Higher Linear Algebra

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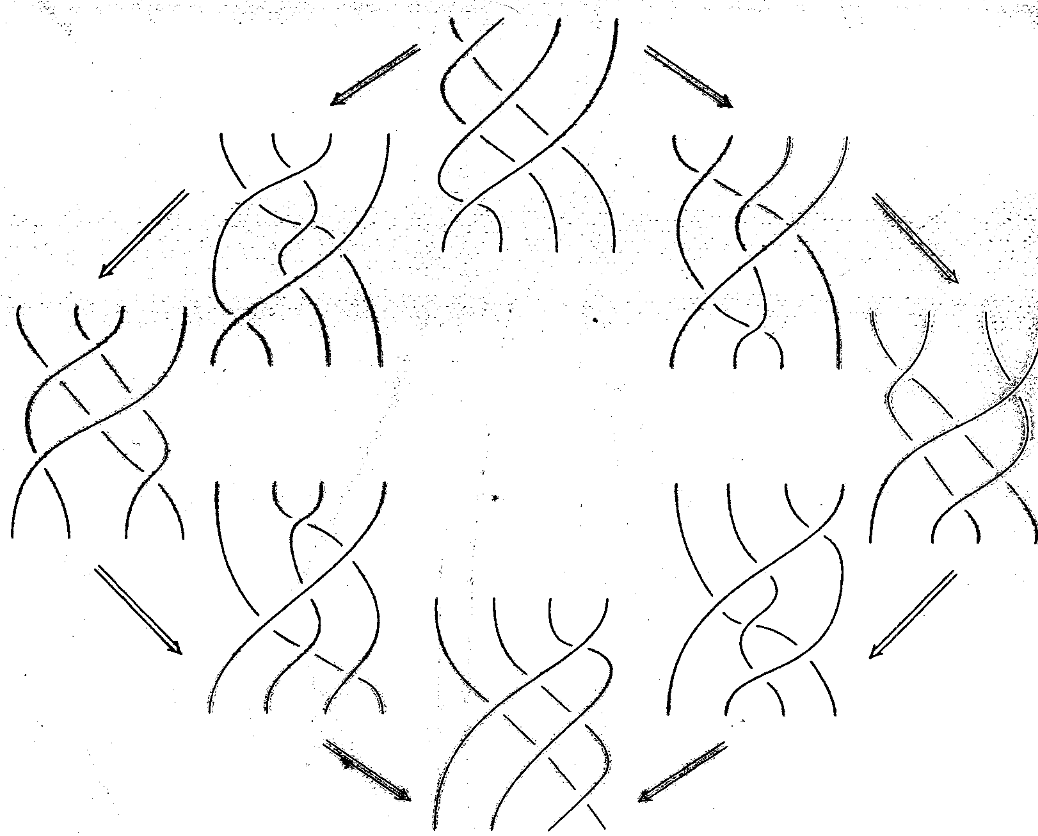

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June 2004

Categorified vector spaces

- Kapranov and Voevodsky defined a finite-dimensional 2-vector space to be a category of the form Vect^n.

- Instead, we define a **2-vector space** to be a category in Vect.

  We have a 2-category, 2Vect whose:

  - objects are 2-vector spaces,
  - 1-morphisms are linear functors between these,
  - 2-morphisms are linear natural transformations between these

- Proposition: $2\mathrm{Vect} \simeq 2\mathrm{Term}$

n-vector spaces

- An $(n+1)$**-vector space** is a strict n-category in Vect.

  We have a strict $(n+1)$-category, nVect whose:

  – objects are n-vector spaces,

  – 1-morphisms are strict linear functors between these,

  – 2-morphisms are strict linear natural transformations between these, ...

  and so on!

- Proposition: $(n+1)\text{Vect} \simeq (n+1)\text{Term}$

- $\bigoplus$, $\bigotimes$ of n-vector spaces

Moral: Homological algebra is secretly categorified linear algebra!

Questions still remain:

- What about weak n-categories in Vect?
- What about weakening laws governing addition and scalar multiplication?

Categorified Linear Algebra

Associative algebras	A_∞-algebras
Commutative algebras	E_∞-algebras
Lie algebras	L_∞-algebras
Coalgebras	?
Hopf Algebras	?
$\vdots$	$\vdots$

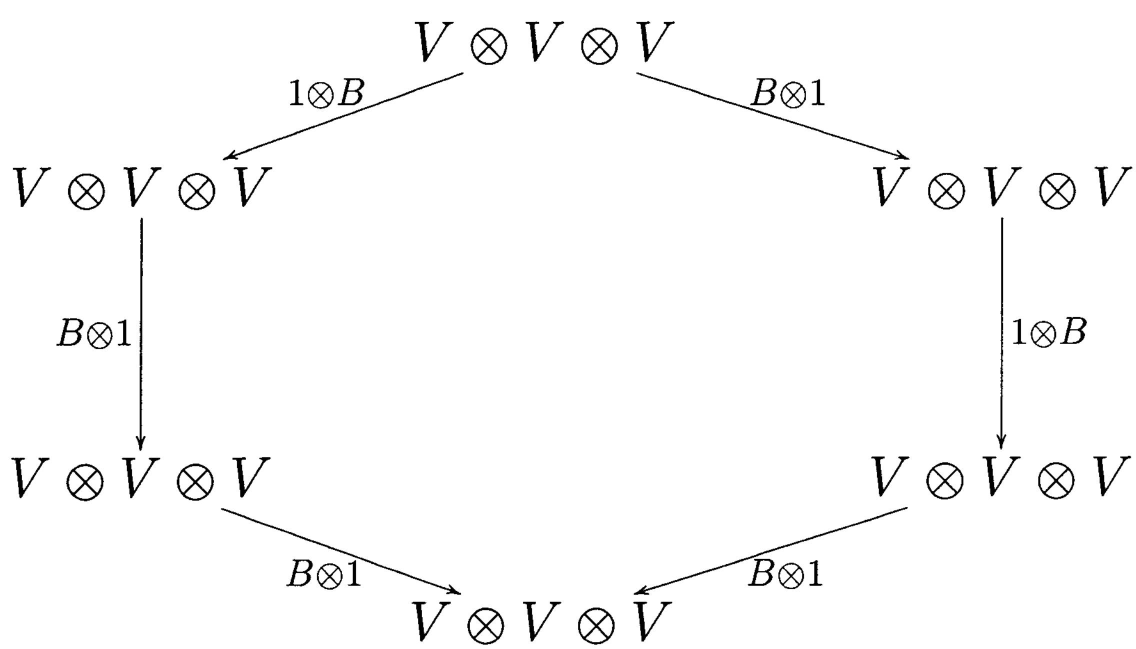
Given a vector space V and an isomorphism

$$B\colon V \otimes V \to V \otimes V,$$

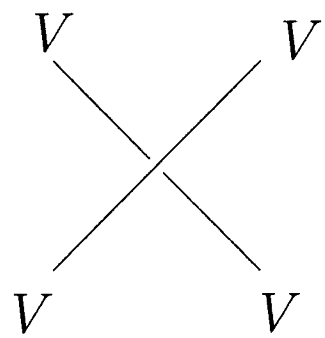
we say B is a **Yang–Baxter operator** if it satisfies the **Yang–Baxter equation**, which says that:

$$(B \otimes 1)(1 \otimes B)(B \otimes 1) = (1 \otimes B)(B \otimes 1)(1 \otimes B),$$

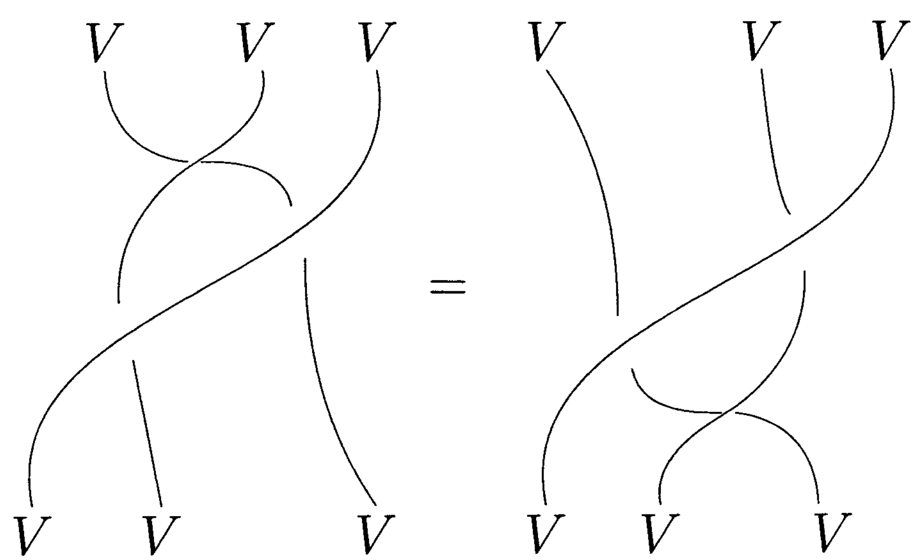
or in other words, that this diagram commutes:



If we draw $B\colon V \otimes V \to V \otimes V$ as a braiding:

$B =$ 

the Yang–Baxter equation says that:



Proposition: Let L be a vector space over k equipped with a skew-symmetric bilinear operation

$$[\cdot,\cdot]\colon L \times L \to L.$$

Let $L' = k \oplus L$ and define the isomorphism

$$B\colon L' \otimes L' \to L' \otimes L' \text{ by}$$

$$B((a,x)\otimes(b,y)) = (b,y)\otimes(a,x)+(1,0)\otimes(0,[x,y]).$$

Then B is a solution of the Yang–Baxter equation if and only if $[\cdot,\cdot]$ satisfies the Jacobi identity.

Goal

Develop a higher-dimensional analogue, obtained by categorifying everything in sight!

A **semistrict Lie 2-algebra** consists of:

- a 2-vector space L

equipped with:

- a skew-symmetric bilinear functor, the **bracket**,

$$[\cdot,\cdot]\colon L \times L \to L$$

- a completely antisymmetric trilinear natural isomorphism, the **Jacobiator**,

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y],$$

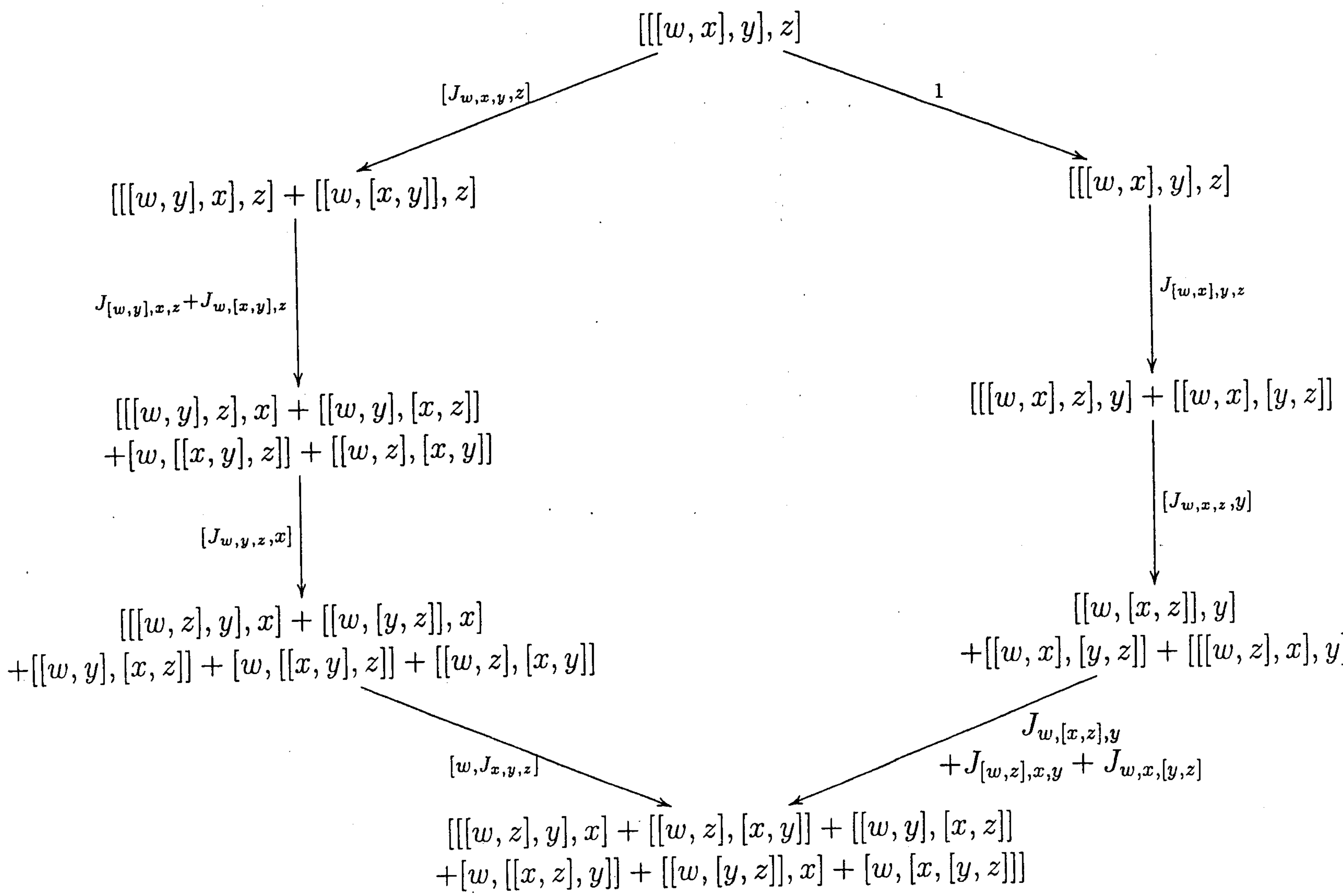
that is required to satisfy:

- the **Jacobiator identity**:

$$J_{[w,x],y,z} \circ [J_{w,x,z}, y] \circ (J_{w,[x,z],y} + J_{[w,z],x,y} + J_{w,x,[y,z]}) =$$

$$[J_{w,x,y}, z] \circ (J_{[w,y],x,z} + J_{w,[x,y],z}) \circ [J_{w,y,z}, x] \circ [w, J_{x,y,z}]$$

for all $w, x, y, z \in L_0$.



[[[w, x], y], z]
[J_{w,x,y,z}]
1
[[[w, y], x], z] + [[w, [x, y]], z]
[[[w, x], y], z]
J_{[w,y],x,z}+J_{w,[x,y],z}
J_{[w,x],y,z}
[[[w, y], z], x] + [[w, y], [x, z]]
+[w, [[x, y], z]] + [[w, z], [x, y]]
[[[w, x], z], y] + [[w, x], [y, z]]
[J_{w,y,z,x}]
[J_{w,x,z,y}]
[[[w, z], y], x] + [[w, [y, z]], x]
+[[w, y], [x, z]] + [w, [[x, y], z]] + [[w, z], [x, y]]
[[w, [x, z]], y]
+[[w, x], [y, z]] + [[[w, z], x], y]
[w, J_{x,y,z}]
J_{w,[x,z],y}
+J_{[w,z],x,y} + J_{w,x,[y,z]}
[[[w, z], y], x] + [[w, z], [x, y]] + [[w, y], [x, z]]
+[w, [[x, z], y]] + [[w, [y, z]], x] + [w, [x, [y, z]]]

Zamolodchikov tetrahedron equation

Given a 2-vector space V and an invertible linear functor $B\colon V \otimes V \to V \otimes V$, a linear natural isomorphism

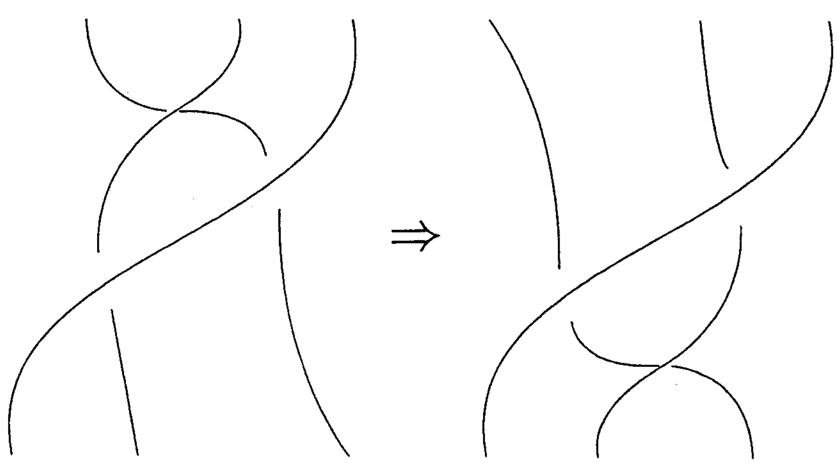
$$Y\colon (B \otimes 1)(1 \otimes B)(B \otimes 1) \Rightarrow (1 \otimes B)(B \otimes 1)(1 \otimes B)$$

satisfies the **Zamolodchikov tetrahedron equation** if:

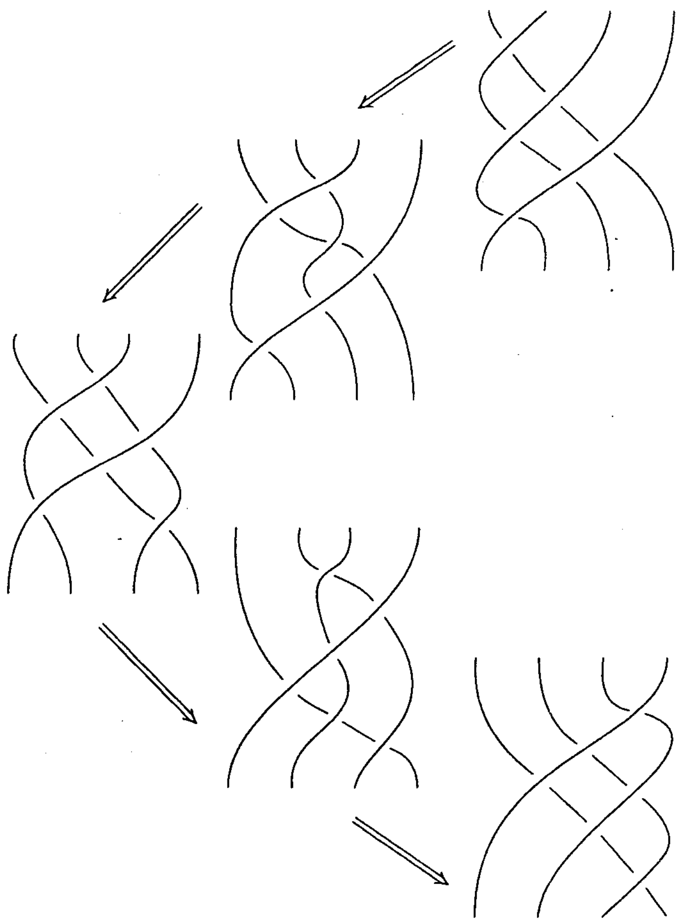
$$[Y \circ (1\otimes 1\otimes B)(1\otimes B\otimes 1)(B\otimes 1\otimes 1)][(1\otimes B\otimes 1)(B\otimes 1\otimes 1) \circ Y \circ (B\otimes 1\otimes 1)]$$
$$[(1\otimes B\otimes 1)(1\otimes 1\otimes B) \circ Y \circ (1\otimes 1\otimes B)][Y \circ (B\otimes 1\otimes 1)(1\otimes B\otimes 1)(1\otimes 1\otimes B)]$$
$$=$$
$$[(B\otimes 1\otimes 1)(1\otimes B\otimes 1)(1\otimes 1\otimes B) \circ Y][(B\otimes 1\otimes 1) \circ Y \circ (B\otimes 1\otimes 1)(1\otimes B\otimes 1)]$$
$$[(1\otimes 1\otimes B) \circ Y \circ (1\otimes 1\otimes B)(1\otimes B\otimes 1)][(1\otimes 1\otimes B)(1\otimes B\otimes 1)(B\otimes 1\otimes 1) \circ Y]$$

We should think of Y as the surface in 4-space traced out by the *process of performing* the third Reidemeister move:

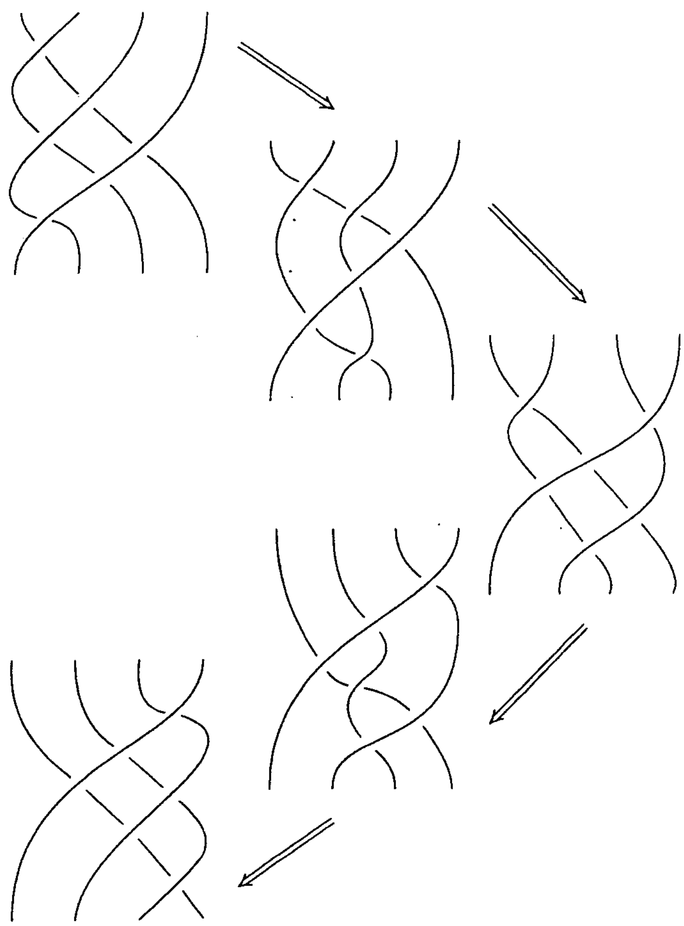
Y:



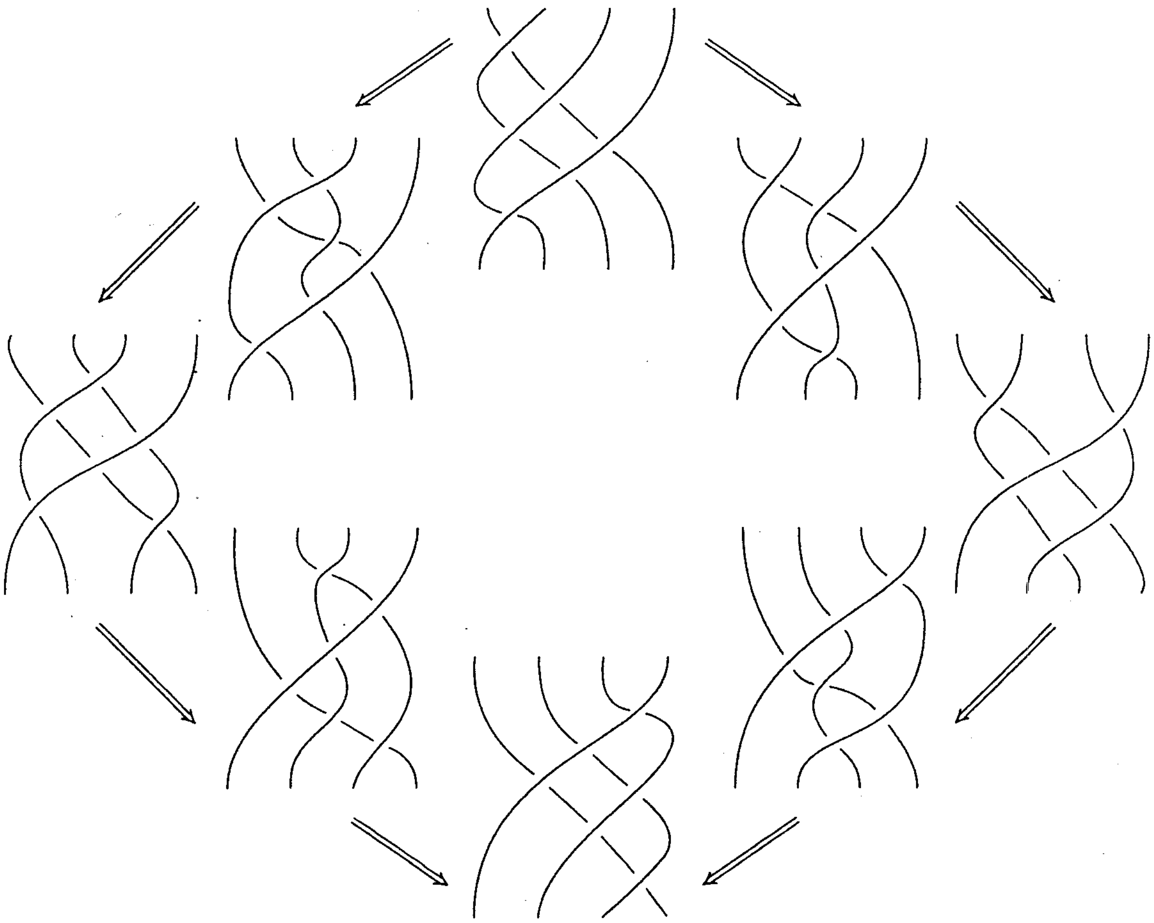
Left side of Zamolodchikov tetrahedron equation:



Right side of Zamolodchikov tetrahedron equation:



In short, the Zamolodchikov tetrahedron equation is a formalization of this commutative octagon:



Theorem: Let L be a 2-vector space, let $[\cdot,\cdot]\colon L \times L \to L$ be a skew-symmetric bilinear functor, and let J be a completely antisymmetric trilinear natural transformation with

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y].$$

Let $L' = K \oplus L$, where K is the categorified ground field.
Let $B\colon L' \otimes L' \to L' \otimes L'$ be defined as follows:

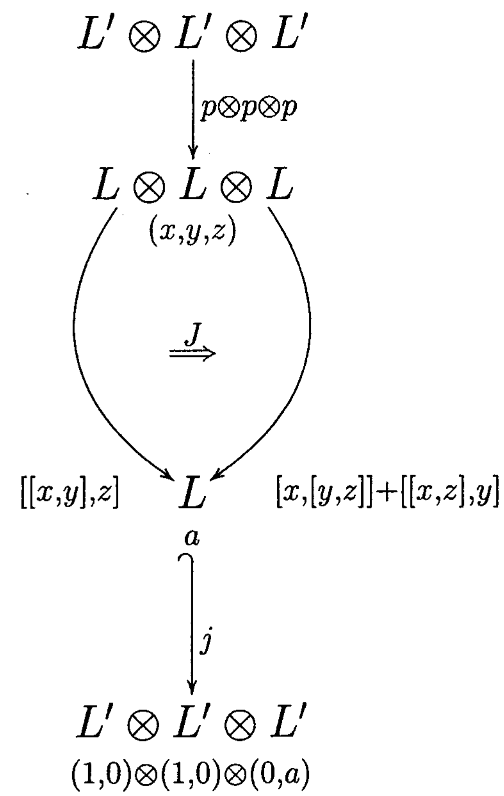
$$B((a,x)\otimes(b,y)) = (b,y)\otimes(a,x) + (1,0)\otimes(0,[x,y])$$

whenever (a,x) and (b,y) are both either objects or morphisms in L'. Finally, let

$$Y\colon (B\otimes 1)(1\otimes B)(B\otimes 1) \Rightarrow (1\otimes B)(B\otimes 1)(1\otimes B)$$

be defined as follows:

$Y =$



where a is either an object or morphism of L. Then Y is a solution of the Zamolodchikov tetrahedron equation if and only if J satisfies the Jacobiator identity.

Hierarchy of Higher Commutativity

Topology	**Algebra**
Crossing	Commutator
Crossing of crossings	Jacobi identity
Crossing of crossing of crossings	Jacobiator identity
⋮	⋮

Examples

Theorem: There is a one-to-one correspondence between equivalence classes of Lie 2-algebras (where equivalence is as objects of the 2-category Lie2Alg) and isomorphism classes of quadruples consisting of a Lie algebra $\mathfrak{g}$, a vector space V, a representation ρ of $\mathfrak{g}$ on V, and an element of $H^3(\mathfrak{g}, V)$.

Example: Given a finite-dimensional, simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, $H^3(\mathfrak{g}, \mathbb{C}) = \mathbb{C}$, so we can construct a Lie 2-algebra with nontrivial Jacobiator consisting of:

- $L_0 = \mathfrak{g}$
- $L_1 =$ identity morphisms
- $J_{x,y,z} = f(x,y,z) 1_{[[x,y],z]}$ where $f \colon \mathfrak{g}^3 \to \mathbb{C}$ is a nontrivial 3-cocycle

In fact, we can take $J_{x,y,z} = \hbar \langle x, [y,z] \rangle 1$ for $\hbar \in \mathbb{C}$.

Let $\mathfrak{g}$ be a Lie algebra and ρ a representation of $\mathfrak{g}$ on the vector space V. Then,

$$\begin{aligned}(\delta\omega)(w,x,y,z) \;=\; & \rho(w)\omega(x,y,z)-\rho(x)\omega(w,y,z)+\rho(y)\omega(w,x,z)\\ & -\rho(z)\omega(w,x,y)-\omega([w,x],y,z)+\omega([w,y],x,z)\\ & -\omega([w,z],x,y)-\omega([x,y],w,z)+\omega([x,z],w,y)\\ & -\omega([y,z],w,x)\end{aligned}$$

$$[J_{w,x,y},z]+[J_{w,y,z},x]+J_{[w,y],x,z}+J_{[x,z],w,y}=$$
$$[J_{w,x,z},y]+[J_{x,y,z},w]+J_{[w,x],y,z}+J_{[w,z],x,y}+J_{[x,y],w,z}+J_{[y,z],w,x}$$

Lie 2-groups

A **coherent 2-group** is a weak monoidal category C in which every morphism is invertible and every object $x \in C$ is equipped with an adjoint equivalence $(x, \bar{x}, i_x, e_x)$.

A **Lie 2-group** is a coherent 2-group object in DiffCat.