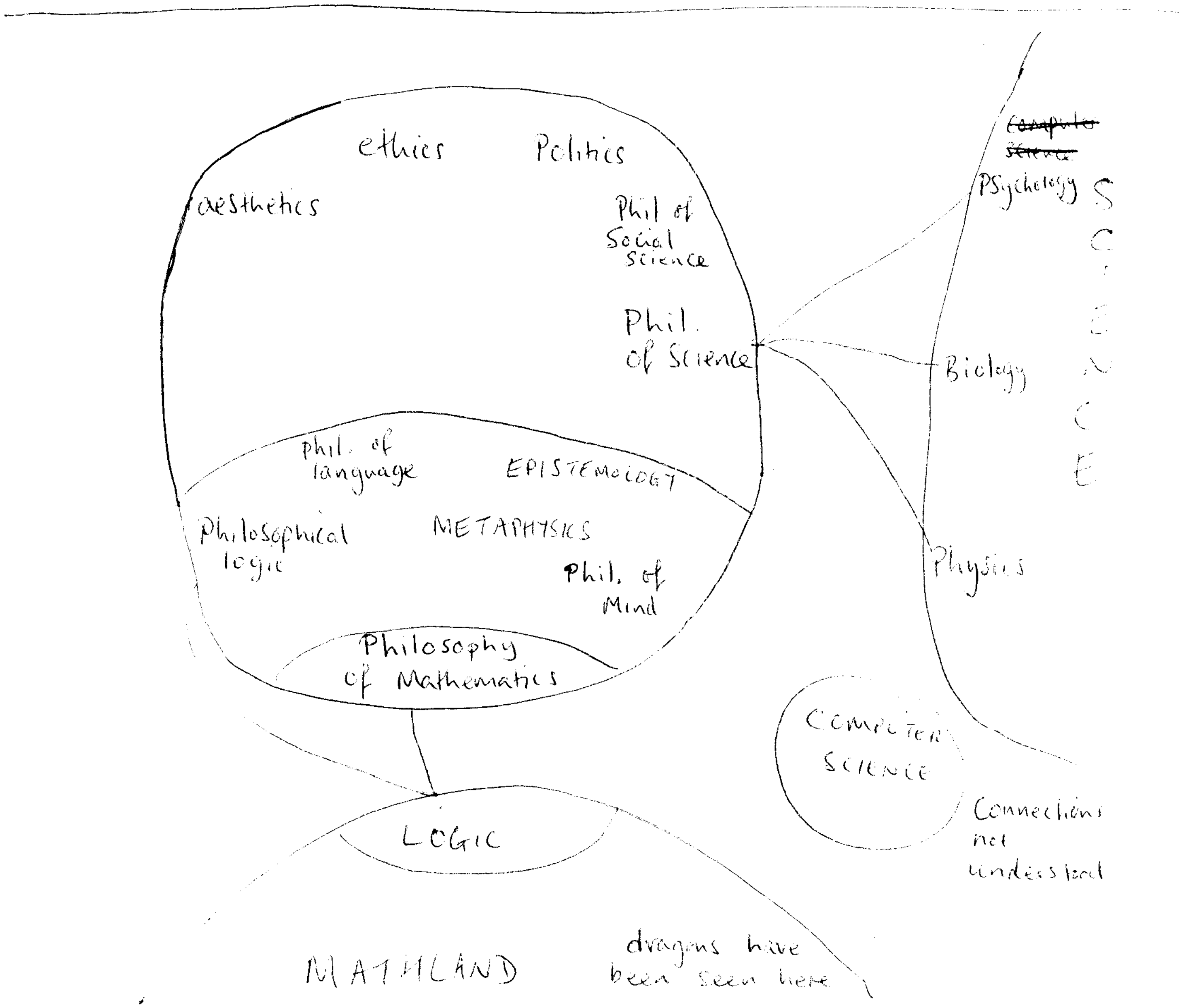
N-category theory as a catalyst for change in philosophy

David Corfield

IMA, University of Minnesota, Friday 11 June 2004

(notes for the talk, including some audience comments and further comments of my own)

In recent times, the trade between the two nations, philosophy and mathematics, has been rather meagre, after a history stretching back over a couple of millennia of powerful interaction. I have drawn a map (see MAP) representing my view of the way philosophical activity is arranged in the English-speaking world. Trading between the two nations occurs largely between mathematical logic and either philosophical logic or philosophy of mathematics. But even these trade routes are becoming less frequently used.



In the central core are branches where the big names of the 20th century have made contributions (Russell, Quine, Kripke, etc.)

Metaphysics studies: being, the substance-property relation, necessity, identity of objects, events and persons, causality, part-whole relations, space, time, and so on. In our terms, it searches for some kind of ultimate category whose objects might be all possible things or, if a physicalist, all possible space-time points or regions.

Epistemology studies knowledge: knowledge is cast as objectively correct belief held for objectively right kind of reason. The sceptic is never satisfied that these conditions hold or can be known to hold. Mathematical and logical truths are often put forward as things we can know with certainty.

Analytic philosophers often recast discourse using some preferred form of logic to reveal 'ontological commitments' (i.e., the things which the speaker must believes exists). Mathematicians appear to commit themselves to the existence of numbers, sets, etc., but then if they are right, the philosopher asks, where are they? If they are abstract objects (i.e., not in space-time), then how can we come into causal contact with them (a requirement made by many theories of knowledge).

To avoid these problems, philosophers engage in rewriting mathematics and its usage, often using 2nd order logic, in such a way that no mention is made of mathematical entities. One major school is known as neo-Fregeanism.

During my wanderings in Mathland, I've never heard a mathematician express any interest in this work. Even those philosophers who have realized that mathematicians deal with structures have made very little impression. Many philosophers are happy to engage 'modestly' in this activity – Who are we to dictate to the mathematicians how they should think?

Personally, I would be upset if no mathematician had not profited in some way from my work. It is possible for this to happen without dictating a change in allowable reasoning.

The islands have reconfigured over the centuries
The islands should reconfigure once more
Mathematics could be a catalyst for change effecting all of philosophy, and n-category theory could act as a spark for this process.

In military terms, I envisage a two-pronged approach, one sweeping along the existing routes equipped with the ideas of n-category theory and forcing metaphysics into much closer proximity with all the sciences from sociology to physics, the other approach is to force epistemology to relate to aesthetics, ethics and politics.

Method 1: (Russellian) Logic and category theory and n-category theory merge, feeding their conceptions into the central core along existing trade routes. This might at first leave things little changed, but would have major long term ramifications.

(a) Every interesting equation is a lie (Baez, Dolan, Makkai)

This claim encapsulates the drive to use n-category theory to replace equality by equivalence. Identity is a crucial notion in metaphysics. 'The Evening Star = The Morning Star' or 'Hesperus = Phosphorus' are frequently treated when analyzing necessity, reference, meaning, etc. We are encouraged then to regard this equation as a false rendition of the fact that there are different parts of the trajectory of Venus (visible in the morning from Earth, visible in the Evening, not visible). 'Cicero = Tully' (2 names for the same individual) might appear to be an uninteresting equation, but put into your own context of being called 'Mum' or 'Dad' (by children), 'I' (by yourself), your name (by your partner), a nickname (by a friend), the complex structure of personhood is indicated. (cf. "A theory or a person is more real than a cobblestone" Polanyi.)

(b) Reappraisal of property and structure

Steve Awodey has done a good job of conveying the category theoretic notion of structure to philosophers. John Baez and James Dolan have devised a way of discussing a hierarchy (property, structure, stuff) by analogy with homotopy theory (Postnikov towers).

(c) Using (b) to reconfiguring modal logic

Their analysis in (b) suggests a progression from propositional logic, to predicate logic (typed) to modal logic (typed worlds). This could provide new models for thinking about necessity or probability. An interesting project would be to relate these ideas to Judea Pearl's Bayesian networks, entities crying out to be cast as a symmetric monoidal category.

(d) Notion of complex structures (Bas, Brown)

Reductionist approaches to an entity hope to model that entity by looking at its set of interacting component parts (Polanyi argues well against this). But for complex structures, organisms, organisations, etc. a multi-layered approach with interacting layers should prove necessary. On a similar note Ronnie Brown has some interesting ideas about the brain's activity.

(e) Part-whole relations, the nature of space

Obviously n-category theory provides a wide range of tools for these topics. Most metaphysicians do not engage even with physics sufficiently. There needs to be a three way interaction here between mathematics, philosophy and science (e.g., physics – TQFT and gerbes in string theory; computer science).

(f) Diagrammatics to question "transparency" of logic

The boundary between algebra and topology blurs when you think of pieces of notation as topological entities. Peirce's existential graphs (alpha system – propositional, beta system – predicate, not quite worked out gamma system – modal) form an interesting case of this phenomenon, making implausible the idea that 'logical truths', however these are defined, are knowable a priori. Existential graphs and so many other cases of diagrammatic reasoning fit neatly within n-category theory (see work of Trimble).

(h) The sphere spectrum is/are the true integers.

This takes the process of categorification (and stabilization) to its extreme, starting out with the integers. It illustrates the idea that n-category theory can carve out the 'right' concepts.

We've seen Frege's name associated with one of these modern rewriting programmes, but Frege is now being reappraised in a way which throws into question whether the best way to imitate him is neo-Fregeanism. It turns out that Frege was a follower of Riemann's style of mathematics, and disliked Weierstrass'. So, rather than see him as continuing a sequence - logicising arithmetic after the arithmeticisation of analysis, we should know that alongside his desire to provide a secure language for mathematical reasoning, Frege hoped that his new logic could be used to carve out mathematical concepts correctly. This is an issue very much with us today, and in particular at this workshop, i.e., how to carve out fruitful conceptions of n-categories, but also, perhaps closer in spirit to Frege, can n-categories be used methodologically to carve out fruitful concepts, e.g., quantum 2-groups.

We can take Frege to be someone engaged on a foundational project, while recognizing that foundational questions are liable to evolve. As Yuri Manin puts it:

"I will understand 'foundations' neither as the para-philosophical preoccupation with the nature, accessibility, and reliability of mathematical truth, nor as a set of normative prescriptions like those advocated by finitists or formalists. I will use this word in a loose sense as a general term for the historically variable conglomerate of rules and principles used to organize the already existing and always being created anew body of mathematical knowledge of the relevant epoch. At times, it becomes codified in the form of an authoritative mathematical text as exemplified by Euclid's Elements. In another epoch, it is better expressed by the nervous self-questioning about the meaning of infinitesimals or the precise relationship between real numbers and points of the Euclidean line, or else, the nature of algorithms. In all cases, foundations in this wide sense is something which is relevant to a working mathematician, which refers to some basic principles of his/her trade, but which does not constitute the essence of his/her work." (Manin, *Georg Cantor and His Heritage*: 6, AG/0209244)

In fact, Manin goes on to discuss the movement of foundations from sets to categories to n-categories. I think it is no accident that people here feel driven to express their "philosophy" (e.g., Brown, Cheng, Baez), although of course others do too (e.g., Connes, Gowers).

Much of what is stake is the balance between constraint and freedom. Going back to the mid 20th century, we find two great names talking about this. Weyl declares,

"Mathematics is not the rigid and rigidity-producing schema as which the layman views it; rather, we find ourselves in it at exactly that crossing point of constraint and freedom which is the very essence of man's nature."

John von Neumann in 'The Mathematician' (1954) warns about mathematicians following the "path of least resistance", and famously advocates a return to empirical sources. A philosopher might say that this suits them fine, all we need concern ourselves with are the two criteria of logical correctness and physical instantiation (taken in a fairly immediate way).

But is there nothing in between? Of course there is. Von Neumann himself goes on to add the clause "unless developed by men of exquisite taste". The idea that a strong aesthetic faculty is necessary is nicely expressed by the Hungarian scientist turned philosopher, Michael Polanyi.

"External experience is indispensable both to mathematics and art, *as their theme*, but to a person prepared to inhabit their framework, mathematics or art convey their own internal thought, and it is for the sake of this internal experience that his mind accepts their framework as its dwelling place." (Polanyi, Personal Knowledge: 283)

We're closing in on a word that I've heard some of you utter.

"We should declare … candidly that we dwell on mathematics and affirm its statements for the sake of its intellectual beauty, which betokens the reality of its conceptions and the truth of its assertions. For if this passion were extinct, we would cease to understand mathematics; its conceptions would dissolve and its proofs carry no conviction. Mathematics would become pointless and would lose itself in a welter of insignicant tautologies and of Heath Robinson operations, from which it could no longer be distinguished." (Polanyi)

[Heath Robinson is the British equivalent of Rube Goldberg.]

Rota tells us that Ulam would tease von Neumann about the needless development of continuous geometries and von Neumann algebras. We now know he needn't have worried. But elitism now rears its head. Whose taste? The values of the few? Is it that the elite see the way before anyone else what the many will come to agree is good, and even foresee values that future mathematicians will hold dear? Do the elite dictate or just steer the future direction of mathematics?

Summing up where we have got so far, an (ahistorical, asocial, impersonal) objectivism which can only deal with the values of truth, in the sense of logical correctness, and direct physical instantiation cannot capture fully (can barely begin to capture) what is at stake in mathematics. This represents something close to a *reductio* (always remember, however, that "one person's *reductio* is another person's interesting consequence").

Of course, many have argued that even the notion of truth in mathematics cannot be taken in a completely objective sense, but it does feel as though considerations of beauty in most walks of life are highly subjective.

By thinking through these issues, mathematics could provide a way for philosophy to put its house in order, not through its being the most certain form of knowledge, but because even in this extreme corner of human activity, it is unthinkable that it be sustainable without value judgements.

Method 2: (bypassing the central core to connect with branches of philosophy on the Northern and Western fringes)

Instead, philosophers need to engage with mathematics as a living activity full of passion, joy and strife. Instead of seeing n-category theorists as a weird subset of a weird profession, we can take you as engaging in a form of activity of an extreme kind.

Philosophers can participate in the articulation of values of mathematics: Moral truth, naturalness, fruitfulness, being interesting, having good properties. Here we can borrow from similar disputes in ethics and aesthetics. Charles Taylor has argued convincingly against the reduction of values to a single one, such as utility. Something about this articulation resembles the work of a mathematician carving out a new concept, but I think here we're dealing with something less definable. We won't end with anything formalisable, but rather devise new forms of expression, make the tacit articulate.

"Our attempts to formulate what we hold important must, like descriptions, strive to be faithful to something. But what they strive to be faithful to is not an independent object with a fixed degree and manner of evidence, but rather a largely inarticulate sense of what is of decisive importance. An articulation of this 'object' tends to make it something different from what it was before." (Charles Taylor, Philosophical Papers I: 38)

There are internal and external aspects of this value articulation. Representing you to the outside and contributing to internal disputes.

Some philosophers fret that while other values evidently exist, truth can be treated mathematically, and so much more progress made. Yet many mathematicians who write philosophically are not at all put off by this, e.g., Rota in 'Indiscrete Thoughts' treats beauty. It is very noticeable that practitioners of a scientific discipline, e.g., neuroscience, who feel they'd like to engage with philosophy, commonly turn to phenomenology rather than analytic philosophy. Rota upset many philosophers by likening their use of formalism in many situations to paying for groceries with banknotes (bills) from a game of Monopoly.

I want now to turn to the philosophy of science, where we find some interesting stirrings of new ways of thinking. Van Fraassen in 'The Empirical Stance' while criticizing analytic metaphysics, invokes Sartre's theory of emotions to explain how it is possible for scientists trained in a tradition to contemplate radically new ideas. Michael Friedman (The Dynamics of Reason) and Helen Longino invoke Habermas to discuss communication within science.

Problems similar to the ones we encountered for philosophy of mathematics earlier were also encountered in philosophy of science. The dream of an impersonal, asocial, ahistorical appraisal of scientific knowledge, e.g., through inductive logic, came to little. Besides the problem that confirmation is not unequivocal, (e.g., tacit knowledge involved in X-ray diagnosis), it matters in varying degrees that a theory be of intrinsic interest, fit in systematically with the rest of scientific theory, and be beautiful. This judgement requires plenty of tacit awareness.

In the mid-20th century there arose historically-sensitive philosophy of science. Many were interested in the social organization of science. Neurath, Popper – Open Society, Lakatos – Honest Scorekeeping between rival research programmes, more skeptically Kuhn – paradigms are like dogmas, Feyerabend – science as a whole is like a religion.

Summarising the problem in a single question:
What kind of a community which evidently relies on ultimately unformalisable assessment by people who understand only a fraction of the scope and details of its workings, can cherish its tradition, and yet at the same time encourage the kind of work Kuhn labelled revolutionary?

Communications between philosophy of mathematics and philosophy of science aren't as good as they should be. Philosophy of science generally buys into logical reductionism of math – even some philosophers of physicists – as explicitly stated by Hempel in 1945 in a paper on mathematical truth.

Still there is plenty to pull back and push forward:

E.g., Poincaré on laws,
"...the mathematical facts worthy of being studied are those which, by their analogy with other facts, are capable of leading us to the knowledge of a mathematical law, just as experimental facts lead us to the knowledge of a physical law. They are those which reveal to us unsuspected kinship between other facts, long known, but wrongly believed to be strangers to one another."

Traditional philosophy of science has devoted much effort to an articulation of what is a law, but perhaps the situation is easier to study in mathematics. As one of us here said 'Behind every analogy there lies a functor'. An easy extrapolation suggests that behind every analogy between analogies there lies a natural transformation. (Exercise: iterate to omega.)

This value that Poincaré alludes to has changed over time. Maybe a thousand years from know we'll look back at analogy formation as we do now on the obsession with dualities in 19th century projective geometry. (I discuss a potential counterexample to Poincare's claim arising curiously from the Poincare conjecture on my webpage, see end of talk). In the lecture we discussed the interesting question "Do miracles occur in mathematics?". Suggestions: algebraic completeness of complex numbers, Bott periodicity,...

A more radical borrowing from history and philosophy of science involves ideas from anthropology. Not Victorian anthropology, how far have you primitives advanced along the scale which leads to the most aware kind of thinking, "They think they're talking about real things – poor fools – but really they're talking about fictions."

This borrowing comes from Harvard historian of science Peter Galison ('Image and Logic'), where he likens communication between instrument constructors, experimentalists, and theoreticians as involving the development of pidgins, which may be elaborated to creoles. Each party does not thoroughly understand each other, but they can convey what they need. Interesting, then, to think of this meeting in such terms, perhaps also with the notion of dialects.

Continuing with the theme of communication, there's something strange in the naming of our more open evening sessions as 'Russian seminars'. Two philosophers who emphasized the role of communication in science were Imre Lakatos and Michael Polanyi, both Hungarians and both reacted against Soviet interference into science. Polanyi quotes Bukharin telling him that "pure science was a morbid symptom of a class society; under socialism the conception of science pursued for its own sake would disappear, for the interests of scientists would spontaneously turn to the problems of the current five year plan" (The Tacit Dimension, 3). There is no way of discerning from this Bukharin's views on the subject, but it's reproduction gives a flavour of Polanyi's philosophy as aimed towards promoting values proper to a field of endeavour, with no interference towards foreign ends, as in the case of Lysenko.

There's one very good idea in Plato that mathematical definitions are to be improved by discussion. In fact, Plato (The Republic) suggests a division of labour between reasoning from first principles and questioning those principles. Surely these are two parts of mathematical activity and there is no reason a philosopher should have a special angle on what is a space, or what is a n-category. In the 20th century the philosopher of mathematics who most clearly understood the role of dialogue was Imre Lakatos (Proofs and Refutations). However, he never overcome an obstacle – one of his own making – that dialectic progress could not be made in a field which had been axiomatised, cf. chap. 6 of my book.

[Politics and mathematics interact in subtle ways. The joint rise of proof and democracy has been noted by historians such as Szabo, but then the Soviet Union over several decades has produced many great mathematicians. Russians here have suggested that when there are so few opportunities for a career which allows you to think freely, mathematics becomes a very attractive career.]

While both Lakatos and Polanyi believed in the importance of dialogue in science for idea formation, in Polanyi's case this extends explicitly to value formation. Such is the fragility of a society such as mathematics that:

"The transmission of mathematics has today been rendered more precarious than ever by the fact that no single mathematician can fully understand any longer more than a tiny fraction of mathematics. Modern mathematics can be kept alive only by a large number of mathematicians cultivating different parts of the same system of values: a community which can be kept coherent only by the passionate vigilance of universities, journals and meetings, fostering these values and imposing the same respect for them on all mathematicians. Such a far-flung structure is highly vulnerable and, once broken, impossible to restore. Its ruins would bury modern mathematics in an oblivion more complete and lasting than that which enveloped Greek mathematics twenty-two centuries ago." (PK 192-3)

There are many ways you could use current technology to enhance this communication: far more exposition on the web (clone Baez a few times); encouragement of the recording by senior mathematicians of the course of their research (cf. Ross Street's fascinating account).

Turning now to ethics, as a profession you cannot avoid the ethical dimension of commitment. "The freedom of the subjective person to do as he pleases is overruled by the freedom of the responsible person to act as he must." (PK, 309). "...the original mind takes a decision on grounds which are insufficient to minds lacking similar powers of creative judgment. The active scientific investigator stakes bit by bit his whole professional life on a series of such decisions and this day-to-day gamble represents his most responsible activity." (PK: 309-10). All the time you are making choices as to what to study, which papers to accept, which avenues to steer students down. Polanyi would encourage you all to be as open and forthright as possible.

To conclude:

Connections need to forged between philosophy of mathematics and many other areas of philosophy (epistemology, metaphysics, politics, ethics, aesthetics, philosophy of science). Cooperation can bring about major changes in the landscape of philosophy and perhaps even the practice of mathematics.

There is something in particular about the vision of n-category theory I find appealing. Where else can you find such monumental visions as those at stake in the field of mathematics. Here is a field of great maturity and yet upstarts are suggesting an enormous rethinking of its framework. What kind of community can support the furtherance of such visions, and even, with due skepticism, encourage them to disrupt old meanings?

The current state of affairs is such that educated lay people are deprived of any conception of mathematics. You know it's beautiful or you wouldn't spend half your life studying it. You shouldn't have to justify the activity via some value you know isn't primary, like usefulness. A society should be proud of its capacity to support such work without needing to interfere.

PK – 'Personal Knowledge: Towards a Post-Critical Philosophy' by Michael Polanyi (London, Routledge: 1958).

For more ideas on philosophy of mathematics see my webpages
http://users.ox.ac.uk/~sfop0076/phorem.htm/
http://users.ox.ac.uk/~sfop0076/Towards.htm/ (concerning my book).