

Geometric Partial Differential Equations and Image Analysis*

Guillermo Sapiro
University of Minnesota
Short Course on
Mathematical Methods in Speech and Image Analysis
Institute of Mathematics and It's Applications
University of Minnesota

September 14th, 2000

Here we will discard the previous train of thoughts that only way to do image analysis is to think in the “discrete” frame point. Instead we will try to think of these problems as continuous, and then solve these problems, and then figure out how to figure out the discrete properties that the computers limit us. We will attack this problem in a series of steps: curves, curves & images, images, Vector Images and Geoneal Manifolds.

1 Curves & Images

The typical structure to the solution of this problem is to remove noise with a linear filter such

$$C(P) : R_{[0,1]} \rightarrow \mathbb{R}^2 \quad (1)$$

we estimate this mapping, or filter as $\hat{C}, \sigma = C(p, 0)Filter(p, \sigma)$ where σ denotes the amount of noise to remove. Suppose that we have two filters $C(p, 0) \rightarrow C(P, \sigma_1)$ and $C(p, 0) \rightarrow C(P, \sigma_2)$ then we should expect that if $\sigma_2 > \sigma_1$ then $C(p, \sigma_1) \rightarrow C(P, \sigma_2)$. This is the scale space where it is a memoryless system. This was the first requirement that was needed.

Other requirements were added such as shift and scale invariant. It can be shown that only a Gaussian filter satisfies these requirements. This was stated in 1983. The gaussian filter is denoted as $G(0, \sigma)$. This idea was explored by looking at multiple version of the pictures that have been smoothed to various degrees, hence getting rid of more and more noise but blurring more and more details.

*Thanks to Brad Love for typing up these notes, his trip was supported by the Max Planck Institute for Demographic Research, located in Rostock, Germany.

It was commented that if you are going to filter a picture, $C(p, \sigma) = C(p, 0) \otimes G(0, \sigma)$ is equivalent to solving the heat flow equation,

$$\frac{\partial C}{\partial t} = \Delta C \quad (2)$$

where $C(x, y)$ satisfies

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial p^2} \quad (3a)$$

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial p^2} \quad (3b)$$

From these equations we see that maximal principle is equivalent to a scale-space. This suggests that any PDE can translate into a state-space and to analysis images. The goal of this method was to get rid of the assumption of linearity, i.e. only using the Gaussian filter, and this convolution method gives rise to many other PDE solutions.

Why do we need more? Well consider an image as,

$$I(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R} \quad (4)$$

or with noise we would have,

$$I(x, y, \sigma) = I(x, y) * G(x, y, \sigma) \quad (5)$$

or in other words solving the following equation,

$$\frac{\partial I}{\partial t} = \Delta I = \frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} \quad (6)$$

The problem of course, is that we get rid of noise, but we lose information as well. Note that we apply the heat equation as given in (3) with the parametrization $C(p) = [x(p), y(p)]$. Without this parametrization, then the equation can not be applied. The heat equation is not intrinsic to the image.

Thus if many people were to take the same picture and apply the heat equation to the picture, they would all have a different picture, since the parameterization would lead to different results. But if a parameterization is agreed upon, say $C(p, 0)$ is the arc length say $S_{C(p)}$, then we will get the same result. Then the heat equation can be boiled down to,

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial S_{C(p)}^2} \quad (7)$$

but this violates the assumptions of the State-Space, in attempt to nail down the parametrization problem. To solve both of these problems we re-express the above equation as,

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial S_{C(t)}^2} \quad (8)$$

therefore getting,

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial S^2} = K \vec{N} \quad (9)$$

called the geometric heat flow. It can be proved that this equation can define a state-space.

Another formula that becomes important, if we are discussing moving curves in time, then a basic formula is given by,

$$\frac{\partial C}{\partial t} = \vec{N} \quad (10)$$

This equation basically states that a curve is moving inwards or outwards at a constant velocity. But this method can create shocks if run long enough by Huygen's principle. These shocks are important for defining skeletons.

These two basic equations are complementary together. If we only want to look at moving curves in binary pictures, then we will get a lot of mileage out of these equations.

Things become more interesting when we want to move curves on non-binary curves. Then we might consider the curves as given by (10) will continue to go inwards, but if we want the curves to stop at the edges of a curve, we want to stop the curves at the edges then we will use the following manipulation of (10),

$$\frac{\partial C}{\partial t} = gk\vec{N} + g\vec{N} + \dots \quad (11)$$

where the external force $g \propto |\nabla I|$.

Now what happens if we move a Gaussian filter in time? Since images can be expressed $I(x, y) : R^2 \rightarrow R$ and apply the heat equation,

$$\frac{\partial I}{\partial t} = \Delta I \quad (12)$$

Gabor suggested in the 70's that

$$\frac{\partial I}{\partial t} = \text{div}(g\nabla I) \quad (13)$$

where details are not smoothed out, but noise is. In the 80's, Perona and Malik suggested the same equation, without knowing of the paper by Gabor. At the same time Rabor and Oster also presented similar ideas in in Rabor's dissertation at Cal Tech. They suggested using the following functions for looking at the boundary,

$$g = \frac{1}{\epsilon + |\nabla I|} \quad (14a)$$

or

$$g = \rho^{-|\nabla I|} \quad (14b)$$

Thus using the isotropic equation as given in (3) loses a lot of information but the adjusted heat equation, the anisotropic heat equation above denoises an image and retains the details.

The link is that the gaussian filter is related to the heat flow, and the heat flow is related to $\in |\nabla I|^2$ via the gradient descent of energy. Since gaussian filter is a PDE, it is actually a variational problem. Making all of these processes related to each other. If we put an L^1 norm then we get the following,

$$\frac{\partial I}{\partial t} = \text{div} \left(\frac{\nabla I}{|\nabla I|} \right) \quad (15)$$

This gives the nice property that the L^1 norm is more robust than the L^2 norm. So a better norm,

$$\int \log(1 + |\nabla I|^2) \quad (16)$$

instead of the L^1 norm, but the variational equation that this solves is not straight forward.

Suppose that the observed image is given by $\hat{I} = I_{\text{original}} + \text{Noise}$ so you want to minimize the following

$$\min \int \phi(|\nabla I|) + \alpha \int (I - \hat{I})^2 \quad (17)$$

This idea can be complicated more by considering blurring that needs to be removed as well, suppose that the picture is not in focus then we can take care of this by considering the following equation representation $\hat{I} = I_{\text{original}} * G + \text{Noise}$ where G is the blurring effect or the non-focus effect.

Consider an image, and conceptually take all of the boundaries of the image as curves and move them to the snakes. If this is done for all of the boundaries then this is the same as applying the isotropic heat equation to all of the boundaries at the same time, hence the loss of the information of the details.

Consider the following equation,

$$\frac{\partial C}{\partial t} = k \vec{N} \quad (18)$$

but here we will put the following conditions on the boundaries and the smoothing direction,

$$\begin{aligned} \frac{\partial I}{\partial t} &= \frac{\partial^2 I}{\partial \xi^2} \\ &= \frac{I_x^2 I_{yy} - 2I_x I_y I_{xy} + I_y^2 I_{xx}}{|\nabla I|^3} \end{aligned} \quad (19)$$

this equation is got by making the following assumptions that $\xi \perp \frac{\nabla I}{|\nabla I|}$ and define $\rho_\lambda = \{(x, y) : I(x, y) = \lambda\}$ as the level lines. So this gives rise to looking at,

$$\frac{\partial \phi}{\partial t} = \beta |\nabla \phi| \quad (20a)$$

instead of,

$$\frac{\partial C}{\partial t} = \beta \vec{N} \quad (20b)$$

thus if we apply this idea to each individual level line, and then reconstruct the image, this is the same as running the original equation on the entire picture at the same time.

2 Vector Images

Suppose that we have an image, $I : R^2 \rightarrow R^n$ i.e. scalar to the vector case. The anitropic equation that was used above is given by,

$$\frac{\partial I}{\partial t} = \text{div} (g(|\nabla I|)\nabla I) \quad (21)$$

and the A.M.L. is given from,

$$\frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial \xi^2} \quad (22)$$

when $\xi \perp \frac{\nabla I}{|\nabla I|}$. Then we can look at the gradient, i.e. $\nabla I = (I_x, I_y)$ with the same intreptation, that the gradient will find the direction of the quickest jump to decrease. We then can calculate this by calculating the matrix,

$$\begin{pmatrix} I_x \cdot I_x & I_x \cdot I_y \\ I_x \cdot I_y & I_y \cdot I_y \end{pmatrix} \quad (23)$$

and find the eigen values and vectors of the above matrix. The eigen vectors are going to be orthogonal. We can use the snake methods using the eigen values as stopping functions.

3 Contrast Enhancement

Although we are mainly focusing on problems of denoising or deblurring, but other directions are available. This type of problem is similar to saying that here is a problem, what PDE solves it, versus deriving PDE's to address imaging. Consider the problem of contrast enhancement. This can be argued to be a more important problem than denoising. This is because with denoising, the picture can still be made out, although if there picture is lost to noise no image techniques will recover the picture, but contrast enhancement will take a picture that often takes nothing into something. The goal is not to make the picture nicer, but to make the picture recognizable. A pretty picture is not the goal. Contrast enhancement, is a way of stretching the pixels from the coverage that are observed to all of the pixels possible. There are many techniques for this, such as a uniform stretch. One can also look at the histogram distribution of the pixels, and then transform the distribution into the desired distribution, i.e. this is called global histogram modification. But small patterns are hurt or lost in the enhancement. So

address this problem we will try doing local histogram modification, so that small objects are not lost.

PDE's allow the user to look at objects that are not squares, rather connected objects, and stretching will stretch these objects in the same manner, so that they are not lost. These techniques are called shape preserving contrast enhancement.

4 Image Inpainting

Image inpainting is fundamentally different than denoising since denoising is information plus noise, but inpainting is missing information. So how can we correct for this, but still maintain the picture? Furthermore we can use similar techniques to remove objects from the picture and replace the background.

Of course these techniques are often done in the movie industry. This is easily done in movies so that we basically copy the missing part of the frame before and after and put it to the missing picture. We can also replace unwanted information in a series of pictures. We can also try to fill in the wanted area by texture copying.

We can do these techniques by using the propagation information

$$\nabla L \cdot \vec{N} = 0 \quad (24)$$

and the evolutionary form,

$$\frac{\partial I}{\partial t} = \nabla L \cdot \vec{N} \quad (25)$$

and then we can fill in the information. Here L is the smoothness estimator the Laplacian and N is the isophote direction and it is time variant and is the normal vectors. So we use,

$$\frac{\partial I}{\partial t} = \nabla(\Delta I) \cdot \nabla^\perp I \quad (26)$$

plus numerical schemes by Osher and with boundary conditions.

An extension is the variational form

$$\min_{u, \theta} \int_{\Omega \cup \text{Band}} \text{div}(\theta)^p (a + b \|\nabla * u\|) + c(\|\nabla u\| - \theta \cdot \nabla u) \quad (27)$$

solved via E-L: coupled 2nd order PDE's, and the full theory is given. similar practical results than 3rd order equations.

5 Variational formulations and PDE's on or onto Implicit Surfaces, A manifold approach

Solving PDE's and variational problems for data defined from a generic surface onto a generic surface. Variational problems and PDE's on the surface? You get, and it may spell the end to triangulated surfaces solutions. The main goal is to solve the PDE's and variational problems to fix the data defined from a generic surface onto to a generic surface. Consider the following problems:

- Mean curvature motion (Ilmanen,etc.)
- Mathematical Physics
- Computer Graphics
- Image processing
- Regularized of inverse problems (e.g. EEG+MRI, e.g. Faugeras
- 3-D surface mapping

The classical approach to these types of problems, is to work with triangulated/meshed surfaces and doing discretization on non-uniform grids and projections onto triangulated surfaces. Doing this limits the user to functions (e.g. Kimmel) and has a very limited framework. Work on the surface is mapped to the plane, but loose geometry adds to the complexity to the problem. Furthermore no work on target surfaces reported. In the purposed PDE approach, will try to address these problems.

Here we will look at variational problems and PDE's through the theory of Harmonic maps, which is a well defined framework. This represents the surface in implicit form. There is work in classical numerics on Cartesian grids, no projections. These problems have been motivated by Osher-Sethian level sets and Osher variational level-sets.

First we have to find the map between manifolds (M, g) and (N, h)

$$\min_{I:M \rightarrow N} \int_M \|\nabla_M I\|^p dvol_M \quad (28)$$

with the graident descent (p=2) is given by,

$$\frac{\partial I}{\partial t} = \Delta_M I + A_N(I) \langle \nabla_M I, \nabla_M I \rangle \quad (29)$$

this is the Laplace-Bethrami second fundamental form, one can see further examples in Perona, Chan-Shen, Sochen et al., Hoppe et al, Zorin et al., etc.. We can use this equation to denoise color images.

Here we need to think about the embedding, so we have the framework $I : M \rightarrow N$ where M is a generic surface, the domain (Bertalmino, Cheng, Osher), and N is a generic surface, the target, (Memoli, Osher).

We will start embedding the domain surface, the simplist problem to tackle is running the heat equation on the surface. A map from a generic domain surface $I : M \rightarrow R$, where we look at,

$$\min_{I:M \rightarrow R} \int_M \|\nabla_M I\|^2 dvol_M \quad (30)$$

and

$$\frac{\partial I}{\partial t} = \Delta_M I \quad (31)$$

where M is the level set of $\Psi = \{x : \Psi(x) = 0\}$.

We need to solve the following problem,

$$\begin{aligned} \int_M \|\nabla_M I\|^2 dvol_M &= \int_M \|P_{\nabla\Psi} \nabla I\|^2 dvol_M \\ &= \int_{R^3} \|P_{\nabla\Psi} \nabla I\|^2 \delta(\Psi) \|\nabla\Psi\| dx \end{aligned} \quad (32)$$

to take the intrinsic gradient, take the gradient in R^3 then project to the tangent plane, to the level sets, so that the derivatives are taken in Euclidean space.

The do the gradient descent flow, i.e. the heat flow on an intrinsic surfaces,

$$\frac{\partial I}{\partial t} = \frac{1}{\|\nabla\Psi\|} div (P_{\nabla\Psi} \nabla I \|\nabla\Psi\|) \quad (33)$$

all the computations are done on the cartesian grid. The PDE should only run in a band around the surface, since only needed results are on the surface, then extend the results to the surrounding surface using normals to project out.

To denosing on implicit surfaces with an L^1 norm requires looking at,

$$\begin{aligned} \int_M \|\nabla_M I\| dvol_M &= \int_M \|P_{\nabla\Psi} \nabla I\| dvol_M \\ &= \int_{R^3} \|P_{\nabla\Psi} \nabla I\| \delta(\Psi) \|\nabla\Psi\| dx \end{aligned} \quad (34)$$

and we get the following gradient descent flow,

$$\frac{\partial I}{\partial t} = \frac{1}{\|\nabla\Psi\|} div \left(\frac{P_{\nabla\Psi} \nabla I}{\|P_{\nabla\Psi} \nabla I\|} \|\nabla\Psi\| \right) \quad (35)$$

We can also get the unit vector/color denoising on implicit surfaces, consider that I is a map from the 3-D surface to the 3-D unit sphere

$$\frac{\partial I}{\partial t} = \frac{1}{\|\nabla\Psi\|} div \left(\frac{P_{\nabla\Psi} \nabla I}{\|P_{\nabla\Psi} \nabla I\|} \|\nabla\Psi\| \right) + I \|P_{\nabla\Psi} \nabla I\| \quad (36)$$

Now we can also consider a pattern formulation on implicit 3-D surfaces, which follow Turing, Kass-Witkin, Turk

$$\frac{\partial a}{\partial t} = f(a, b) + \alpha \Delta_M a \quad (37a)$$

$$\frac{\partial b}{\partial t} = g(a, b) + \beta \Delta_M b \quad (37b)$$

and then modified to

$$\frac{\partial a}{\partial t} = f(a, b) + \alpha \frac{1}{\|\nabla\Psi\|} \|P_{\nabla\Psi} \nabla I\| \quad (38a)$$

$$\frac{\partial b}{\partial t} = g(a, b) + \beta \frac{1}{\|\nabla \Psi\|} \|P_{\nabla \Psi} \nabla I\| \quad (38b)$$

Vector field visualization on a surface, a classical problem, that can be easily done within this framework.

Embedding in the target manifold, so again $I : M \rightarrow N$, the let N be the level set of $\phi = \{x : \phi(x) = 0\}$. Where we,

$$\min_{I: M \rightarrow \{\phi=0\}} \int_M \|\nabla_M I\|^2 dx \quad (39)$$

and the equation gives,

$$\frac{\partial I}{\partial t} = \Delta_M I + A_{\{\phi=0\}}(I) \langle \nabla_M I, \nabla_M I \rangle \quad (40)$$

so we must minimize,

$$\frac{\partial I}{\partial t} = \Delta_M I + \left(\sum_k H_{\Phi} \left\langle \frac{\partial I}{\partial x_k}, \frac{\partial I}{\partial x_k} \right\rangle \right) \|\nabla I\| \quad (41)$$

where H denotes the Hessian

Concluding remarks are that there is no more need for triangulated surfaces for variational problems and PDE's. The results are locally independent of embeddin function. The extended to open domain and target surfaces. There are of course many open theory problems left to solve.