Complete Segal Spaces, Segal Categories, and S-Categories

Problem: Would like Quillen equivalences of model category structures between:

$$\begin{array}{c} \mathcal{S}\text{-Categories} \\ \uparrow\downarrow \\ \text{Segal categories} \\ \uparrow\downarrow \\ \text{Complete Segal spaces} \end{array}$$

Complete Segal Spaces

Definition: A bisimplicial set is a functor $X: \Delta^{op} \to SSets$.

Definition: Let W be a (Reedy fibrant) bisimplicial set. W is a Segal space if for each $k \geq 2$ the Segal map

$$W_k \longrightarrow \underbrace{W_1 \times_{W_0} \cdots \times_{W_0} W_1}_{k}$$

is a weak equivalence of simplicial sets.

Given a Segal space, we can apply "categorical" terms to it.

Let W be a Segal space.

Define $\mathrm{Ob}(W) = W_{0,0}$
(0-set of the simplicial set $W_\bullet$)

If $x, y \in \mathrm{Ob}(W)$, define the mapping space
$\mathrm{map}_W(x,y)$
to be the fiber over (x,y) in the map
$(d_1, d_0): W_1 \longrightarrow W_0 \times W_0$.

(The Reedy fibrant condition guarantees that this definition is homotopy invariant.)

Given two "maps" $f, g \in \mathrm{map}_W(x,y)$
they are homotopic $(f \sim g)$ if they lie in the same component

There is a notion of "composition" of maps.

A map $g \in \mathrm{map}_W(x,y)_0$ is a <u>homotopy equivalence</u> if there exist maps $f, h \in \mathrm{map}_W(y,x)_0$ such that $g \circ f \sim id_x$ and $h \circ g \sim id_y$.

Proposition: (Rezk)
Any map in the same component as a homotopy equivalence is itself a homotopy equivalence.

Thus, we can define the space of homotopy equivalences

$$W_{hoequiv} \subset W_1$$

Since identity maps are homotopy equivalences, the map

$$s_0 : W_0 \longrightarrow W_1$$

factors through $W_{hoequiv}$:

$$\begin{tikzcd} W_0 \arrow[rr, "s_0"] \arrow[dr] & & W_1 \\ & W_{hoequiv} \arrow[ur, "\text{inclusion}"'] & \end{tikzcd}$$

Definition: A Segal space W is a <u>complete Segal space</u> if the map

$$W_0 \longrightarrow W_{hoequiv}$$

is a weak equivalence of simplicial sets.

Let $\mathcal{C}$ be a simplicially enriched category.

Taking its nerve yields a Segal category.

Can "localize" to obtain a complete Segal space.

Example: Let $\mathcal{C}$ be a (discrete) category.

The resulting complete Segal space looks like:

$$\begin{tikzcd} & \vdots \\ & \coprod_{\langle x\rangle,\langle y\rangle} B\,\mathrm{Aut}(\mathrm{Hom}(x,y)) \arrow[d, shift left] \arrow[d, shift right] \\ \text{equivalence classes of objects} \arrow[r] & \coprod_{\langle x\rangle} B\,\mathrm{Aut}(x) \end{tikzcd}$$

Theorem: (Rezk)

There is a model category structure on the category of bisimplicial sets such that:

- the weak equivalences are the maps $f: X \to Y$ such that for any complete Segal space W, the map $f^*: \mathrm{Hom}(Y, W) \longrightarrow \mathrm{Hom}(X, W)$ is a weak equivalence of simplicial sets
- the cofibrations are inclusions
- fibrant objects = complete Segal spaces

Recall: There is a model category structure on the category of Segal precategories such that the fibrant objects are Segal categories.

There is a Quillen equivalence of model categories:

$$\begin{tikzcd} \text{"Segal category" model category} \arrow[r, hook, shift left, "\text{inclusion}"] & \text{"complete Segal space" model category} \arrow[l, shift left, "\substack{\text{right adjoint} \\ \text{"discretizes"}}"] \end{tikzcd}$$

S-Categories - Model Category Structure

Let $\mathcal{C}$ be an S-category.

Define $\pi_0\mathcal{C}$ to be the category of components of $\mathcal{C}$.

A morphism $g \in \mathrm{Hom}_{\mathcal{C}}(x,y)_0$ is a <u>homotopy equivalence</u> if there is a morphism $g' \in \mathrm{Hom}_{\mathcal{C}}(y,x)_0$ such that $g'g$ is in the same component as $\mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x,x)_0$ (and similarly for gg').

Theorem: There is a model category structure on $\mathcal{S}$-$\mathcal{C}at$ in which:

- the weak equivalences $f: \mathcal{C} \to \mathcal{D}$ satisfy
  - for any $a_1, a_2 \in \mathrm{Ob}(\mathcal{C})$, the map $\mathrm{Hom}_{\mathcal{C}}(a_1, a_2) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(fa_1, fa_2)$ is a weak equivalence of simplicial sets, and
  - $\pi_0 f: \pi_0 \mathcal{C} \to \pi_0 \mathcal{D}$ is an equivalence of categories
- the fibrations $f: \mathcal{C} \to \mathcal{D}$ satisfy
  - for any $a_1, a_2 \in \mathrm{Ob}(\mathcal{C})$, the map $\mathrm{Hom}_{\mathcal{C}}(a_1, a_2) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(fa_1, fa_2)$ is a fibration of simplicial sets, and
  - for any $a_1 \in \mathcal{C}$, $b \in \mathcal{D}$, and homotopy equivalence $g: fa_1 \to b$, there is a homotopy equivalence $d: a_1 \to a_2$ in $\mathcal{C}$ such that $fd = g$.

$$\begin{tikzcd} \mathcal{C} \arrow[d] & a_1 \arrow[d] \arrow[r, dashed, "\exists d"] & a_2 \arrow[d, dotted] \\ \mathcal{D} & fa_1 \arrow[r, "g"] & b \end{tikzcd}$$

Hope: There is a Quillen equivalence

$\mathcal{S}\text{-Cat} \rightleftarrows$ "Segal categories"

Problem: The model category structure on Segal precategories does not work.