Optimization models for the financial valuation of supply chain risks

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Supply Chain Risk

- Demand/supply is variable in time, quantity, and quality.
- Inventories accumulate — product, equipment, personnel, even financial obligations.
- Profits are variable, ie RISKY.

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Semiconductor Fab:

- 0-3 month WIP forecast has st. dev. $\sim 75\%$ of mean, and order/qualify machine tools takes 1-2 years.
- How to measure and compensate Fab's risk exposures?

Financial Valuation

Basic approach to financial valuation:

- Observe market prices in related contracts.
- Hedge contract risk by trading “equivalents”.
- Market price of risk ($\sim$ shareholders’ risk).

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Complications for Supply Chain risk valuation:

- Role of “non-market” information.
- Non-financial incentives and risks (market share, service-levels,...)
- Negotiation.
- Meta-effects: state-transitions due to scale, chunking and complexity, stochastics, industrial organization, etc.

Outline

OBJECTIVE: Present *optimization approach* to financial valuation and explore its use as tool in valuation of supply chain risks.

- “Supply chain” view of options pricing.
- Application to pricing risk in quantity-flexible contracting.
- Calibration to market.
- Final words

Options in Finance

Options are used to manage financial risks...
can one apply similar techniques to supply chain risk?

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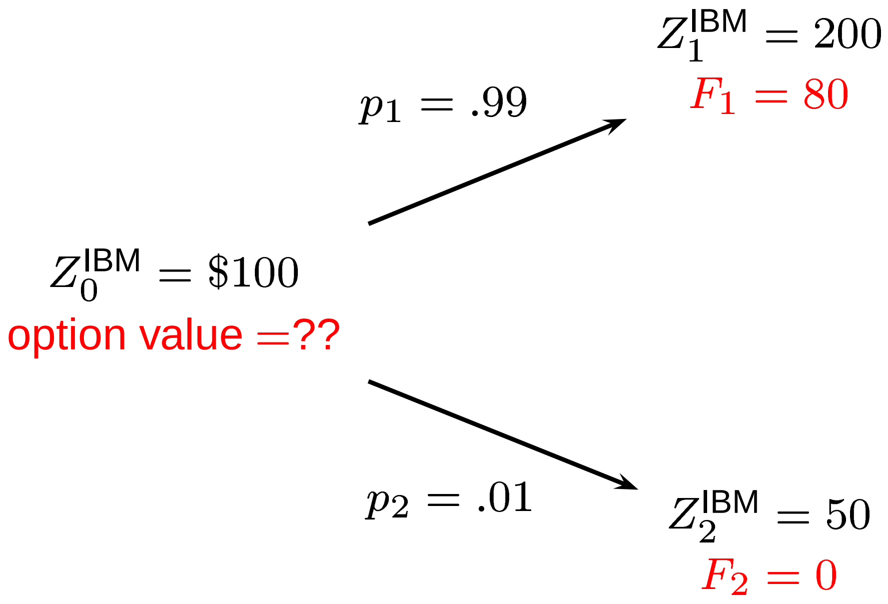
Supply chain view of options pricing theory

- Option “payout” is function of “underlying”
- Trading in underlying can “(super)replicate” option payouts
- Price of option equals “cost of manufacturing” option payouts through trading.

Simple Options Pricing Example

- IBM stock price takes values $\{Z_n^{\text{IBM}}\}$
- IBM call option w/strike \$120 pays $F_n = [Z_n^{\text{IBM}} - 120]_+$



$Z_1^{\text{IBM}} = 200$

$F_1 = 80$

$p_1 = .99$

$Z_0^{\text{IBM}} = \$100$

option value $=$??

$p_2 = .01$

$Z_2^{\text{IBM}} = 50$

$F_2 = 0$

Key Idea (Black-Scholes-Merton 1972/3)

Option price equals cost of trading strategy that “replicates” payoffs $F_n = [Z_n^{\mathsf{IBM}} - 120]_+$

Formulation as LP:

asset	*price*	*position*
stock:	Z_n^{IBM}	θ_n^{IBM}
bond:	$Z_n^{\mathsf{B}} = 1$	θ_n^{B}

$$\begin{array}{llll} \min\limits_{(\theta)} & F_0 & & \\ \texttt{st:} & Z_0 \cdot \theta_0 & = F_0 & \\ & Z_n \cdot \theta_0 & \geq F_n & (n = 1, 2) \end{array}$$

Dual of LP (Ross, 1996)

$$
\begin{array}{rlll}
\max_{(q)} & \sum_{n=1}^{2} q_n F_n & & \\
\texttt{st:} & q_0 & = 1 & \\
 & q_n & \geq 0 & (n = 1, 2) \\
 & q_0 & = \sum_{n=1}^{2} q_n & \text{(bond)} \\
 & q_0 Z_0^{\text{IBM}} & = \sum_{n=1}^{2} q_n Z_n^{\text{IBM}} & \text{(stock)}
\end{array}
$$

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- q are the weights of a probability measure Q
- Q makes stock price $\{Z_n^{\text{IBM}}\}$ into “martingale”

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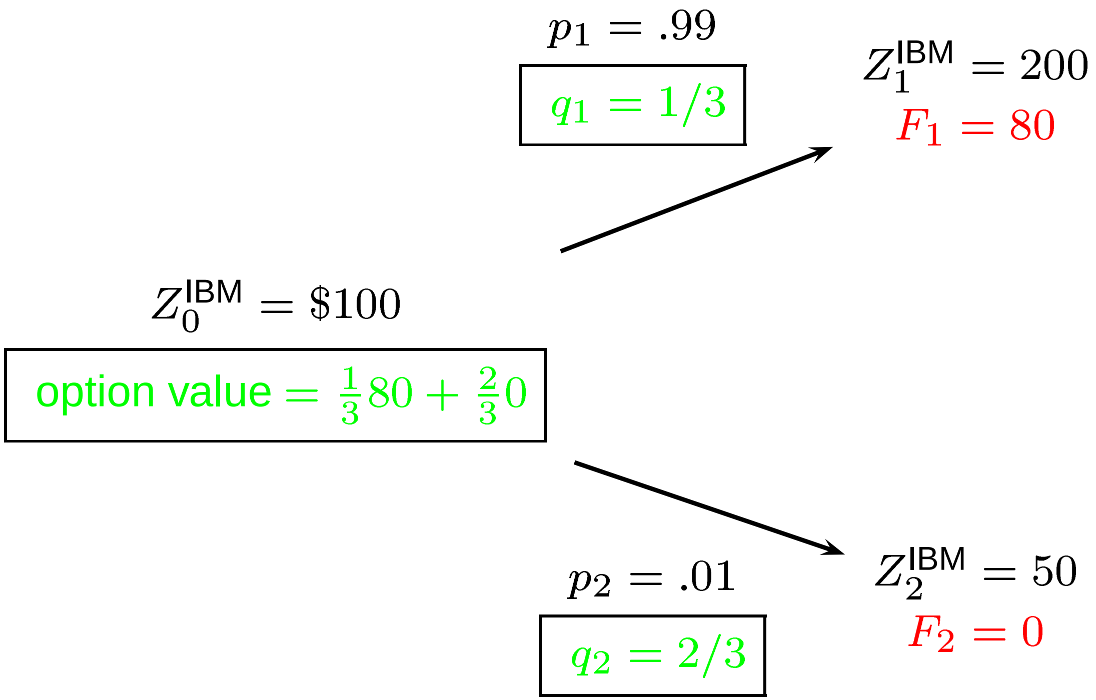
- q are the weights of a probability measure Q
- Q makes stock price $\{Z_n^{\text{IBM}}\}$ into “martingale”
- Option value is $\max_Q \ E^Q [F_T]$

Martingale for Simple Example

There is only one possible solution to Dual.



$p_1 = .99$

$\boxed{q_1 = 1/3}$

$Z_1^{\text{IBM}} = 200$

$F_1 = 80$

$Z_0^{\text{IBM}} = \$100$

$\boxed{\text{option value} = \frac{1}{3}80 + \frac{2}{3}0}$

$p_2 = .01$

$\boxed{q_2 = 2/3}$

$Z_2^{\text{IBM}} = 50$

$F_2 = 0$

Replicating Portfolio for Simple Example

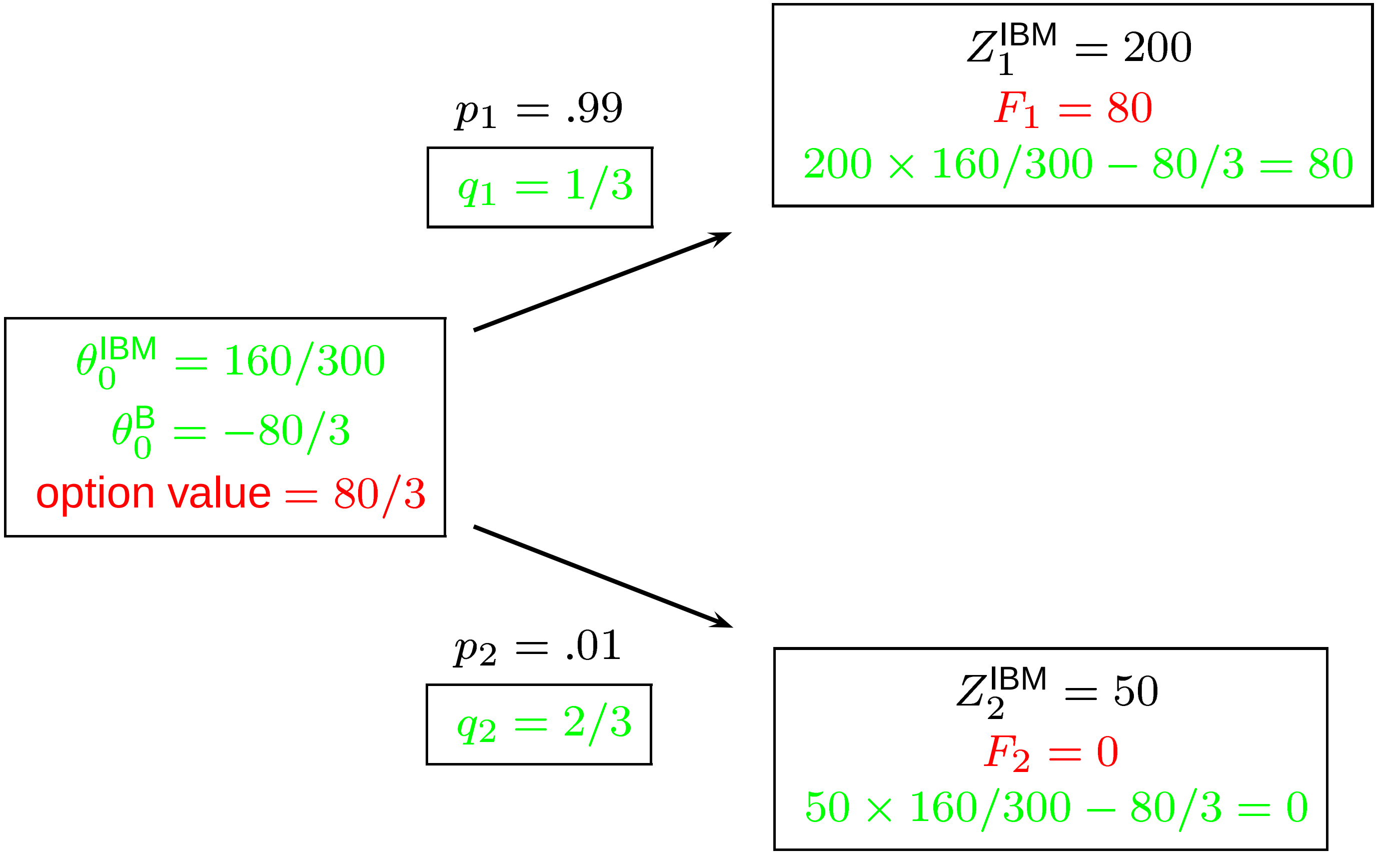
Determine replicating portfolio from knowledge of dual optimal value

$$
\begin{aligned}
\theta_0^{\mathrm{B}} + 100\ \theta_0^{\mathrm{IBM}} &= 80/3 \\
\theta_0^{\mathrm{B}} + 200\ \theta_0^{\mathrm{IBM}} &\geq 80 \qquad (n=1) \\
\theta_0^{\mathrm{B}} + 50\ \theta_0^{\mathrm{IBM}} &\geq 0 \qquad (n=2)
\end{aligned}
$$

Solution:

asset	*position*
stock:	$\theta_0^{\mathrm{IBM}} = 160/300$ shares
bond:	$\theta_0^{\mathrm{B}} = -80/3$ loan

Replicating Portfolio for Simple Example



$\theta_0^{\text{IBM}} = 160/300$

$\theta_0^{\text{B}} = -80/3$

option value $= 80/3$

$p_1 = .99$

$q_1 = 1/3$

$Z_1^{\text{IBM}} = 200$

$F_1 = 80$

$200 \times 160/300 - 80/3 = 80$

$p_2 = .01$

$q_2 = 2/3$

$Z_2^{\text{IBM}} = 50$

$F_2 = 0$

$50 \times 160/300 - 80/3 = 0$

Using Stochastic Programming to (Super-)Replicate Claims

General setup for discrete probability space

- Vector stochastic price process $\{Z_t\}_{t=0}^{T}$, with $Z_t^0 := 1$ as the “bond”.
- Distinct paths at time t correspond to nodes $n \in \mathcal{N}_t$, with *parent node* $a(n) \in \mathcal{N}_{t-1}$ and *child nodes* $\mathcal{C}(n) \subset \mathcal{N}_{t+1}$.
- Expectation: $E^Q\,[\,Z_t\,] := \sum_{n \in \mathcal{N}_t} q_n Z_n$

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Optimization model (super-)replicates cash flows F_n at minimum price

- portfolio holdings θ_n
- trades $\Delta\theta_n := \theta_n - \theta_{a(n)}$.

$$\boxed{\begin{array}{llll} \min\limits_{(\theta)} & F_0 & & \\ \texttt{st:} & Z_0 \cdot \theta_0 & = F_0 & \\ & Z_n \cdot \Delta\theta_n & = -F_n & (n \in \mathcal{N}_t, t \geq 1) \\ & Z_n \cdot \theta_n & \geq 0 & (n \in \mathcal{N}_T) \end{array}} \qquad (1)$$

This is called a “stochastic (linear) program”.

Fundamental Pricing Theorem

Theorem 1 *There is a trading strategy that super-replicates F_n iff there exists a martingale probability measure Q, in which case the option price is*

$$F_0^* = \max_{Q \in \mathit{MPM}} E^Q \left[\sum_{t=1}^{T} F_t \right] \tag{2}$$

Proof is by strong LP duality. Here is the dual:

$$\begin{array}{lll} \max_{(q)} & \sum_{n=1}^{\mathcal{N}} q_n F_n & \\ \texttt{st:} & q_0 = 1 & \\ & q_n \geq 0 & (n \in \mathcal{N}_T) \\ & q_n Z_n = \sum_{m \in \mathcal{C}(n)} q_m Z_m & (n \in \mathcal{N}_t, t = 0, \ldots, T-1) \end{array}$$

Details in (King, 1998). Continuous state extensions in (King-Korf, 2002).

Options Pricing is a Dual Method

The basic method as outlined in Harrison-Pliska (1981):

1. Describe stochastic price process $\{Z_t\}$, with prices normalized so that the bond's value is 1 in all states.
2. Find a *dual solution* Q, a probability measure
   satisfying martingale equalities $Z_t = E^Q\,[\,Z_{t+1}|Z_t\,]$
3. Calculate payouts $\{F_t\}$
   — may need to know Q, eg, when F is an American-style option.
4. Option price is $F_0 = E^Q\,[\,\sum_t F_t\,]$
5. (Super-)Replicating portfolio is $\theta_0 = \partial_{Z_0} E^Q\,[\,\sum_t F_t\,]$

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This describes the *seller*, who receives F_0 in order to generate F_n.

. . . but what about the buyer?

The Arbitrage Interval

Theorem 2 *The maximum price that the buyer will pay for stochastic cashflows F_n is*

$$F_0^* = \min_{Q \in \mathit{MPM}} E^Q \left[\sum_{t=1}^{T} F_t \right] \tag{3}$$

Proof is by reversing signs in (2).

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Theorem 3 *The maximum price that the buyer will pay for stochastic cashflows* F_n *is*

$$F_0^* = \min_{Q \in MPM} E^Q \left[\sum_{t=1}^{T} F_t \right] \tag{4}$$

Proof is by reversing signs in (2).

Buyers and sellers have different prices:

- Seller's minimum offering price is $F_0^w := \max_{Q \in \text{MPM}} \; E^Q \left[\sum_{t=1}^{T} F_t \right]$
- Buyer's maximum acceptable price is $F_0^b := \min_{Q \in \text{MPM}} \; E^Q \left[\sum_{t=1}^{T} F_t \right]$
- One has $F_0^b < F_0^w$, unless Q is *unique*.

Arbitrage interval: $[F_0^b, F_0^w]$.

The Arbitrage Interval

Theorem 4 *The maximum price that the buyer will pay for stochastic cashflows F_n is*

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- One has $F_0^b < F_0^w$, unless Q is *unique*.

"Arbitrage interval": $[F_0^b, F_0^w]$.

. . . we conclude that neither seller nor buyer want to trade options!

Reasons to Trade Options

Let F_0 be the market price of the claim.

- Investor *buys* if $F_0 < E^Q\left[\sum_{t=1}^{T} F_t\right]$
- Investor *sells* if $F_0 > E^Q\left[\sum_{t=1}^{T} F_t\right]$

Buyers and Sellers have *different* martingale measures.

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Trading occurs because buyers and sellers differ:

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   . . . only makes the Arbitrage Interval *wider*.
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Trading occurs because buyers and sellers differ:

1. Transactions costs and/or taxes.
2. Views of the risk of the option.
   . . . no difference unless seller is willing to take a loss.
3. Endowments and liabilities.

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2. Views of the risk of the option.
3. Endowments and liabilities.

...the most important reasons to buy/sell options are

- *differences in endowments: initial portfolios*
- *differences in liabilities: future cash flows correlated with the value of the underlying*

Optimization Models for Options Pricing — Summary

This optimization approach to options pricing appears in *(King, Math. Programming, 2002)*

General features are:

1. Options can be replicated iff the price process supports a Martingale Probability Measure.
2. Buyers and Sellers of options must have different MPM
3. Differences in MPM arise from differences in endowments and future liabilities.

Quantity-Flexible Supply Contracts

Risk analysis of Quantity-Flexible Supply Contracts (Ahmed and King, 2002).

- QFS contract between single buyer, single seller.
- Fixed price per unit; quantity demanded is variable.
- Seller may pay penalties if out-of-stock.
- Examples: IBM Printer Division, Sun, HP.

Buyer pays “Franchise Fee” to compensate for seller’s risk and loss of pricing power.

...options pricing provides a guide to how to compute this fee.

Options Pricing Approach to QFS

Two parts to modeling QFS franchise fee:

1. Production to meet demand or pay unmet demand penalty
2. Trading in correlated securities to hedge risks

We will consider supplier's point-of-view, but for simplicity we model only capacity expansion decisions.

QFS Model Development

Production model:

Buyer's demand	$\{d_t\}_{t=1}^T$
Supplier's marginal profit	s_t
Shortage penalty	γ_t
Capacity expansion charge	α_t
Capacity maintenance	δ_t
Capacity expansion decision	X_t
Unmet demand variable	U_t

Market model: Z_t and θ_t, etc.

Supplier's Stochastic Program

$$\begin{array}{lll}
\min_{(\theta,X,U,V)} & V & \\
\texttt{st:} & & \\
& Z_0\theta_0 = V - \alpha_0 X_0 & \\
& Z_n\theta_n = Z_n\theta_{a(n)} + s_n d_n - \alpha_n X_n & \\
& \qquad\qquad -\gamma_n U_n - \delta_n \sum_{m\in\mathcal{A}(n)} X_m & n \in \mathcal{N}_t, 1 \le t \le T-1 \\
& Z_n\theta_n = Z_n\theta_{a(n)} + s_n d_n & \\
& \qquad\qquad -\gamma_n U_n - \delta_n \sum_{m\in\mathcal{A}(n)} X_m & n \in \mathcal{N}_T \\
& Z_n\theta_n \ge 0 & n \in \mathcal{N}_T \\
& \sum_{m\in\mathcal{A}(n)} X_m + U_n \ge d_n & n \in \mathcal{N}_t, 1 \le t \le T \\
& X_n \ge 0 & n \in \mathcal{N}_t, 0 \le t \le T-1 \\
& U_n \ge 0 & n \in \mathcal{N}_t, 1 \le t \le T
\end{array} \tag{6}$$

Martingale Assumption

Assume martingale measure Q is known.

Theorem 5 *The franchise fee is* $F_0 = \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} d_n [v_n^* - q_n s_n]$ *where* v^* *is the optimal solution of the linear program*

$$\begin{array}{lll} \max & \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} d_n v_n & \\ \text{s.t.} & \sum_{m \in \mathcal{D}(n)} v_n \le C(n) & n \in \mathcal{N}_t, 0 \le t \le T-1 \\ & 0 \le v_n \le D(n) & n \in \mathcal{N}_t, 1 \le t \le T \end{array} \tag{7}$$

with $C(n) := (q_n \alpha_n + \sum_{m \in \mathcal{D}(n)} q_m \delta_m)$*, and* $D(n) = q_n \gamma_n$.

Dual Algorithm for QFS

1. Initialize $v_n^* = 0$ for all n.
2. Sort $\{d_n\}$ such that $d_{n_1} > d_{n_2} > \cdots > d_{n_K}$.
3. Repeat the following steps for $k = 1, \ldots, K$:
    - Let $\mathcal{N}_k = \{n | d_n = d_{n_k}\}$,
    - Repeat the following steps until $\mathcal{N}_k = \emptyset$:

      For each $n \in \mathcal{N}_k$, calculate

      $$v'_n = \min\{D(n), \min_{m \in \P(n)} \{C(m) - \sum_{q \in \mathcal{D}(m)} v_q^*\}\}.$$

      Let $m = argmax_{n \in \mathcal{N}_k} \{v'_n\}$, set $v_m^* = v'_m$ breaking ties arbitrarily, set $\mathcal{N}_k \leftarrow \mathcal{N}_k \setminus \{m\}$.
4. Return v_n^*, optimal solution.

Primal view of QFS

Take dual of in “market” variables, to isolate role of martingale measure

$$
\begin{array}{lll}
\min_{X,U} & \max_{q\in\mathcal{Q}} & \beta_0 C_0 X_0 + \sum_{t=1}^{T}\sum_{n\in\mathcal{N}_t} q_n[C_n X_n + \gamma_n U_n - s_n d_n] \\
\text{s.t.} & & \\
& & \sum_{m\in\mathcal{A}(n)} X_m + U_n \geq d_n \qquad n\in\mathcal{N}_t, 1\leq t\leq T \\
& & X_n \geq 0 \qquad n\in\mathcal{N}_t, 0\leq t\leq T-1 \\
& & U_n \geq 0 \qquad n\in\mathcal{N}_t, 1\leq t\leq T
\end{array} \tag{8}
$$

Shows role of Q in “risk-neutral discounting” of risky profits.

Note that fixing Q results in an *underestimate* of franchise fee.

Calibration of Martingale Measure

Where may we look for probability measure for this analysis?
... calibration (King and Penannen, 2002; see also Avellenadas and collaborators (1999-2002)) F^i, $i = 1, \ldots, k$ have bid/ask prices $F_b^i < F_a^i$, payoffs F_n^i.
Let L_n be the “liability” cash flow in state n.

$$
\begin{array}{llll}
\min_{q,v_+,v_-} & \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} q_n L_n & +(v_+ + v_-) & \\
\texttt{st:} & q_0 & = 1, & \\
 & q_n & \geq 0 & n \in \mathcal{N}_T, \\
 & \sum_{m \in \mathcal{C}(n)} y_m Z_m & = q_n Z_n & n \in \mathcal{N}_t,\ t = 1, \ldots, T-1, \\
 & \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} q_n F_n & \leq F_a + v_+, & \\
 & \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} q_n F_n & \geq F_b - v_-, & \\
 & v_+, v_- & \geq 0 &
\end{array}
$$

Primal View of Calibration Model

$$\begin{array}{lll} \min_{\theta,\xi_+,\xi_-} & V & \\ \texttt{st:} & Z_0 \cdot \theta_0 + (F_a \cdot \xi_+ - F_b \cdot \xi_-) & = V \\ & Z_n \cdot (\theta_n - \theta_{a(n)}) - F_n \cdot (\xi_+ - \xi_-) & = L_n \quad n \in \mathcal{N}_t,\ t = 1, \ldots, T, \\ & Z_n \cdot \theta_n & \geq 0 \qquad n \in \mathcal{N}_T, \\ & \xi_+, \xi_- & \in [0,1]. \end{array}$$

1. The primal hedges the liability with smallest cash.
2. Market-traded options can be bought/sold, but positions are bounded.
3. The dual variable is a martingale probability measure.
4. The dual maximizes the integrated cash flow of the liabilities plus penalties for market-traded option prices being too far over the ask or too far under the bid.

Combined Calibration and QFS Pricing

Investing in market traded options automatically calibrates Q — so long as these positions are bounded.

Surprise: there is no need to specify probability measure!

May not be able to correlate all QFS risks with market.

- Q is martingale on *market* $\mathcal{M} \subset \mathcal{N}$
- could average production flows conditional on $\mathcal{M}$ and solve discounted calibration/QFS problem
- or could simply fix $q_n = q_m / |\mathcal{N}_t(m)| \forall n \in \mathcal{N}_t(m)$ for all “market” nodes $m \in \mathcal{M}$

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