-Chen 01/03*)

Let $D \subset \mathbb{R}^n$ be bounded $C^{1,1}$ with characteristics $r_0 < 1$ and Λ , and $1 < \alpha < 2$. Let $Q \in \partial D$ and $r \in (0, r_0)$. Assume that $u \geq 0$ vanishes continuously on $\partial D \cap B(Q, r)$ and is Y -harmonic in $D \cap B(Q, r)$. Then there is a constant $K = K(n, \alpha, \Lambda) > 1$ such that

$$\frac{u(x)}{u(y)} \leq K \frac{\delta_D(x)^{\alpha-1}}{\delta_D(y)^{\alpha-1}} \quad \text{for } x, y \in D \cap B(Q, r/2).$$

Questions: What are the structure of the spaces of harmonic functions and superharmonic functions of Y ?

Fix $x_0 \in D$ and set

$$M(x, y) := \frac{G(x, y)}{G(x_0, y)}.$$

Theorem 5 (*Chen-Kim 02*)

Let $D \subset \mathbb{R}^n$ be bounded $C^{1,1}$ and $\alpha > 1$. Then

(i) For each $z \in \partial D$, $M(x, z) := \lim_{y \rightarrow z \in \partial D} M(x, y)$ exists for every $z \in \partial D$ and $M(x, z)$ is jointly continuous in $D \times \partial D$;

(ii) $x \mapsto M(x, z)$ is a minimal harmonic function of Y and

$$M(x, z) \approx \frac{\delta_D(x)^{\alpha-1}}{|x - z|^{n-2+\alpha}}.$$

Thus the Martin boundary and the minimal Martin boundary of Y can all be identified with ∂D .

From now on, we assume $D \subset \mathbf{R}^n$ is bounded $C^{1,1}$ and $1 < \alpha < 2$.

Martin Representation

For every Y -superharmonic function $u \geq 0$ in D that is not identically infinite, there is a unique Radon measure μ_1 in D and a unique finite measure μ_2 on ∂D such that

$$u(x) = \int_D G(x, y) \mu_1(dy) + \int_{\partial D} M(x, z) \mu_2(dz). \quad (1)$$

Furthermore, u is Y -harmonic iff $\mu_1 = 0$.

Conversely if μ_1 is a Radon measure in D such that $G\mu_1 \not\equiv \infty$ and μ_2 is finite measure on ∂D , then the function u given by (1) is Y -superharmonic in D .

Fatou Theorem (Kim 03): Any Y -harmonic function $u \geq 0$ has nontangential limit at z for σ -a.e. $z \in \partial D$.

Non-local Feynman-Kac Transform

Let q and F be functions on D and $D \times D$ respectively, where $F(x, x) = 0$. Define

$$e_{q+F}(t) = \exp \left(\int_0^t q(X_s) ds + \sum_{0 < s \leq t} F(Y_{s-}, Y_s) \right).$$

It defines a Schrödinger semigroup

$$T_t f(x) = \mathbf{E}_x \left[e_{q+F}(t) f(X_t) \right].$$

The above transform is called non-local since $t \mapsto e_{q+F}(t)$ is discontinuous.

Formally, the generator of $\{T_t, t \geq 0\}$ is

$$\tilde{\mathcal{L}}f(x) = \mathcal{L}f(x) + c \int_D \frac{e^{F(x,y)} - 1}{|x - y|^{n+\alpha}} f(y) dy + q(x) f(x).$$

The above q can be replaced by a Revuz measure μ but conditions need to be imposed on q (or μ) and F . New Kato class $\mathbf{S}_\infty(Y)$ and $\mathbf{A}_\infty(Y)$ are introduced in Chen-Song (02) and in Chen (02).

Sufficient conditions:

- $q \in \mathbf{S}_\infty(Y)$ if $q = q_1 + q_2$ with $q_1 \in L^{p/\alpha}(D)$ for some $p > n$ and

$$|q_2(x)| \leq c \delta_D(x)^{-\beta}$$

for some $\beta < 2(\alpha - 1) + \frac{2-\alpha}{n}$.

- $F \in \mathbf{A}_\infty(Y)$ if $|F(x, y)| \leq c|x - y|^\gamma$ for some $\gamma > \alpha$.

For $y \in D$, let Y^y denote the conditional process obtained from Y through Doob's h -transform with $h(\cdot) = G(\cdot, y)$ and let \mathbf{E}_x^y denote the expectation for Y^y starting from $x \in D$. That is,

$$\mathbf{E}_x^y [f(Y_t^y)] = \frac{1}{G(x, y)} \mathbf{E}_x [G(Y_t, y) f(Y_t)].$$

Theorem 6 Assume that $q \in \mathbf{S}_\infty(Y)$ and $F \in \mathbf{A}_\infty(Y)$. Then

(1) (Chen-Song 02) (GT)

The gauge function $g(x) = \mathbf{E}_x [e_{q+F}(\zeta)]$ is either bounded on D or identically ∞ on D .

(2) (Chen-Song 02) (CGT)

The conditional gauge function

$$u(x, y) = \mathbf{E}_x^y [e_{q+F}(\zeta^y)]$$

is either bounded or identically ∞ on $(D \times D) \setminus d$.

(3) (Chen 02) (Equivalence Theorem) TFAE

(3a) The gauge function g is bounded on D .

(3b) The conditional gauge function $u(x, y)$ is bounded on $(D \times D) \setminus d$;

Let $V(x, y)$ be the Green function for Schrödinger semigroup $\{T_t, t \geq 0\}$:

$$\int_D V(x, y)\varphi(y)dy = \mathbf{E}_x \left[\int_0^\zeta e_{q+F}(t)\varphi(Y_t)dt \right].$$

Theorem 7 (*Chen 02*)

$$\frac{V(x, y)}{G(x, y)} = u(x, y).$$

Assume that (q, F) is **gaugeable**, that is, g is bounded. Then u is bounded between two positive constants and so V is comparable to G .

Question: How do we know if (q, F) is gaugeable?

Theorem 8 (Chen 03)

Suppose that $q \in \mathbf{S}_\infty(Y)$ and $F \in \mathbf{A}_\infty(Y)$ is symmetric. Let $F_1(x, y) := e^{F(x, y)} - 1$ and

$$m_{F_1^+}(x) := c \int_D \frac{F_1^+(x, y)}{|x - y|^{n+\alpha}} dy.$$

Then (q, F) is gaugeable if and only if

$$\begin{aligned} & \inf \left\{ \mathcal{E}(u, u) - c \int_{D \times D} \frac{u(x)u(y)F_1(x, y)}{|x - y|^{n+\alpha}} dy dx \right. \\ & \quad \left. + \int_E u(x)^2 (q^-(x) + m_{F_1^+})(x) dx : \right. \\ & \quad \left. \text{for } u \in C_c^\infty(D) \text{ with} \right. \\ & \quad \left. \int_E u(x)^2 (q^+(x) + m_{F_1^+}(x)) dx = 1 \right\} \\ & > 1. \end{aligned}$$

One can similarly define functions that are (q, F) -harmonic and (q, F) -superharmonic.

What is the Martin boundary for $\{T_t, t \geq 0\}$?

Theorem 9 (Chen-Kim 03)

Suppose (q, F) is gaugeable.

(1) The conditional gauge function $u(x, y)$ is continuous on $D \times D \setminus d$, so is $V(x, y)$.

(2) For $w \in \partial D$ and $x \in D$, define $u(x, w) = \mathbf{E}_x^w [e_{q+F}(\zeta^w)]$. Then $\lim_{y \rightarrow w, y \in D} u(x, y) = u(x, w)$ and $u(x, w)$ is jointly continuous on $D \times \partial D$.

(3) For every $x \in D$ and $w \in \partial D$,

$$K(x, w) := \lim_{y \rightarrow w, y \in D} \frac{V(x, y)}{V(x_0, y)}$$

exists and is finite. Furthermore,

$$K(x, w) = M(x, w) \frac{u(x, w)}{u(x_0, w)}.$$

So $K(x, w)$ is jointly continuous on $D \times \partial D$ and is comparable to $M(x, w)$.

Theorem 10 (*Chen-Kim 03*)

Suppose (q, F) is gaugeable. Then the Martin boundary and the minimal Martin boundary for the Schrödinger semigroup $\{T_t, t \geq 0\}$ can all be identified with ∂D . Furthermore for every (q, F) -superharmonic function $u \geq 0$ that is not identically infinite, there is a unique Radon measure μ_1 on D and a unique finite measure μ_2 on ∂D such that

$$u(x) = \int_D V(x, y) \mu_1(dy) + \int_{\partial D} K(x, z) \mu_2(dz). \quad (2)$$

u is (q, F) -harmonic if and only $\mu_1 = 0$.

Conversely, if μ_1 is a Radon measure in D such that $V\mu_1$ is not identically infinite and μ_2 is finite measure on ∂D , then the function u given by (2) is (q, F) -superharmonic.

So there is one-to-one correspondence between the space of Y -harmonic (Y -superharmonic) functions and the space of (q, F) -harmonic ((q, F) -superharmonic, respectively) functions through μ_1 and μ_2 in (1) and (2).

Example: Relativistic Stable Process

For $0 < \alpha < 2$, a relativistic α -stable process U in \mathbf{R}^n is a Lévy process whose characteristic function is given by

$$\mathbf{E} \left[e^{i\xi \cdot (U_t - U_0)} \right] = e^{-t \left((|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m \right)},$$

where $m > 0$ is a constant and $\xi \in \mathbf{R}^n$. In other words, the relativistic α -stable process in \mathbf{R}^n has infinitesimal generator $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$.

For $D \subset \mathbf{R}^n$ bounded and $C^{1,1}$, one can construct a censored relativistic α -stable process Z in D from U in the same way as Y is constructed. It can be shown that Z is related to Y by a non-local Feynman-Kac transform for some (q, F) and that (q, F) is gaugeable. Hence Theorems 9 and 10 apply.

Reflected α -Stable Process

Recall that the generator of Y is $\mathcal{L} = \Delta^{\alpha/2} + \kappa_D(x)$ with Dirichlet condition. How about $\Delta^{\alpha/2} + \kappa_D(x)$ with “Neumann” boundary condition?

The corresponding process Y^* is called reflected α -stable process on \overline{D} , which is introduced in Bogdan-Burdzy-Chen (01/03) as a tool to investigate Y . Intuitively, Y^* is the maximum Markovian extension of Y .

$D \subset \mathbf{R}^n$ is called an n -set if there is a constant $C > 0$ such that

$$|D \cap B(x, r)| \geq C r^n$$

for all $x \in D$ and $0 < r \leq 1$.

For $\alpha \in (0, 2)$ and $D \subset \mathbf{R}^n$ an n -set, define

$$B_{\alpha/2}^{2,2}(D) = \left\{ u \in L^2(D) : \mathcal{E}(u, u) < \infty \right\},$$

where

$$\mathcal{E}(u, u) = c \int_{D \times D} \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} dx dy$$

$(\mathcal{E}, B_{\alpha/2}^{2,2}(D))$ is a regular Dirichlet form on \bar{D} . Its associated process Y^* is called the reflected α -stable process on D .

Theorem 11 (*Chen-Kumagai 03*) *There is a Feller process Y^* on \bar{D} associated with the Dirichlet form $(\mathcal{E}, B_{\alpha/2}^{2,2}(D))$ and Y^* has a Hölder continuous transition density function $p(t, x, y)$. Furthermore,*

$$p(t, x, y) \approx \min \left\{ t^{-d/\alpha}, \frac{t}{|x - y|^{d+\alpha}} \right\}$$

on $[0, k] \times \bar{D} \times \bar{D}$ for every $k > 0$.