Rough Paths & approximations

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- $x_t \in R^d$ path of finite variation enhance to

  $$X_t := (X_t^1, X_t^2) = (x_t - x_0, \int_0^t (x - x_0) \otimes dx)$$

- β_t BM^d has finite p-variation whenever $2 < p$

  $$B_t := (\beta_t, \int_o^t \beta \otimes d\beta) \quad \text{(Ito- or Stratonovich-)}\textit{EBM}$$

- Get object $Y_t \in R^d \oplus (R^d)^{\otimes 2} =: T^2$ (tensoralgebra)

  $$Y_s^{-1} \otimes Y_t =: Y_{s,t} \Longrightarrow Y_{s,t} = Y_{s,r} \otimes Y_{r,t} \quad \text{(Chen)}$$

  Have finite $p-$variation on both "levels",

  $$|Y|_p = \max_{k=1,\dots,2} \left(\sup_D \sum_l |Y^k_{t_{l-1},t_l}|^{p/k} \right)^{k/p} < \infty$$

  $\Longrightarrow Y$ called a *rough path*, write $Y \in \Omega_p(R^d)$

- If $Symm(Y^2) = \frac{1}{2} Y^1 \otimes Y^1$...*geometric rough path*

- Lyons (95) proves existence & uniqueness for purely deterministic *rough ODE,*

  $$dZ = V(z_0 + Z^1_{0t})dY$$

  with $V = (V_1, ..., V_d)$ (non-commuting) vectorfields

  Generalizes ODE, Stratonovich- and Ito-SDE, e.g. choose $Y = B^{Strato}$

  $$z_0 + Z^1_{0t} \text{ solves } dz = V_i(z) \circ d\beta^i$$

- [L98] Universal Limit Theorem: Ito-map $Y \to Z$ is continuous w.r.t.

  $$d_p(Y, Y') = |Y - Y'|_p$$

  $\Rightarrow$2-parameter picture, $Y \in \Omega_p(R^d) \subset C(\Delta, T^2)$

- [F03] Ito-map Lipschitz continuous under

$$|Y|_\alpha := \max_{k=1,2} \sup_{0\le s<t\le 1} \frac{|Y^k_{s,t}|}{|t-s|^{k\alpha}},\ \alpha = 1/p < 1/2.$$

- Different view: A geometric rough path Y is a genuine G-valued path, Lie-algebra of G is

$$R^d \oplus so(d), \quad [a,b] = (0, a^1 \otimes b^1 - b^1 \otimes a^1)$$

  [F Victoir03] Define homog. subadditive norm $||Y_t|| := \max(|Y^1_t|, |Y^2_t|^{1/2})$ on G. Then Ito-map is continuous under

$$d_\varphi(Y, Y') := \sup_{0\le s<t\le 1} \frac{||Y^{-1}_{st} \otimes Y'_{st}||}{\varphi(t-s)}$$

  where $\varphi(t-s) = |t-s|^{1/p}$ or (better) of Levy-modulus type

- Application: simple proof that EBM has finite "2-level" Hoelder norm, Kolmogorov or GRR works with $(G, \otimes, e)$ instead of $(R^d, +, 0)$ ...

- Application: dyadic approximations lifted to geometric rough paths $B(n)$ converge to B in $d_{\tilde{\varphi}}$ where

  provided $\varphi = o(\tilde{\varphi})$.Sketch of proof:

  a) conditioning, uniform boundedness of $d_{\varphi}(B(n), 0)$

  b) equicontinuity, martingale convergence, by A.A. $B(n) \to B$ uniformly

  c) compactness argument $B(n) \to B$ in $d_{\tilde{\varphi}}$

- Corollary: $B^{ad}(n) \to B$ in $d_{\tilde{\varphi}}$

  Proof: metric invarint by time-shift

- With a little work: $T_{h-B^{ad}(n)}[B] \to h$ in $d_{\tilde{\varphi}}$ for any (lifted) Cameron-Martin path $h \in H$

  Background: $T_f[Y] = (y + f, \int_0 (y + f) d(y + f))$ and expand with Young-integrals

- This implies

$$\text{supp (law of } B)=d_{\tilde{\varphi}}\text{-closure of } H$$

(soft argument by Millet, Sanz-Sole using Girsanov's)

- Using ULT & projecting to first level, recover Stroock-Varadhan's support theorem in Orlicz-Besov-type topology (choose $\varphi(t) \sim_{t\to 0} \sqrt{-t\ln t}$) or simply in α-Hoelder for $\alpha < 1/2$.

- Remark: Support theorem has been proven with rough paths before in p-variation topology only & using correlation inequalities, [6].

References

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