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Harmonic Functions of Non-local Operators of Variable Order

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UCONN

Definition of $\mathcal{L}$, Assumptions

$$\mathcal{L}u(x) = \int_{\mathbb{R}^d\setminus\{0\}} \Big[u(x+h) - u(x) - 1_{|h|\le 1} h \cdot \nabla u(x)\Big]\, n(x,h)\, dh$$

Assumptions:

(1) $0 < \alpha < \beta < 2$ and $\beta - \alpha < 1$

(2) $c_1\, |h|^{-d-\alpha} \le n(x,h) \le c_2 |h|^{-d-\beta} \quad for |h| \le 2$

(3) $\int_{|h|>1} n(x,h)\, dh \le c_3$

(4) $n(x, z-x) \le c_4 n(y, z-y)$
if $|z-x| > 1, |z-y| > 1, |x-y| \le 1$

Note: $n(x,h) = |h|^{-d-\alpha} \;\Rightarrow\; \mathcal{L} = \Delta^{\alpha/2}$.
Assumption includes $n(x,h) = |h|^{-d-\alpha(x)}$.
$n(x,h)$ may be anisotropic.

Detour: Jump-Diffusions and PDE's ($\alpha = \beta$)

$\mathcal{A} = a_{ij}(x)D_{ij}$ or $\mathcal{A} = D_i\big(a_{ij}(x)D_j\big)$

Solutions for PDE's, $\mathcal{A}u + \mathcal{L}u = f$ or nonlinear

Lenhart(81), Bensoussan/J.-L. Lions(84), Gimbert/P.-L. Lions(84), K.(2002)
Krylov, Safonov, Pragarauskas, Mikulevicius (1979-1993)
Jacob-Hoh-Schilling, Taira, ... (1990-2003)

Martingale Problem: Komatsu(73), Stroock(75), Le-Peltier/Marchal(77), Bass(88), Hoh(95) ... (various conditions)

Hölder estimates, Harnack:

$\mathcal{A}u + \mathcal{L}u = 0$: Mikulevicius/Pragarauskas(88), K.(2003)

$\mathcal{L}u = 0$: Komatsu(88)
Bass/Levin (2002)

Main Result

Definition: u is $\mathcal{L}$-harmonic in a domain Ω if $u(X_{t\wedge\tau_\Omega})$ is a martingale, we write: $\mathcal{L}u = 0$ in Ω.

Theorem: Assume that $\mathcal{L}u = 0$ in $B(x_0, 2R)$. Assume u is positive in $B(x_0, 2R)$ and bounded in $\mathbb{R}^n$.

- Then there exists a constant $C(R)$, independent of u and x_0 such that

$$u(x) \leq C(R)\, u(y) \quad \forall\, x, y \in B(x_0, R)\ .$$

Discussion: R-dependence of Harnack-constant and Assumption 4.

History: $(C(R) = \text{const.})$: Cordes, Landis, Moser, Krylov-Safonov, Safonov, Fabes/Stroock, Bass/Levin

Main Ingredients in the Proof

Assume $\beta > 1$

Lemma 1: For $r \leq 1$

$$\mathbb{E}^x \tau_{B(x,r)} \geq c r^\beta, \quad \mathbb{E}^x \tau_{B(x,r)} \leq c r^\alpha.$$

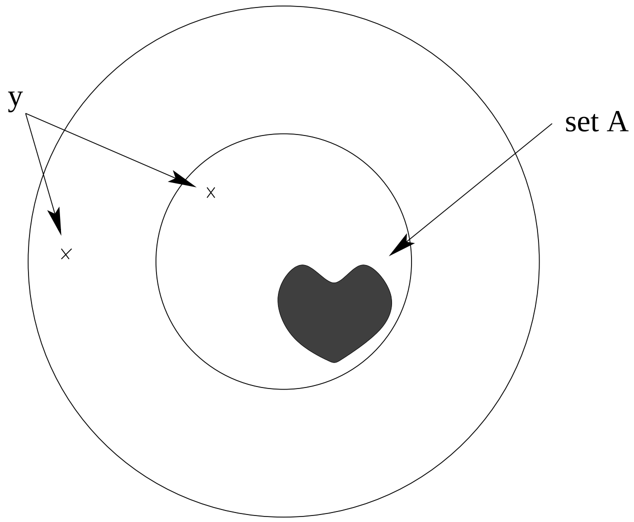
Lemma 2: Suppose $r < 1$.

- For $A \subset B(x_0, r/2)$ and $y \in B(x_0, r/2)$:

$$\mathbb{P}^y(T_A < \tau_{B(x_0,r)}) \geq c r^{\beta-\alpha} |A|/|B(x_0,r)|.$$

- For $A \subset B(x_0, r/2)$ and $y \in B(x_0, r)$:

$$\mathbb{P}^y(T_A < \tau_{B(x_0,r)}) \geq c[\text{dist}(y, \partial B(x_0,r))]^\beta r^{-\alpha} |A|/|B(x_0,r)|.$$



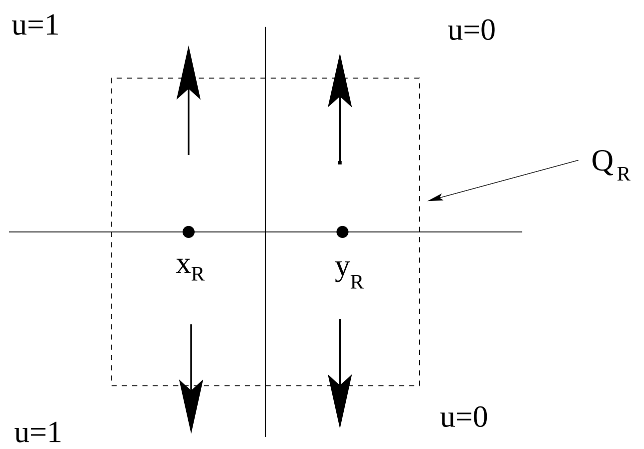
Note: Degenerate Version of Krylov-Safonov estimate.

Discussion

Example 1: Let $0 < \alpha < \beta < 2$. There exists a function $n(x,h)$ satisfying our Assumptions with the following property.

- For $R < 1$ there exist functions u_R that are nonnegative and harmonic on $B(0,R)$ and points $x_R, y_R \in B(0,R/2)$ such that $u_R(y_R)/u_R(x_R) \to \infty$ as $R \to 0$.

Idea: Build a process with high speed in x_2-direction and slow speed in x_1-direction.



$$\mathbb{P}^{x_R}(X^1_{\tau_{Q_R}} > 0) \to 0, \qquad \mathbb{P}^{y_R}(X^1_{\tau_{Q_R}} > 0) \to 1 \qquad \text{as } R \to 0.$$

Consequence: $u_R(x_R) \to 1$, $u_R(y_R) \to 0$

$\Rightarrow$ $u_R(x_R)/u_R(y_R) \to \infty$ as $R \to 0$.

Discussion

Example 2: There exists a function $n(x, h)$ satisfying Assumptions (1)-(3) (but not (4)) for which the Harnack inequality fails for the corresponding operator.

Outlook:

Symmetric Case
Parabolic Case
Continuity of Harmonic Functions
Random Media