Symmetry in Computer Vision

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Basic Issues in Computer Vision

- multi-scale resolution
- denoising/smoothing
- image enhancement
- edge detection
- segmentation
- geometric attributes
  lengths, areas, volumes,
  relative positions, etc.
- object recognition
- invariant signatures
- occlusion

Symmetry Groups

Euclidean *Length–preserving*

Translations

Rotations

Reflections

Similarity *Preserves length ratios*

Euclidean + Scaling

Equi-affine *Area–preserving*

$A\,\mathbf{x} + \mathbf{b} \qquad \det A = 1$

Translations + Unimodular linear

Symmetry Groups

Affine *Preserves volume ratios*

$$A\,\mathbf{x} + \mathbf{b}$$

Equiaffine + Scaling

Projective *Preserves cross-ratios*

$$\left(\frac{ax + by + c}{gx + hy + j}, \frac{dx + ey + f}{gx + hy + j} \right)$$

Camera Rotations

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} \in \mathrm{SO}(3)$$

An Invariant Image Processing System

- Smoothing
  - Geometric diffusion
  - Level set curve shortening
- Segmentation
  - Conformal snakes
- Object recognition
  - Signature curve or surface
- Symmetries of objects
- Invariant Numerical Implementation

Evolutionary Smoothing

Multi-scale resolution provided by evolutionary partial differential equation

$$\Phi_t = F(\mathbf{x}, \Phi, \nabla\Phi, \nabla^2\Phi, \ldots)$$

$$\Phi(\mathbf{x}, 0) = I(\mathbf{x})$$

$$\begin{aligned} \mathbf{x} &= \text{spatial position} \\ t &= \text{scale parameter} \\ &= \text{degree of smoothing} \\ I(\mathbf{x}) &= \text{raw gray-scale image} \\ \Phi(\mathbf{x}, t) &= \text{smoothed image} \end{aligned}$$

Gaussian Smoothing

$\Longrightarrow$ *Simplest model*

Heat equation = Gaussian convolution

$$\Phi_t = \Delta\Phi \qquad \Phi(\mathbf{x}, 0) = I(\mathbf{x}).$$

$$\Phi(\mathbf{x}, t) = \mathcal{G}(\mathbf{x}, t) * I(\mathbf{x})$$

Problems:

- Smooths out both noise and relevant features indiscriminantly
- Isotropic process

$\Longrightarrow$ Need an anisotropic (nonlinear) diffusion process which eliminates noise but retains edges and other features.

Level Set Evolution

Idea:

Use geometric diffusion to smooth

Evolve individual level sets

Theorem. The level sets

$$C_k(t) = \{ \, (x, y) \mid \Phi(x, y, t) = k \, \}$$

evolve according to the normal flow

$$C_t = -\alpha \, \mathbf{N}$$

if and only if Φ satisfies the evolution equation

$$\Phi_t = \alpha \, \| \nabla \Phi \|$$

Osher–Sethian

$\mathbf{N}$ — outward normal to level set

$$\Phi_t = \alpha(\Phi, \nabla\Phi, \nabla^2\Phi)\,\|\nabla\Phi\|$$

- Smoothing of level sets *only*
- Level sets move independently of each other
- Can continue after crossing/ separation/singularities
- Readily implementable in both 2D and 3D

$\Longrightarrow$ *Concentrate on 2D images from now on.*

Curve Evolution

$C(q,t)$ — parametrized family of (closed) curves in $\mathbb{R}^2$

$\mathbf{T}$ — unit tangent

$\mathbf{N}$ — unit (outward) normal

General curve evolution

$$\frac{dC}{dt} = \alpha\,\mathbf{N} + \beta\,\mathbf{T}$$

By reparametrizing, can assume

$$\beta = 0$$

No tangential component:

$$\frac{dC}{dt} = \alpha\,\mathbf{N}$$

Symmetry Groups

Euclidean *Length–preserving*

Translations

Rotations

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Similarity *Preserves length ratios*

Euclidean + Scaling

Invariant Curve Flows

Assume, for simplicity, that C is a graph:

$$y = u(x, t)$$

Wave front flow

$$C_t = -\mathbf{N} \qquad\qquad u_t = -\sqrt{1 + u_x^2}$$

$$\Phi_t = \|\nabla \Phi\| = \sqrt{\Phi_x^2 + \Phi_y^2}$$

- Simplest Euclidean invariant flow
- Formation of caustics

Euclidean Curve Shortening

$$C_t = -\kappa\,\mathbf{N} \qquad\qquad u_t = -\frac{u_{xx}}{1+u_x^2}$$

$$\Phi_t = \frac{\Phi_y^2\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + \Phi_x^2\Phi_{yy}}{\Phi_x^2+\Phi_y^2}$$

$$= \|\nabla\Phi\|\ \mathrm{div}\ \frac{\nabla\Phi}{\|\nabla\Phi\|}$$

- Euclidean invariant flow
- Shortens Euclidean perimeter as rapidly as possible
- $\nabla\Phi$ — characteristic
- Nonconvex curves convexify
- Convex curves shrink to round points

Grayson–Gage–Hamilton

Affine Curve Shortening

$$C_t = -\sqrt[3]{\kappa}\,\mathbf{N} \qquad\qquad u_t = \sqrt[3]{u_{xx}}$$

$$\Phi_t = (\Phi_y^2\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + \Phi_x^2\Phi_{yy})^{1/3}$$

- Simplest affine invariant flow
- Affine curvature flow, in direction of affine normal
- Shortens affine arc length as rapidly as possible
- Curves shrink to elliptical points

Angenent–Sapiro–Tannenbaum

Projective curve flow

$$u_t = \frac{u_{xx}^3}{(9u_{xx}^2 u_{xxxxx} - 45 u_{xx} u_{xxx} u_{xxxx} + 40 u_{xxx}^3)^{2/3}}$$

- Simplest projective invariant flow
- In direction of projective normal, proportional to projective curvature
- Shortens projective arc length as rapidly as possible
- Higher order derivatives; not well defined
- Curves can become singular

Transformation groups

$x = (x^1, \ldots, x^p)$

independent variables

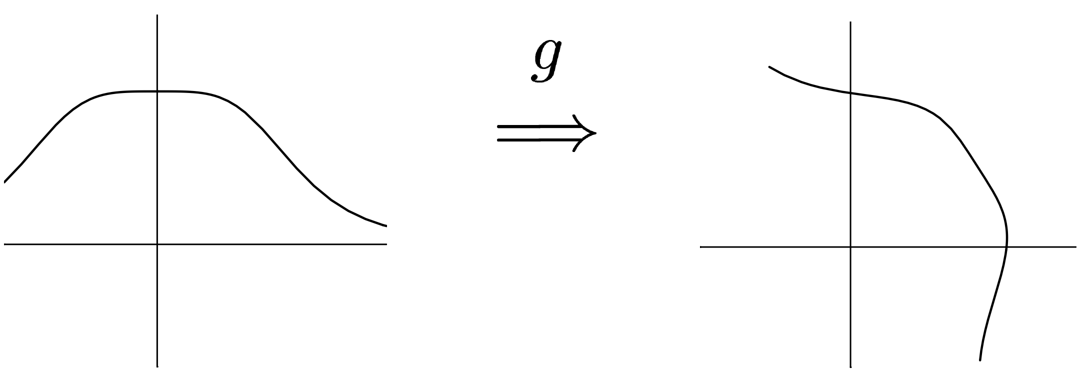
$u = (u^1, \ldots, u^q)$

dependent variables

G transformation group

$$(\bar{x}, \bar{u}) = g \cdot (x, u)$$

G acts on curves and surfaces by pointwise transformation:



Invariants

Definition. An *invariant* is a function $I : M \to \mathbb{R}$ such that

$$I(g \cdot (x, u)) = I(x, u) \qquad g \in G$$

If $I_1, \ldots, I_k$ are invariants, so is $H(I_1, \ldots, I_k)$.

$\Longrightarrow$ Classify invariants up to functional independence.

$m = p + q = \#$ variables

$s =$ dimension of orbits of G (regular $\Longleftrightarrow$ constant)

$k = m - s = \#$ functionally independent invariants.

Invariant Shape Descriptors

- Recognition of geometric objects
- Characterize using geometric invariants
- Joint invariants
  - distances
  - areas
  - cross ratios
- Differential invariants
  - curvatures
  - invariant derivatives

Joint Invariants

If G acts on M, then it acts on the Cartesian product

$$M \overbrace{\times \cdots\cdots \times}^{k\ times} M$$

by simultaneous transformations

$$g \cdot (z_1, \ldots, z_k) = (g \cdot z_1, \ldots, g \cdot z_k)$$

A *joint invariant* is an invariant for this product action:

$$J(g \cdot z_1, \ldots, g \cdot z_k) = J(z_1, \ldots, z_k)$$

Theorem. Every joint invariant $J(z_1, \ldots, z_k)$ of the Euclidean group is a function of the distances

$$|\, z_i - z_j \,|$$

Theorem. Every joint invariant $J(z_1, \ldots, z_k)$ of the equi-affine group is a function of the areas

$$[ijk] = (z_i - z_j) \wedge (z_i - z_k)$$

Prolongation

Represent a curve or surface as the graph of a function

$$S: \qquad u = f(x)$$

$u^{(n)} = (\ldots, u_J^\alpha, \ldots)$

derivative coordinates

G acts on S via

$$(x, u) \longmapsto g \cdot (x, u)$$

hence G acts on the derivative coordinates

$$(x, u^{(n)}) \longmapsto g^{(n)} \cdot (x, u^{(n)})$$

$\Longrightarrow$ *prolonged action*

Rotation group — SO(2)

$$(x,u) \longmapsto (x\cos t - u\sin t,\ x\sin t + u\cos t)$$

Transformed function $\bar u = \bar f(\bar x)$

$$\bar x = x\cos t - f(x)\sin t,$$
$$\bar u = x\sin t + f(x)\cos t,$$

Second prolongation

$$\bar x = x\cos t - u\sin t$$
$$\bar u = x\sin t + u\cos t$$

$$\bar u_{\bar x} = \frac{\sin t + u_x\cos t}{\cos t - u_x\sin t}$$

$$\bar u_{\bar x\bar x} = \frac{u_{xx}}{(\cos t - u_x\sin t)^3}$$

$\cot t \neq u_x$

Differential Invariants

A *differential invariant* is an invariant $I(x, u^{(n)})$ for the prolonged group action:

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

A *semi-differential invariant* is a joint differential invariant:

$$K(g^{(n)} \cdot (x_1, u_1^{(n)}), \ldots, g^{(n)} \cdot (x_k, u_k^{(n)})) =$$
$$= K((x_1, u_1^{(n)}), \ldots, (x_k, u_k^{(n)}))$$

- Classify differential invariants up to functional independence
- # functionally independent differential invariants depends on dimension of the orbits of prolonged action of $G^{(n)}$
- Use invariant differentiation operators to generate higher order ones

$$I \longmapsto \mathcal{D} I$$

Computation of Invariants

- Classical invariant theory
- Geometric invariant theory
- Infinitesimal methods (Lie)
- Invariant differentiation
- Equivalence methods (Cartan)
- Moving frame (repère mobile) (Cartan, Fels-Olver)

Invariant Differential Eqs.

Theorem. A system of n^{th} order differential equations admits G as a regular symmetry group if and only if it can be rewritten in terms of n^{th} order differential invariants:

$$F_\nu(I_1(x, u^{(n)}), \ldots, I_k(x, u^{(n)})) = 0$$

Rotation group — SO(2)

$$(x,u) \longmapsto (x\cos t - u\sin t,\ x\sin t + u\cos t)$$

Transformed curve $\bar{u} = \bar{f}(\bar{x})$

$$\bar{x} = x\cos t - f(x)\sin t, \quad \bar{u} = x\sin t + f(x)\cos t,$$

Second prolongation

$$\bar{u}_{\bar{x}} = \frac{\sin t + u_x \cos t}{\cos t - u_x \sin t} \qquad \bar{u}_{\bar{x}\bar{x}} = \frac{u_{xx}}{(\cos t - u_x \sin t)^3}$$

Order	Differential Invariant
0	$r = \sqrt{x^2+u^2}$
1	$w = \tan\phi = \dfrac{xu_x - u}{x + uu_x}$
2	$\kappa = \dfrac{u_{xx}}{(1+u_x^2)^{3/2}} = \left(\dfrac{d\phi}{dr} + \dfrac{\tan\phi}{r}\right)\sec\phi$

Invariant differential equation

$$\frac{xu_x - u}{x + uu_x} = H(\sqrt{x^2+u^2}\,) \qquad \frac{d\theta}{dr} = \frac{H(r)}{r}$$

$\Longrightarrow$ *Lie's Integration Theory*

Differential Invariants in $\mathbb{R}^2$

G — ordinary transformation group

$$r = \dim G$$

G-invariant arc length

$$ds = P(x, u^{(k)})\, dx, \qquad k \leq r - 2$$

- simplest G-invariant one-form

G-invariant curvature

$$\kappa(x, u^{(r-1)})$$

- simplest differential invariant

Theorem. All other differential invariants are functions of

$$\kappa, \quad \frac{d\kappa}{ds}, \quad \frac{d^2\kappa}{ds^2}, \quad \ldots$$

Euclidean Invariants

Joint Euclidean invariant:

$$\mathbf{d}(A, B) = |A - B|$$

Euclidean curvature:

$$\kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}$$

Euclidean arc length:

$$ds = \sqrt{1 + u_x^2}\, dx$$

Higher order Euclidean invariants:

$$\kappa_s = \frac{d\kappa}{ds} \qquad \kappa_{ss} = \frac{d^2\kappa}{ds^2} \qquad \ldots$$

Equi–Affine Invariants

Joint affine invariant:

$$[ijk] = (P_i - P_j) \wedge (P_i - P_k)$$

Affine curvature

$$\kappa = \frac{3u_{xx}u_{xxxx} - 5u_{xxx}^2}{9(u_{xx})^{8/3}}$$

Affine arc length

$$ds = \sqrt[3]{u_{xx}}\; dx$$

Higher order affine invariants:

$$\kappa_s = \frac{d\kappa}{ds} \qquad \kappa_{ss} = \frac{d^2\kappa}{ds^2} \qquad \ldots$$

Vision Groups

Group	*Arc Length*	*Curvature*
Euclidean	$\sqrt{1+u_x^2}\,dx$	$\dfrac{u_{xx}}{(1+u_x^2)^{3/2}}$
Similarity	$\dfrac{u_{xx}\,dx}{(1+u_x^2)}$	$\dfrac{P_3}{u_{xx}^2}$
Special Affine	$\sqrt[3]{u_{xx}}\,dx$	$\dfrac{P_4}{(u_{xx})^{8/3}}$
Affine	$\dfrac{\sqrt{P_4}}{u_{xx}}\,dx$	$\dfrac{P_5}{(P_4)^{3/2}}$
Projective	$\dfrac{(P_5)^{1/3}}{u_{xx}}\,dx$	$\dfrac{P_7}{(P_5)^{8/3}}$

$$P_3 = (1+u_x^2)u_{xxx} - 3u_x u_{xx}^2$$

$$P_4 = 3u_{xx}u_{xxxx} - 5u_{xxx}^2$$

$$P_5 = 9u_{xx}^2 u_{xxxxx} - 45u_{xx}u_{xxx}u_{xxxx} + 40u_{xxx}^3$$

$$P_7 = u_{xx}^2[2P_5D_x^2P_5 - \tfrac{7}{3}(D_xP_5)^2] + 2u_{xx}u_{xxx}P_5D_xP_5 - (9u_{xx}u_{xxxx} - 7u_{xxx}^2)P_5^2$$

Curves in Space

$$C \subset \mathbb{R}^n$$

$\kappa_1, \ldots, \kappa_{n-1}$

fundamental differential invariants

(curvature, torsion, etc.)

$ds = P(x, u^{(n)})\, dx$

invariant arc length

Theorem. All other differential invariants are functions of

$$\kappa_\alpha, \quad \frac{d\kappa_\alpha}{ds}, \quad \frac{d^2\kappa_\alpha}{ds^2}, \quad \ldots$$

General submanifolds

$$S \subset \mathbb{R}^n \qquad p = \dim S$$

$J_1, \ldots, J_m$ — fundamental differential invariants

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ — invariant differential operators

Theorem. All other differential invariants are functions of

$$\mathcal{D}^K J_\alpha = \mathcal{D}_{k_1} \cdots \mathcal{D}_{k_m} J_\alpha$$

$$m = ??? \qquad \mathrm{order} J_\alpha = ???$$

Euclidean Surfaces

Fundamental differential invariants:

- H — mean curvature
- K — Gaussian curvature

Frenet frame:

$$du = \cdots$$

$$dv = \cdots$$

Invariant differentiations — ∂_u, ∂_v

Third order differential invariants:

$$\frac{\partial H}{\partial u} \quad \frac{\partial H}{\partial v} \quad \frac{\partial K}{\partial u} \quad \frac{\partial K}{\partial v}$$

Invariant Evolution Equations

Theorem. If $G \subset \mathrm{SL}(3)$ is a subgroup of the projective group, then the most general G-invariant evolution equation has the form

$$u_t = \frac{u_{xx}}{P^2}\, J \qquad (*)$$

J — any differential invariant

$ds = P\,dx$ — G-inv. arc length

Theorem. If G is the similarity, special affine, affine, or projective group, then the invariant evolution equation of lowest order is (*) for J constant.

$\Longrightarrow$ Euclidean case:

wave front flow $J = 1/\kappa$

Invariant Evolution Equations

G — transformation group

$dS = P\,dx$ — G-invariant volume

$\mathcal{V}[u] = \int_\Omega dS$ — G-inv. measure

$\mathcal{E} = \delta\mathcal{V} = 0$ — Euler-Lagrange eqs.

G-invariant minimal submanifolds

Theorem. The most general G-invariant evolution equation has the form

$$u_t = \frac{P}{\mathcal{E}}\, I$$

I — any differential invariant

Note: $\mathcal{E} = (P^3/u_{xx})\, J \qquad J = ???$

Euclidean Joint Differential Invariants — Planar Curves

- One–point

  $\Rightarrow$ curvature

$$\kappa = \frac{\dot{z} \wedge \ddot{z}}{\| \dot{z} \|^3}$$

- Two–point

  $\Rightarrow$ distances $\quad \| z^1 - z^0 \|$

  $\Rightarrow$ tangent angles $\quad \phi^k = \measuredangle(z_1 - z_0, \dot{z}_k)$

Equi–Affine Joint Differential Invariants — Planar Curves

- One–point

  $\Rightarrow$ affine curvature

$$\kappa = \frac{(z_t \wedge z_{tttt}) + 4(z_{tt} \wedge z_{ttt})}{3(z_t \wedge z_{tt})^{5/3}} - \frac{5(z_t \wedge z_{ttt})^2}{9(z_t \wedge z_{tt})^{8/3}}$$

- Two–point

  $\Rightarrow$ tangent triangle area ratio

$$\frac{\dot{z}_0 \wedge \ddot{z}_0}{[\,(z_1 - z_0) \wedge \dot{z}_0\,]^3} = \frac{[\,\dot{0}\ \ddot{0}\,]}{[\,0\ 1\ \dot{0}\,]^3}$$

- Three–point

  $\Rightarrow$ triangle area

$$\tfrac{1}{2}\,(z_1 - z_0) \wedge (z_2 - z_0) = \tfrac{1}{2}\,[\,0\ 1\ 2\,]$$

Projective Joint Differential Invariants — Planar Curves

- One–point

  $\Rightarrow$ projective curvature

$$\kappa = \ldots$$

- Two–point

  $\Rightarrow$ tangent triangle area ratio

$$\frac{[\,0\ 1\ \dot{0}\,]^3\,[\,\dot{1}\ \ddot{1}\,]}{[\,0\ 1\ \dot{1}\,]^3\,[\,\dot{0}\ \ddot{0}\,]}$$

- Three–point

  $\Rightarrow$ tangent triangle ratio

$$\frac{[\,0\ 2\ \dot{0}\,]\,[\,0\ 1\ \dot{1}\,]\,[\,1\ 2\ \dot{2}\,]}{[\,0\ 1\ \dot{0}\,]\,[\,1\ 2\ \dot{1}\,]\,[\,0\ 2\ \dot{2}\,]}\,.$$

- Four–point

  $\Rightarrow$ area cross–ratio

$$\frac{[\,0\ 1\ 2\,]\,[\,0\ 3\ 4\,]}{[\,0\ 1\ 3\,]\,[\,0\ 2\ 4\,]}$$

Differential Invariant Signatures

$\mathcal{C} = \{(x(t), y(t))\} \subset \mathbb{R}^2$ closed curve

$\mathcal{S} = \{(\kappa(t), \kappa_s(t))\} \subset \mathbb{R}^2$ signature curve

Theorem. Two curves are equivalent

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical

$$\overline{\mathcal{S}} = \mathcal{S}$$

Theorem. The following are equivalent:

- $\mathcal{S}$ degenerates to a point
- $\mathcal{C}$ has constant curvature
- $\mathcal{C} = \{\exp(t\mathbf{v})x_0\}$ is the orbit of a one-parameter subgroup

Symmetry

Signature map

$$\Sigma \; : N \quad \longrightarrow \quad \mathcal{S}$$

One-point planar curves:

$$\Sigma(z) = (\kappa(z), \kappa_s(z)), \qquad z \in N = \mathcal{C}$$

Multi-point planar curves:

$$\Sigma : \; N = \mathcal{C}^{\times k} \quad \longrightarrow \quad \mathcal{S}$$

Theorem. Let $\mathcal{S}$ denote the signature of the submanifold N. Then the dimension of the symmetry group G of N equals

$$\dim G \;=\; \dim N - \dim \mathcal{S}$$

Maximally Symmetric Curves

Theorem. The following are equivalent:

- The (one-point) signature curve $\mathcal{S}$ degenerates to a point
- The curve $\mathcal{C}$ has constant curvature
- $\mathcal{C} = \{\exp(t\mathbf{v})x_0\}$ is the orbit of a one-parameter subgroup

$\Longrightarrow$ In Euclidean geometry, these are the circles and straight lines.

$\Longrightarrow$ In equi-affine geometry, these are the conic sections.

Discrete Symmetries

Definition. The *index* of a submanifold N equals the number of points in $\mathcal{C}$ which map to a generic point of its signature $\mathcal{S}$:

$$\iota_N = \min \left\{ \# \, \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \right\}$$

$\Longrightarrow$ Self–intersections

Theorem. The cardinality of the symmetry group of N equals its index ι_N.

$\Longrightarrow$ Approximate symmetries

Joint Euclidean Signature

$$z_0, z_1, z_2, z_3 \in \mathcal{C}$$

Joint invariants:

$$a = \| z^1 - z^0 \|, \qquad b = \| z^2 - z^0 \|, \qquad c = \| z^3 - z^0 \|,$$
$$d = \| z^2 - z^1 \|, \qquad e = \| z^3 - z^1 \|, \qquad f = \| z^3 - z^2 \|.$$

$\Longrightarrow$ six functions of four variables

Joint Signature: $\quad \Sigma : \mathcal{C}^{\times 4} \longrightarrow \mathcal{S} \subset \mathbb{R}^6$

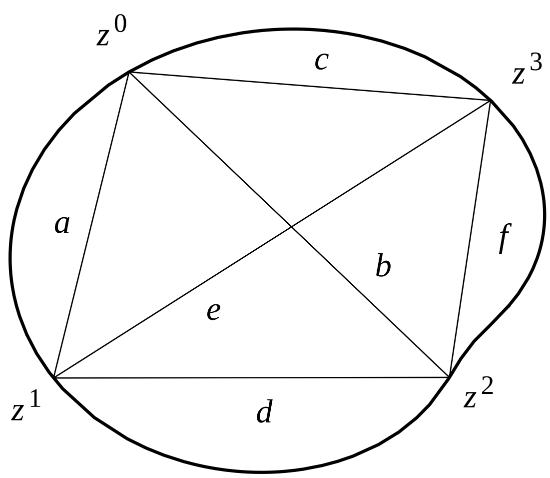
$\dim \mathcal{S} = 4 \quad \Longrightarrow \quad$ two syzygies

$$\Phi_1(a, b, c, d, e, f) = 0, \qquad \Phi_2(a, b, c, d, e, f) = 0,$$

Universal Cayley–Menger syzygy:

$$\det \begin{vmatrix} 2a^2 & a^2 + b^2 - d^2 & a^2 + c^2 - e^2 \\ a^2 + b^2 - d^2 & 2b^2 & b^2 + c^2 - f^2 \\ a^2 + c^2 - e^2 & b^2 + c^2 - f^2 & 2d^2 \end{vmatrix} = 0$$

$\Longleftrightarrow \quad \mathcal{C} \subset \mathbb{R}^2$



Four-Point Euclidean Joint Signature

Invariant Numerical Approximations

Basic Idea:

Every invariant finite difference approximation to a differential invariant must expressible in terms of the joint invariants of the transformation group.

Note: Joint (Semi–) differential invariants must ultimately be approximated by discrete numerical approximations, and so also reduce to joint invariants in any finite difference approximation scheme.

$\Longrightarrow$ *Approximate differential invariants by joint invariants*

Three point approximation to Euclidean curvature

Heron's formula

$$\widetilde{\kappa}(A,B,C) = 4\,\frac{\Delta}{abc} = 4\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}$$

$$s = \frac{a+b+c}{2}$$

Expansion:

$$\widetilde{\kappa} = \kappa + \frac{1}{3}(b-a)\frac{d\kappa}{ds} + \frac{1}{12}(b^2 - ab + a^2)\frac{d^2\kappa}{ds^2} +$$
$$+ \frac{1}{60}(b^3 - ab^2 + a^2 b - a^3)\frac{d^3\kappa}{ds^3} +$$
$$+ \frac{1}{120}(b-a)(3b^2 + 5ab + 3a^2)\kappa^2\frac{d\kappa}{ds} + \cdots .$$

Higher order invariants

$$\kappa_s = \frac{d\kappa}{ds}$$

Invariant finite difference approximation:

$$\widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}) = \frac{\widetilde{\kappa}(P_{i-1}, P_i, P_{i+1}) - \widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{\mathbf{d}(P_i, P_{i-1})},$$

Unbiased centered difference:

$$\widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_{i+2}) = \frac{\widetilde{\kappa}(P_i, P_{i+1}, P_{i+2}) - \widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{\mathbf{d}(P_{i+1}, P_{i-1})}.$$

Better approximation (M. Boutin):

$$\widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}) = 3 \ \frac{\widetilde{\kappa}(P_{i-1}, P_i, P_{i+1}) - \widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{\mathbf{d}_{i-2} + 2\mathbf{d}_{i-1} + 2\mathbf{d}_i + \mathbf{d}_{i+1}},$$

$$\mathbf{d}_j = \mathbf{d}(P_j, P_{j+1})$$

Affine Joint Invariants

$$\mathbf{x} \to A\mathbf{x} + b \qquad \det A = 1$$

Area is the fundamental joint affine invariant

$$\begin{aligned}
[ijk] &= (P_i - P_j) \wedge (P_i - P_k) \\
&= \det \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix} \\
&= \text{Area of parallelogram} \\
&= 2 \times \text{Area of triangle } \Delta(P_i, P_j, P_k)
\end{aligned}$$

Syzygies:

$$[ijl] + [jkl] = [ijk] + [ikl],$$

$$[ijk][ilm] - [ijl][ikm] + [ijm][ikl] = 0.$$

Affine Differential Invariants

Affine curvature

$$\kappa = \frac{3u_{xx}u_{xxxx} - 5u_{xxx}^2}{9(u_{xx})^{8/3}}$$

Affine arc length

$$ds = \sqrt[3]{u_{xx}}\; dx$$

Higher order affine invariants:

$$\kappa_s = \frac{d\kappa}{ds} \qquad \kappa_{ss} = \frac{d^2\kappa}{ds^2} \qquad \ldots$$

Conic Sections

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

Affine curvature:

$$\kappa = \frac{S}{T^{2/3}}$$

$$S = AC - B^2 = \det \begin{vmatrix} A & B \\ B & C \end{vmatrix}$$

$$T = \det \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}$$

Ellipse:

$$\kappa = (\pi / \mathbf{A})^{2/3}$$

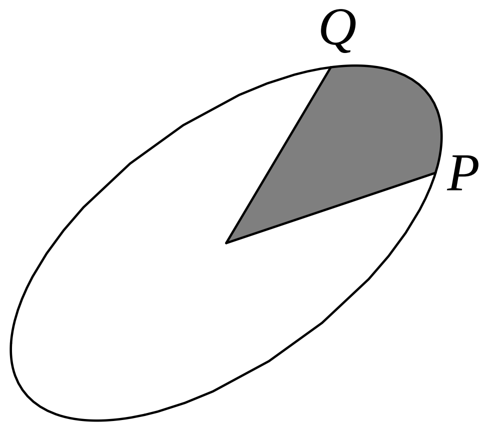
$$\mathbf{A} = \pi \frac{T}{S^{3/2}} = \text{Area}$$

Affine arc length of ellipse:

$$\int_P^Q ds = \frac{T^{1/3}}{S^{1/2}} \arcsin \sqrt{\frac{-CT}{S^2}} \left(x + \frac{CD - BE}{S} \right) \Bigg|_P^Q$$

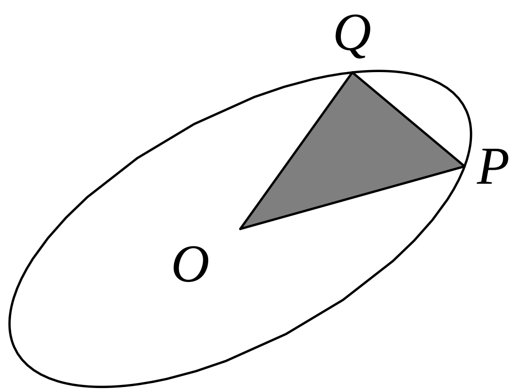
$$= 2ST^{-2/3}\mathbf{A}(P, Q)$$

$\mathbf{A}(P, Q)$:



Triangular approximation:

$\Delta(O, P, Q)$:



Total affine arc length:

$$\mathbf{L} = 2\sqrt[3]{\mathbf{A}} = -2\pi\,\frac{\sqrt[3]{T}}{\sqrt{S}}$$

Conic through five points $P_0, \ldots, P_4$:

$$[013][024][\mathbf{x}12][\mathbf{x}34] = [012][034][\mathbf{x}13][\mathbf{x}24]$$

$$\mathbf{x} = (x, y)$$

Affine curvature and arc length:

$$\kappa = \frac{S}{T^{2/3}}$$

$$ds = \text{Area } \Delta(O, P_1, P_3) = \tfrac{1}{2}[O, P_1, P_3] = \frac{N}{2S}$$

$$4T = \prod_{0 \leq i < j < k \leq 4} [ijk]$$

$$\begin{aligned} 4S = {} & [013]^2[024]^2([124] - [123])^2 + \\ & + [012]^2[034]^2([134] + [123])^2 - \\ & - 2[012][034][013][024]([123][234] + [124][134]) \end{aligned}$$

$$\begin{aligned} 4N = {} & - [123][134]\,\{[023]^2[014]^2([124] - [123]) + \\ & + [012]^2[034]^2([134] + [123]) + \\ & + [012][023][014][034]([134] - [123])\} \end{aligned}$$

Theorem. P_0, P_1, P_2, P_3, P_4 — points on the convex curve $\mathcal{C}$.

κ — affine curvature of $\mathcal{C}$ at P_2

$\widetilde{\kappa} = \widetilde{\kappa}(P_0, P_1, P_2, P_3, P_4)$

affine curvature of conic

$L_i = \int_{P_2}^{P_i} ds$

affine arc length of conic

Expansion:

$$\widetilde{\kappa} = \kappa + \frac{1}{5}\left(\sum_{i=0}^{4} L_i\right)\frac{d\kappa}{ds} + \frac{1}{30}\left(\sum_{0 \le i \le j \le 4} L_i L_j\right)\frac{d^2\kappa}{ds^2} + \cdots .$$