A Dihomotopy Double Category of a Po-Space

Uli Fahrenberg

Dept. of Mathematical Sciences
Aalborg University
9220 Aalborg East, Denmark

uli@math.auc.dk

Po-Spaces

The category poTop:

Objects: topological spaces X with a partial order $\leq$ which is a closed subset of $X \times X$.

Morphisms: "dimaps", continuous monotone mappings

Examples: $\vec{I}$, the directed (totally ordered) unit interval; $\vec{I}^n$, the unit n-cube; $\vec{\mathbb{R}}$; the "Swiss Cross", etc.

Note that any subset of a po-space inherits a po-space structure.

Applications in Computer Science (cf. Eric's and Lisbeth's talks)

Dipaths and Dihomotopies

A dipath in a po-space X is a dimap $\vec{I} \to X$.

An elementary dihomotopy (rel $\partial\vec{I}$) of dipaths a, b is a dimap $u : \vec{I}^2 \to X$ (a "disquare") such that

$$u(0,t) = a(t) \qquad u(1,t) = b(t)$$
$$u(s,0) = a(0) \qquad u(s,1) = a(1)$$

Dihomotopy of dipaths is the equivalence relation generated by the elementary dihomotopies.

$\rightsquigarrow$ the fundamental category $\vec{\pi}X$. Cf. Eric & Lisbeth.

Note: 1. $\vec{\pi}X$ is a "loop free" category, esp. it is not a groupoid. 2. Unlike the situation for (ordinary) homotopy, there are no "point groups" capturing dihomotopy information; we need "the whole" category $\vec{\pi}_1X$.

One Dimension Up

Want: (strict) double category $\rho_2 X$ capturing dihomotopy information in dimension 2.

Objects: points of X

1-cells: paths of X

2-cells (squares): disquares in X modulo dihomotopy

Why cubical? 1. personal preference, 2. Van Kampen theorem

Why strict? See above.

Note on terminology: What we call double category is generally known as edge symmetric double category.

Warning: Some of this stuff is still rather preliminary. Don't try this at home ☺

References

[BHKP02] Ronald Brown, Keith A. Hardie, Klaus Heiner Kamps, and Timothy Porter. A homotopy double groupoid of a Hausdorff space. *Theory and Applications of Categories*, 10(2):71–93, 2002.

[BKP04] Ronald Brown, Klaus Heiner Kamps, and Timothy Porter. A van Kampen theorem for the homotopy double groupoid of a Hausdorff space. Preprint, 2004.

[BM99] Ronald Brown and Ghafar H. Mosa. Double categories, 2-categories, thin structures and connections. *Theory and Applications of Categories*, 5(7):163–175, 1999.

[HKK00] Keith A. Hardie, Klaus Heiner Kamps, and Rudger W. Kieboom. A homotopy 2-groupoid of a Hausdorff space. *Applied Categorical Structures*, 8:209–234, 2000.

Concatenation

First thought: Regardless how concatenation of dipaths is defined, it is not associative. So 1-cells cannot be dipaths, but must be dipaths modulo reparametrisation.

So say that $a \sim_T b$ if there exist surjective reparametrisations $\varphi, \psi : \vec{I} \to \vec{I}$ such that $a \circ \varphi = b \circ \psi$.

Then concatenation of 1-cells is associative with units $\varepsilon x(t) = x(0)$.

And for the record, we define

$$(a + b)(t) = \begin{cases} a(2t) & \text{if } t \leq \frac{1}{2} \\ b(2t - 1) & \text{if } t \geq \frac{1}{2} \end{cases}$$

$\rightsquigarrow$ the dipath category $\rho_1 X$: objects—points of X, morphisms—equivalence classes of dipaths wrt. $\sim_T$.

The Singular Cubical Set

$R_n X = [\vec{I}^n, X]$

points $x, y, \ldots \in R_0 X = X$
dipaths $a, b, \ldots \in R_1 X$
disquares $u, v, \ldots \in R_2 X$
dicubes $\alpha, \beta, \ldots \in R_3 X$

$$\delta_1^0 u(t) = u(0,t) \qquad \delta_1^1 u(t) = u(1,t)$$
$$\delta_2^0 u(s) = u(s,0) \qquad \delta_2^1 u(s) = u(s,1)$$

$$\delta_1^0 \alpha(s,t) = \alpha(0,s,t) \qquad \delta_2^0 \alpha(r,t) = \alpha(r,0,t)$$
$$\delta_3^0 \alpha(r,s) = \alpha(r,s,0)$$

$$\varepsilon_1 a(s,t) = a(0,t) \qquad \varepsilon_2 a(s,t) = a(s,0)$$

$$(u +_1 v)(s,t) = \begin{cases} u(2s,t) & \text{if } s \le \frac{1}{2} \\ v(2s-1,t) & \text{if } s \ge \frac{1}{2} \end{cases}$$
$$(u +_2 v)(s,t) = \begin{cases} u(s,2t) & \text{if } t \le \frac{1}{2} \\ u(s,2t-1) & \text{if } t \ge \frac{1}{2} \end{cases}$$

etc.

Thin Globular Disquares

The relation $\sim_T$ is generated by thin globular disquares, i.e. disquares $u \in R_2X$ for which the $\delta_2^0 u$, $\delta_2^1 u$ are degenerate, and there is a factorisation

$$u : \vec{I}^2 \to \vec{I} \to X$$

That is, say that $a \sim_T^{\mathsf{e}} b$ if there exists a thin globular disquare $u \in R_2X$ such that $\delta_1^0 u = a$, $\delta_1^1 u = b$, then $\sim_T$ is the equivalence relation generated by $\sim_T^{\mathsf{e}}$.

We want to quotient out dihomotopy in dimension 2, but to get well-defined degeneracies, we need to allow reparametrisations. So say that $\alpha : u \equiv_T^{\mathsf{e}} v$ is an elementary dihomotopy with thin boundary if

- $\delta_1^0 \alpha = u$, $\delta_1^1 \alpha = v$
- $\delta_2^0 \alpha$, $\delta_2^1 \alpha$, $\delta_3^0 \alpha$, $\delta_3^1 \alpha$ are all thin globular disquares.

and let $\equiv_T$ be the equivalence relation generated by $\equiv_T^{\mathsf{e}}$.

$\rho_2 X$

The triple $\rho_2 X = (R_2X/\equiv_T, R_1X/\sim_T, X)$ is a cubical set, with the obvious boundary maps and degeneracies. We also have composition $+$ on $R_1X/\sim_T$, and we need to define compositions $+_1$, $+_2$ on $R_2X/\equiv_T$.

Let $u, v \in R_2X$ and assume that $\delta_1^1\langle u\rangle = \delta_1^0\langle v\rangle$, i.e. $\delta_1^1 u \sim_T \delta_1^0 v$. Want to define $\langle u\rangle +_1 \langle v\rangle$.

Have $\varphi, \psi : \vec{I} \to \vec{I}$ such that $\delta_1^1 u \circ \varphi = \delta_1^0 v \circ \psi$. So define

$$\widehat{u} = u \circ (\mathrm{id} \times \varphi) \qquad \widehat{v} = v \circ (\mathrm{id} \times \psi)$$

and $\langle u\rangle +_1 \langle v\rangle = \langle \widehat{u} +_1 \widehat{v}\rangle$, similar for $+_2$.

Independent of reparametrisations because $u \circ (\varphi_1 \times \varphi_2) \equiv_T u \circ (\varphi_1' \times \varphi_2')$.

Associativity $\surd$, interhcange law $\surd$, identities $\langle \varepsilon_1 a\rangle$, $\langle \varepsilon_2 a\rangle$.
$\rightsquigarrow$ $\rho_2 X$ is a (loop free) double category.

Relation to Dihomotopy

An elementary dihomotopy (rel $\partial\vec{I}^2$) of disquares u, v is a dicube α such that $\delta_1^0\alpha = u$, $\delta_1^1\alpha = v$, and the other four faces are degenerate.

Conjecture: If $u, v \in R_2X$ are globular, i.e. the $\delta_2^0 u$, $\delta_2^1 u$, $\delta_2^0 v$, $\delta_2^1 v$ are degenerate, and $\delta_1^0 u = \delta_1^0 v$, $\delta_1^1 u = \delta_1^1 v$, then u and v are dihomotopy equivalent if and only if $u \equiv_T v$.

True in the non-directed case [HKK00], but appears to be difficult to show in our directed setting.

Thin Structure

A disquare $u \in R_2X$ is ("geometrically") thin if there is a factorisation $u : \vec{I}^2 \to \vec{I} \to X$.

Goal: Relate this to a thin structure on $\rho_2 X$.

Connections $\gamma^0, \gamma^1 : R_1X \to R_2X$:

$$\gamma^0 a(s,t) = a(\max(s,t))$$
$$\gamma^1 a(s,t) = a(\min(s,t))$$

Induce maps $\gamma^i \langle a \rangle = \langle \gamma^i a \rangle$ in $\rho_2 X$. These form a connection pair [BM99]
$\rightsquigarrow$ thin structure on $\rho_2 X$.

Slogan: $\langle u \rangle \in \rho_2 X$ is ("algebraically") thin iff it is a composite of degeneracies and connections.

Properties: Thin squares have commuting boundary. Commuting 2-shells have unique thin fillers.

Thin Structure, 2.

Proposition: $\langle u \rangle \in \rho_2 X$ is algebraically thin iff $\exists v \in R_2 X$ such that $u \equiv_T v$ and v is geometrically thin.

Keys to proof: 1. Geometrically thin disquares have commuting boundary.
2. Composites of geometrically thin disquares are geometrically thin.
3. If $u, v \in R_2 X$ are geometrically thin, and all $\delta_i^j u \sim_T \delta_i^j v$, then $u \equiv_T v$.

The van Kampen Theorem

Let $\mathcal{U}$ be an open cover of X, and denote by $a_{UV} : U \cap V \hookrightarrow U$, $b_{UV} : U \cap V \hookrightarrow V$, $c_U : U \hookrightarrow X$ the inclusions, for $U, V \in \mathcal{U}$. Then $c^* = \text{coeq}(a^*, b^*)$ in the following diagram in the category of double categories:

$$\bigsqcup_{U,V \in \mathcal{U}} \rho_2(U \cap V) \underset{b^*}{\overset{a^*}{\rightrightarrows}} \bigsqcup_{U \in \mathcal{U}} \rho_2 U \xrightarrow{c^*} \rho_2 X$$

Needed in proof: Homotopy Commutativity Lemma:
Given $\alpha \in R_3 X$, then

$$\langle \partial \alpha \rangle = \big(\langle \delta_1^0 \alpha \rangle, \langle \delta_1^1 \alpha \rangle, \langle \delta_2^0 \alpha \rangle, \langle \delta_2^1 \alpha \rangle, \langle \delta_3^0 \alpha \rangle, \langle \delta_3^1 \alpha \rangle \big)$$

is a commuting 3-shell in $\rho_2 X$.