Partial Differential Equations and Multiplicative Processes

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Objectives

- Develop a framework for representing solutions of linear and nonlinear evolution equations as expected values on branching processes.

- Quasilinear equation: Incompressible Navier-Stokes equations.

- Introduce majorizing kernels for the Navier Stokes equations. Existence, uniqueness and regularity of solutions.

- Linear and Semilinear Equations. Representation of solutions in physical and frequency space.

Navier Stokes Equation

$$\frac{\partial \mathbf{u}}{\partial t} = \nu \Delta \mathbf{u} - \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + g(x,t) - \nabla p. \quad \nabla \cdot \mathbf{u} = 0. \quad \text{(NS)}$$

Initial data $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_0(\mathbf{x})$.

LeJan-Sznitman. Probabilistic interpretation in frequency space.

$\hat{f}(\xi) = (2\pi)^{-3/2} \int_{\mathbf{R}^3} f(\mathbf{x}) e^{-i\xi\cdot\mathbf{x}} d\mathbf{x}.$

$$\hat{\mathbf{u}}(\xi, t) = \frac{1}{|\xi|^2} \mathbf{E}_{\xi_\theta = \xi} \left[\chi(\theta, t) \right]$$

χ random multiplicative functional defined on the vertices of a multitype branching tree $\tau_\theta(t)$.

$$\chi(\xi,t) \;=\; e^{-\nu|\xi|^2 t}\;\chi_0(\xi)$$

$$+\;\int_0^t \nu|\xi|^2\; e^{-\nu|\xi|^2 s}\left[\frac{1}{2}\varphi(\xi,t-s)+\frac{1}{2}m(\xi)\right.$$

$$\left.\int_{\mathbf{R}^3}\chi(\eta,t-s)\odot_\xi\chi(\xi-\eta,t-s)\,\frac{h(\eta)h(\xi-\eta)}{(h*h)(\xi)}d\eta\right]ds$$

$$m(\xi) \;=\; \frac{2(h*h)(\xi)}{\nu(2\pi)^{\frac{3}{2}}|\xi|h(\xi)}$$

$$\chi_0(\xi,t) \;=\; \frac{\widehat{\mathbf{u}}_0(\xi,t)}{h(\xi)}$$

$$\varphi(\xi,t) \;=\; \frac{2\widehat{g}(\xi,t)}{\nu|\xi|^2 h(\xi)}.$$

Random Variables

$$S_{\mathbf{v}}, \xi_{\mathbf{v}}, \kappa_{\mathbf{v}}$$

$$\mathbf{v} \in \mathcal{V} = \cup_{n=0}^{\infty} \{1, 2\}^n.$$

$$S_{\mathbf{v}} \stackrel{\mathcal{L}}{=} \mathrm{Exp}(\nu |\xi_{\mathbf{v}}|^2), \quad \kappa_{\mathbf{v}} \stackrel{\mathcal{L}}{=} \mathrm{Bernoulli}(0,1).$$

Times Functional

$$\chi(\theta, t) = \begin{cases} \chi_0(\xi), & \text{if } S_\theta \geq t \\ \varphi(\xi, t - S_\theta), & \text{if } S_\theta < t, \kappa_\theta = 0, \\ m(\xi_\theta)\chi(<1>, t - S_\theta)) & \\ \odot_{\xi_\theta}\chi(<2>, t - S_\theta)) & \text{else} \end{cases}$$

Distribution of types:

$$\xi_{<1>} + \xi_{<2>} = \xi_\theta$$

$$K_\xi(d\xi_1, d\xi_2) = \frac{h(\xi_1)h(\xi_2)}{(h * h)(\xi)} \; \mathbf{1}_{\xi_1 + \xi_2 = \xi} \; d\xi_1 d\xi_2$$

Proposition: If $\mathbf{E}_{\xi_\theta = \xi}\left[|\chi(\theta, t)|\right] < \infty$,

$$\hat{\mathbf{u}}(\xi, t) = h(\xi)\mathbf{E}_{\xi_\theta = \xi}\left[\chi(\theta, t))\right]$$

solves FNS.

Example: Let

$$h_0(\xi) = \frac{\pi^3}{|\xi|^2}, C_\nu = \frac{\nu}{2}(2\pi)^{3/2}$$

Then

$$h_0 * h_0(\xi) = |\xi| h_0(\xi) \text{ and } m(\xi) = 1.$$

If

$$|\hat{\mathbf{u}}_0(\xi)| \leq C_\nu h_0(\xi)$$

$$|\hat{g}(\xi, t)| \leq \frac{\nu}{2} C_\nu \pi^3 \quad \text{for} \quad 0 \leq t \leq T,$$

then

$$|\chi(\theta, t)| \leq 1 \quad \text{for} \quad 0 \leq t \leq T.$$

Definition A non negative locally integrable function $h(\xi)$ on $W_h \subset \mathbf{R}^n$ is a Navier Stokes admissible standardized *majorizing kernel* with exponent θ iff (i) $\bar{W}_h$ is a semigroup, (ii) h is continuous in W_h, (iii) $(h*h)(\xi) > 0$ a.e. in W_h and

$$(h*h)(\xi) \leq |\xi|^{\theta} h(\xi)$$

Function Space $\mathcal{F}_{h,T} = \{v \in \mathcal{S}' : \widehat{v}(\xi,t) = 0, \xi \in W_h^c, \|v\|_{\mathcal{F}_{h,T}} = \text{ess sup}_{\substack{\xi\in W_h \\ 0\leq t<T}} \frac{|\widehat{v}(\xi,t)|}{h(\xi)} < \infty\}$

Theorem Case $\theta = 1$. **Global Solutions for small data** Let $h(\xi)$ be a standardized majorizing kernel with exponent 1. Let $C_\nu = \frac{\nu}{2}(2\pi)^{3/2}$. Suppose that for $0 \leq t \leq T$

$$\|\mathbf{u}_0\|_{\mathcal{F}_{h,T}} \leq C_\nu, \quad \|\Delta^{-1} g(\mathbf{x},t)\|_{\mathcal{F}_{h,T}} \leq \frac{\nu}{2} C_\nu.$$

Then (FNS) has a unique solution such that

$$\|\hat{\mathbf{u}}(\xi,t)\|_{\mathcal{F}_{h,T}} \leq C_\nu.$$

Moreover, $\widehat{u}(\xi,t) = h(\xi) E_{\xi_\theta=\xi} \chi(\theta,t)$

Examples Majorizing Kernels R^3, $\theta = 1$.

LeJan Sznitman

$$h_0(\xi) = \frac{\pi^3}{|\xi|^2}, \qquad h_1(\xi) = \frac{e^{-|\xi|}}{2\pi|\xi|}, \quad \xi \neq 0,$$

Non integrable and integrable radially symmetric majorizing kernels.

Proposition For $0 \leq \beta \leq 1, \alpha > 0$.

$$h_\beta^{(\alpha)}(\xi) = \frac{e^{-\alpha|\xi|^\beta}}{|\xi|^{2-\beta}}, \quad \xi \neq 0$$

are integrable (for $\beta > 0$) radially symmetric majorizing kernels.

Non Radially Symmetric kernels For $\xi \in \{\xi \in R^3 : \sum_{j=1}^3 \delta_{\xi_j,0} < 2\}$,

$$h(\xi) = \int_0^\infty \frac{t^3}{\prod_{j=1}^3 (t^2 + \xi_j^2)} dt.$$

$$h(\xi) = \frac{1}{|\xi|^2} G(\xi)$$

such that

$$\lim_{\xi \to \pm \mathbf{e}_j} G(\xi) = \infty$$

Log convexity property Suppose that for $\xi \in R^n$, $j = 1, 2$,

$$h_j * h_j(\xi) \leq |\xi|^{\theta_j} h_j(\xi)$$

Then, for any $0 \leq \beta \leq 1$, $h = h_1^\beta h_2^{1-\beta}$ satisfies

$$h * h(\xi) \leq |\xi|^{\beta\theta_1 + (1-\beta)\theta_2} h(\xi).$$

Product Property Suppose that $k_1, \ldots, k_m$ is a partition of $n \geq 1$, and h_j is a non-negative function on R^{k_j} such that

$$h_j * h_j(\xi_j) \leq |\xi_j|^{\theta_j} h_j(\xi_j)$$

$\xi_j \in R^{k_j} - \{0\}$, where $\theta_j > 0$, $\sum_j \theta_j = 1$.

$$h(\xi) = \prod_{j=1}^{m} h_j(\xi_j)$$

$\xi = (\xi_1, \xi_2, \ldots, \xi_m)$, $\xi_j \in R^{k_j}$ is a majorizing kernel.

Relation between Fixed Point methods and expected value representation

Iteration Scheme

$$\mathcal{Q}[\hat{\mathbf{u}};\hat{\mathbf{u}}_0,\hat{g}](\xi,t) := e^{-\nu|\xi|^2 t}\hat{\mathbf{u}}_0(\xi) + \hat{B}[\hat{\mathbf{u}},\hat{\mathbf{u}}](\xi,t) + \int_0^t e^{-\nu|\xi|^2 s}\hat{g}(\xi,t-s)ds$$

$$\hat{B}(\hat{\mathbf{u}},\hat{\mathbf{v}}) = \int_0^t e^{-\nu|\xi|^2 s}|\xi|(2\pi)^{-\frac{3}{2}} \int \hat{\mathbf{u}}(\xi-\eta,t-s)\odot_\xi \hat{\mathbf{v}}(\eta,t-s)d\eta ds.$$

Define $\hat{\mathbf{u}}_{n+1}(\xi,t) = \mathcal{Q}[\hat{\mathbf{u}}_n;\hat{\mathbf{u}}_0,\hat{g}](\xi,t)$, $\hat{\mathbf{u}}_1(\xi,t) = \mathcal{Q}[\hat{\mathbf{u}}^{(0)};\hat{\mathbf{u}}_0,\hat{g}](\xi,t)$ $\quad \hat{\mathbf{u}}^{(0)}(\xi,t) = e^{-\nu|\xi|^2 t}\hat{\mathbf{u}}_0(\xi)$.

Replacement time of a vertex $\mathbf{v}$

$$R_{\mathbf{v}} = \sum_{k=0}^{|\mathbf{v}|} S_{\mathbf{v}|k}$$

Introduce

$$\begin{aligned} A_n(\theta,t) &= [|\mathbf{v}| \le n \ \forall \mathbf{v} \in \tau_\theta(t)] \\ &\quad \cap [R_{\mathbf{v}} > t \forall \mathbf{v} \in \{\mathbf{u} \in \tau_\theta(t) : |\mathbf{u}| = n\}], \end{aligned}$$

Let $\mathbf{1}[n;\theta,t]$ indicator of $A_n(\theta,t)$.

Proposition: Let

$$\mathbf{v}_k(\xi,t) = h(\xi)\mathbf{E}_\xi\{\mathbf{1}[k;\xi,t]\chi(\theta,t)\}$$

and $\hat{\mathbf{u}}_k(\xi,t)$ the Fourier transform of the k^{th} iterate of the iteration scheme. Then $\mathbf{v}_k(\xi,t) = \hat{\mathbf{u}}_k(\xi,t)$.

Theorem Case $\theta < 1$. **Local Solutions for arbitrary data** Let $h(\xi)$ be a standardized majorizing kernel with exponent $0 \le \theta < 1$. Suppose $\mathbf{u}_0 \in \mathcal{F}_{h,T}$ and for some $0 \le \beta \le 2$, $-\Delta^{-\beta/2}g \in \mathcal{F}_{h,T}$ Then there exist $T_* \le T$ such that (FNS) has a unique solution $\mathbf{u} \in \mathcal{F}_{h,T}$.

Analyticity of Solutions

Let $\mathcal{F}_{h,1,T} = \{v \in \mathcal{S}' : \widehat{v}(\xi,t) = 0, \xi \in W_h^c, \|v\|_{\mathcal{F}_{h,1,T}} = \operatorname{esssup}_{\substack{\xi \in W_h \\ 0 \le t < T}} \frac{|\widehat{v}(\xi,t)|}{e^{-\sqrt{t}|\xi|}h(\xi)} < \infty\}$

Theorem Let $h(\xi)$ be a standardized majorizing kernel with exponent $\theta = 1$. Fix $0 < T \le +\infty$. Assume $\|e^{\nu t\Delta}u_0(x)\|_{\mathcal{F}_{h,1,T}} \le \frac{(\sqrt{2\pi})^3}{2}\rho\nu e^{-1/2\nu}$ and $\|(-\Delta)^{-1}g(x,t)\|_{\mathcal{F}_{h,1,T}} \le \frac{(\sqrt{2\pi})^3}{4}\rho\nu^2 e^{-1/2\nu}$ for some $0 \le \rho < 1$. Then there is a unique solution u in the ball $\mathcal{B}_1(0,R)$ centered at 0 of radius $R = (\rho/2)(\sqrt{2\pi})^3\nu e^{-\frac{1}{2\nu}}$ in the space $\mathcal{F}_{h,1,T}$.

Burgers Equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\frac{\partial u^2}{\partial x} - u$$

initial data $u(x,0) = u_0(x)$.

Multiplicative process in physical space

$$\begin{aligned} u(x,t) &= e^{-t}\int_{\mathbf{R}} G(x-y,t)u_0(y)dy \\ &\quad + \int_0^t \int_{\mathbf{R}} e^{-s} G(x-y,s)\frac{1}{2}\frac{\partial u^2}{\partial x}(y,t-s)dyds \\ &= e^{-t}\int_{\mathbf{R}} G(x-y,t)u_0(y)dy \\ &\quad + \int_0^t \int_{\mathbf{R}} e^{-s}\frac{x-y}{s} G(x-y,s)\frac{1}{2}u^2(y,t-s)dyds \end{aligned}$$

$G(z,t) = (2\pi t)^{-1/2}\exp(-z^2/2t)$.

Times Functional

$$\chi(x,t) = \begin{cases} u_0(x+B_t), & \text{if } S_\theta \geq t \\ \frac{B_{S_\theta}}{2S_\theta}\chi^{(1)}(B_{S_\theta}, t-S_\theta)\chi^{(2)}(B_{S_\theta}, t-S_\theta) & \text{if } S_\theta < t \end{cases}$$

Multiplicative process in frequency space.

$$\lambda(\xi) = (1 + \frac{1}{2}|\xi|^2), m(\xi) = \frac{\text{sign}(\xi)(h * h)(\xi)}{ih(\xi)\lambda(\xi)},$$

$$\chi_0(\xi, t) = \frac{\widehat{u}_0(\xi, t)}{h(\xi)}.$$

$$\begin{aligned} \chi(\xi, t) \;&=\; e^{-\lambda(\xi)t}\chi_0(\xi) + \\ &\int_0^t e^{-\lambda(\xi)s}\lambda(\xi)m(\xi)\int_{-\infty}^{\infty}\chi(\eta, t-s)\chi(\xi-\eta, t-s)\mathcal{K}_\xi d\eta d \end{aligned}$$

Times Functional

$$\chi(\theta, t) = \begin{cases} \chi_0(\xi), & \text{if } S_\theta \geq t \\ 0, & \text{if } S_\theta < t, \kappa_\theta = 0, \\ m(\xi_\theta)\chi(< 1 >, t - S_\theta) & \\ \chi(< 2 >, t - S_\theta)) & \text{else} \end{cases}$$

Distribution of types:

$$\mathcal{K}_{\xi_\theta}(d\xi_1, d\xi_2) = \frac{h(\xi_1)h(\xi_2)}{(h * h)(\xi)}\mathbf{1}_{\xi_1+\xi_2=\xi}d\xi_1 d\xi_2$$

$$\widehat{u}(\xi, t) = h(\xi)\mathbf{E}\left[\chi(\tau_\theta(\xi, t))\right].$$

Conclusions - Some Open questions

• Method described is applicable to evolution equations that involve fractional derivatives

• In many examples, such us KPP, Burgers, linear problems; the representation of solutions as expected values in both real and Fourier space are available. An outstanding open case is to establish a duality for the Navier-Stokes equations in any dimensions.

• Periodic problems can be treated analogously. Numerical methods.

• Use of other majorizing kernels (e.g. time dependent).

• Establishes a probabilistic interpretation to Fixed Point methods.