TOWARDS A HOMOTOPY THEORY OF HIGHER DIMENSIONAL AUTOMATA
(29 TRANSPARENCIES)
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Abstract. We give an overlook of our work about globular complexes and flows. Globular complexes, flows, S-homotopy and T-homotopy are explained by examples. Several model categories are presented. And some open questions are discussed.

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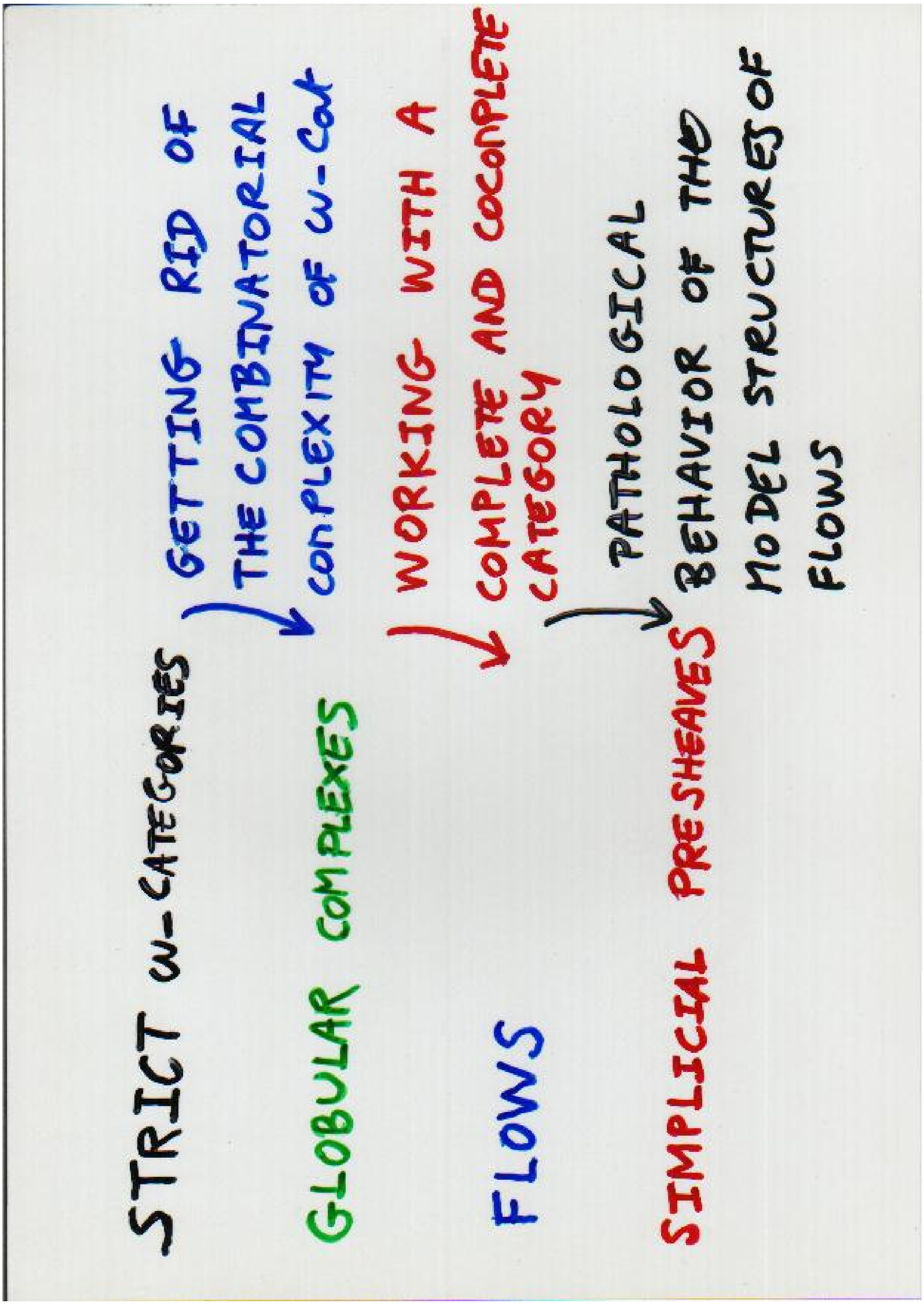

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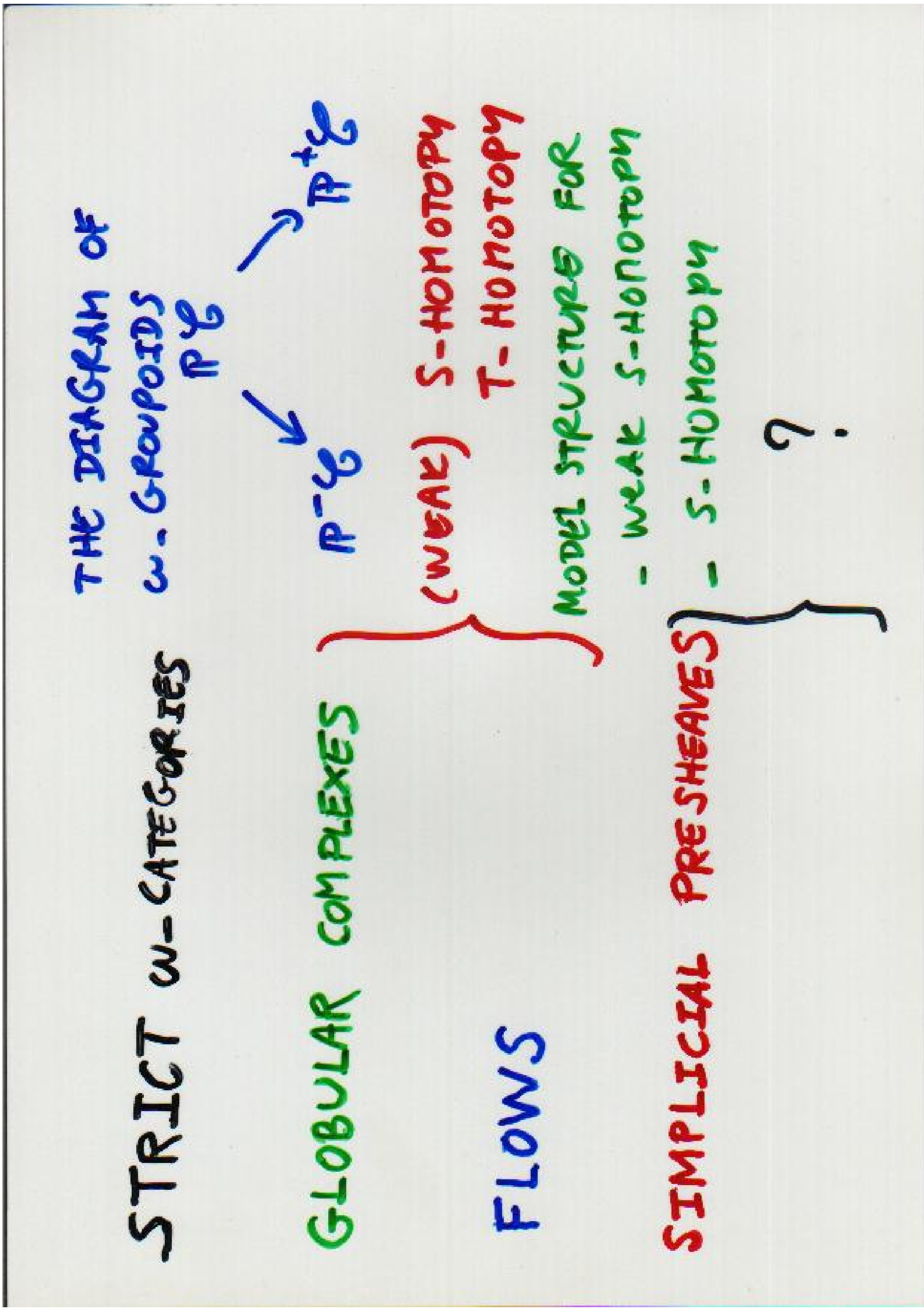

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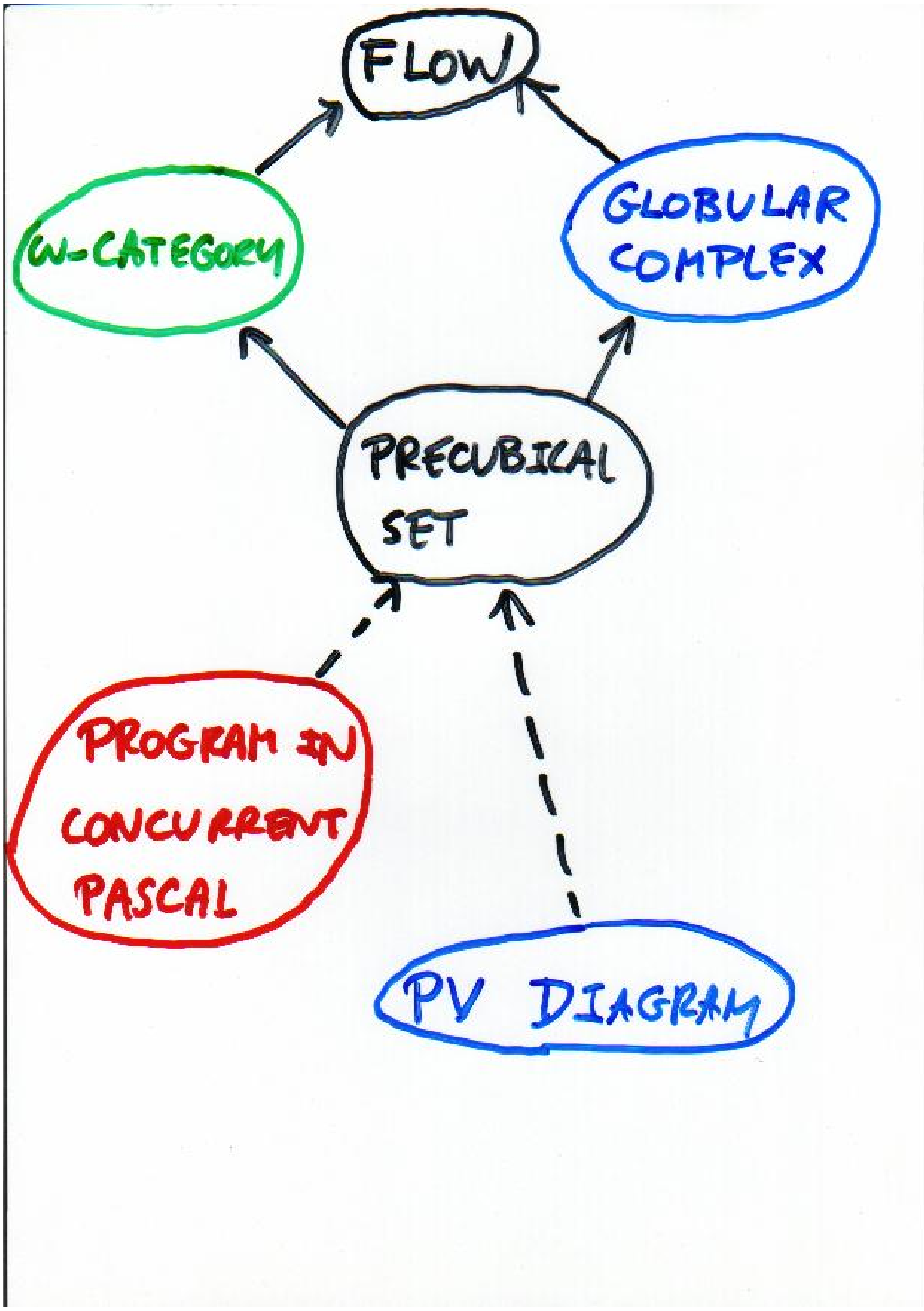

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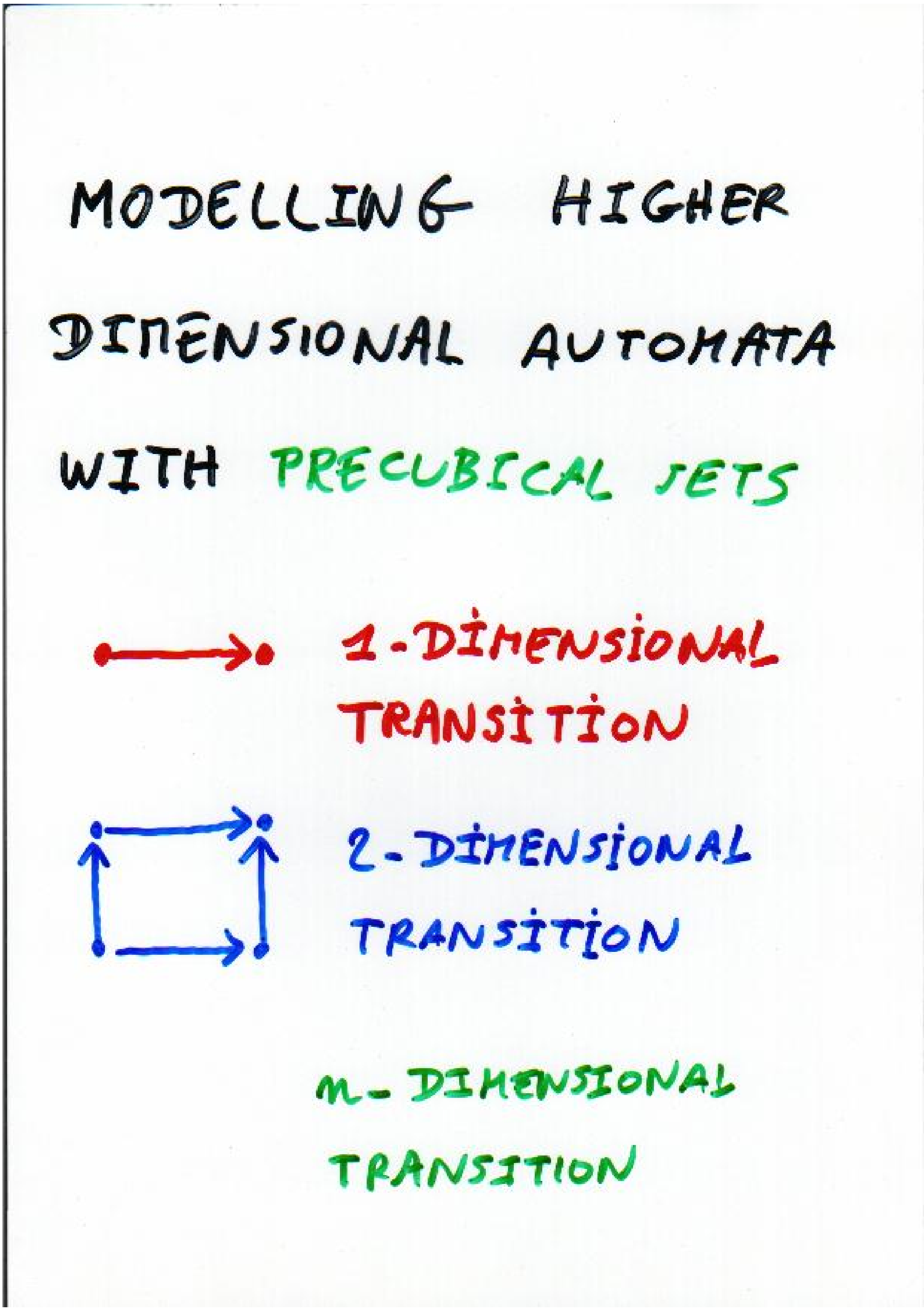

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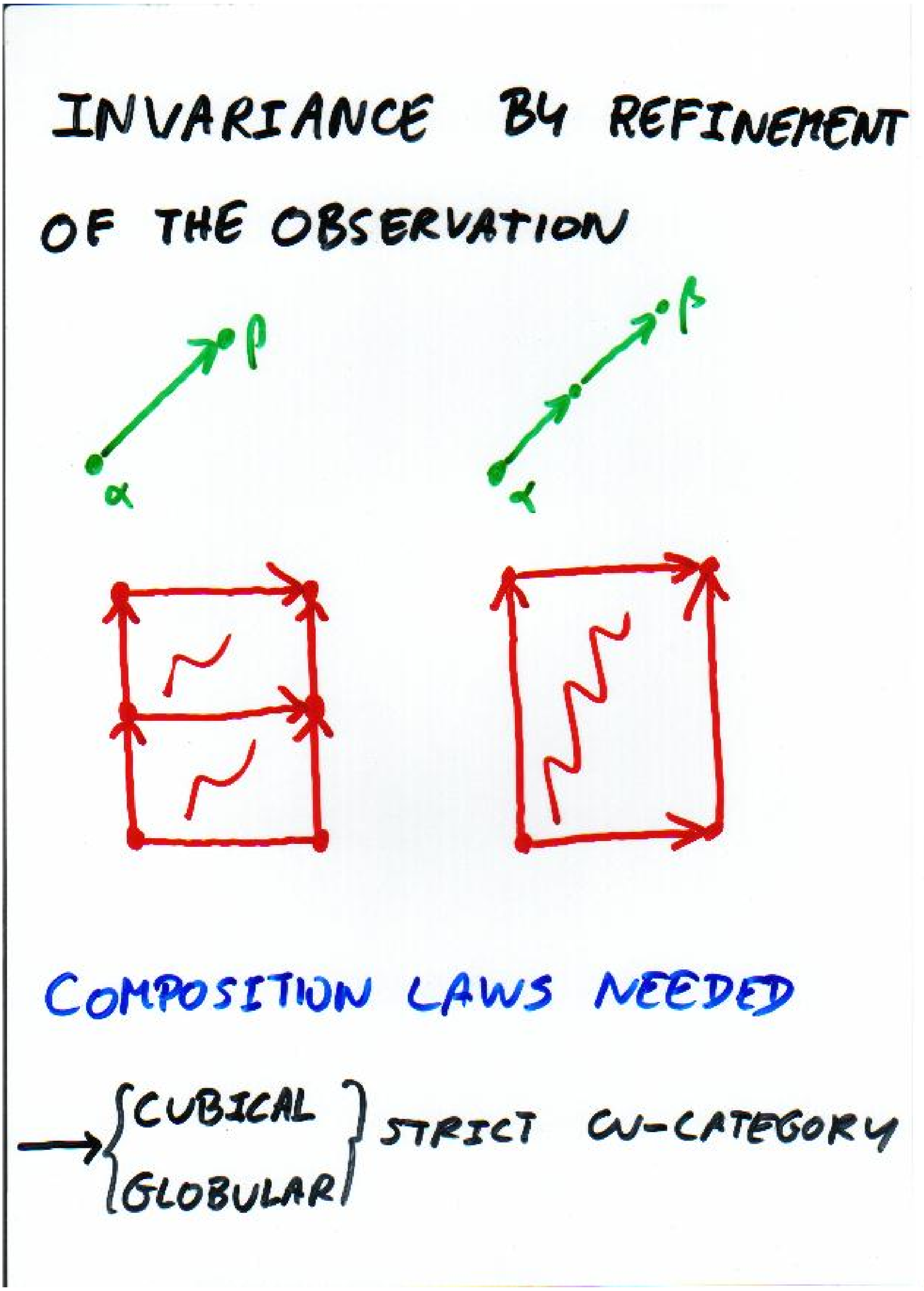

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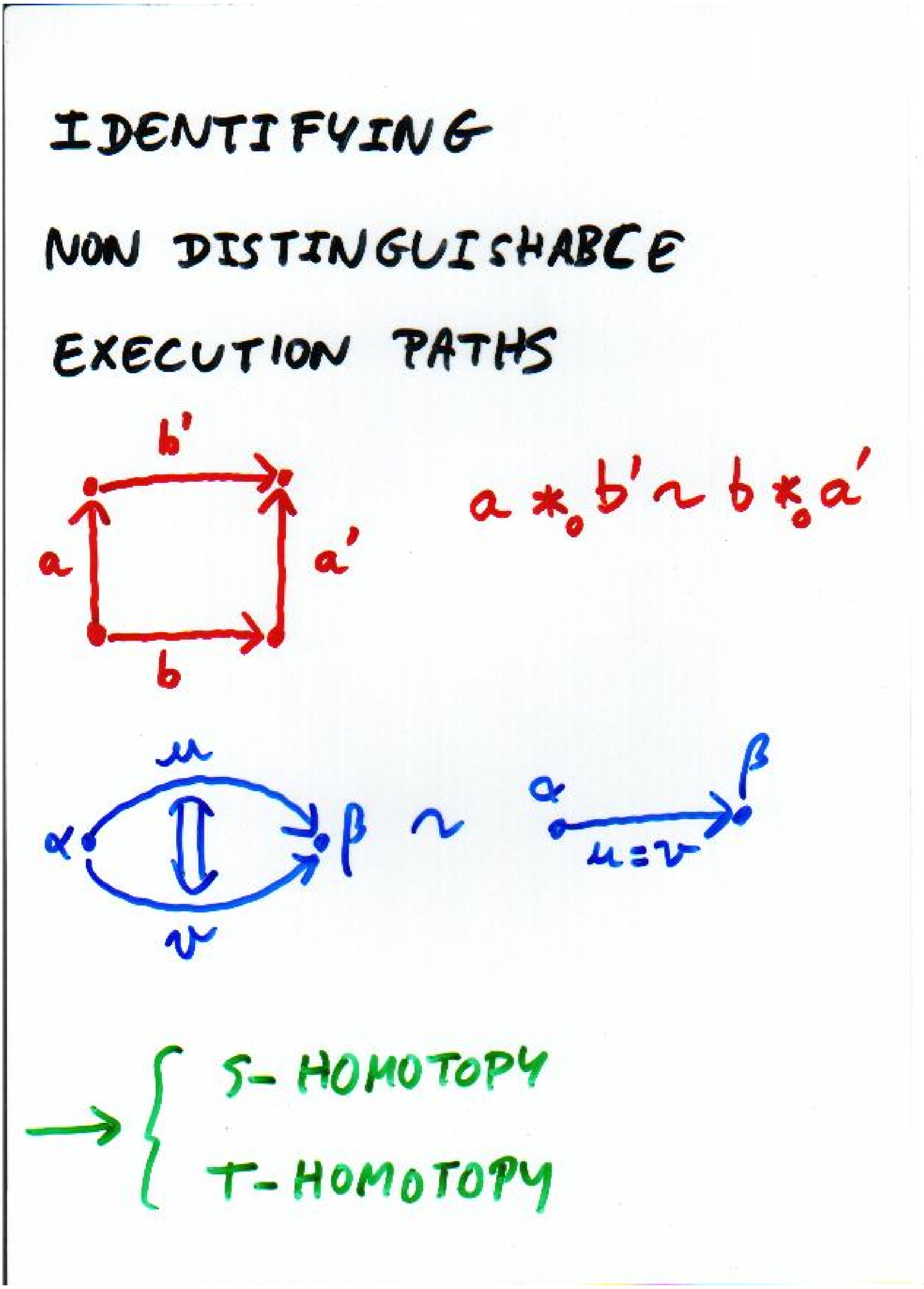

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WORKING WITH STRICT

ω-CATEGORIES $\mathcal{C}$ SUCH THAT

$$\left.\begin{matrix} x \in \mathcal{C} \\ \dim(x) \geq 2 \end{matrix}\right\} \Longrightarrow$$ x INVERTIBLE W.R.T. ALL COMPOSITION LAWS

$$(\mathbb{P}\mathcal{C})_n := \mathcal{C}_{n+1}$$

$\mathbb{P}\mathcal{C}$ ω-GROUPOID

$F : \mathcal{C} \longrightarrow \mathcal{D}$ NON-CONTRACTING FUNCTOR IF F INDUCES

$\mathbb{P}F : \mathbb{P}\mathcal{C} \to \mathbb{P}\mathcal{D}$

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$$\begin{tikzcd} & \mathbb{P}\mathcal{C} \arrow[dl, "h^-"'] \arrow[dr, "h^+"] & \\ \mathbb{P}^-\mathcal{C} & & \mathbb{P}^+\mathcal{C} \end{tikzcd}$$

DIAGRAM OF ω-GROUPOIDS

$$\mathbb{P}^-\mathcal{C} := \text{"} \mathbb{P}\mathcal{C} / _{x = x *_0 y} \text{"}$$

$$\mathbb{P}^+\mathcal{C} := \text{"} \mathbb{P}\mathcal{C} / _{y = x *_0 y} \text{"}$$

THREE INTERESTING AUGMENTED SIMPLICIAL NERVES

$$\begin{tikzcd} \operatorname{Sing} \mathbb{P}\mathcal{C} \arrow[d, "(s,t)"] & \operatorname{Sing} \mathbb{P}^-\mathcal{C} \arrow[d, "s"] & \operatorname{Sing} \mathbb{P}^+\mathcal{C} \arrow[d, "t"] \\ \mathcal{C}_0 \times \mathcal{C}_0 & \mathcal{C}_0 & \mathcal{C}_0 \end{tikzcd}$$

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ω-CATEGORY $\to$ GLOBULAR COMPLEX

TOP = COMPACTLY GENERATED TOPOLOGICAL SPACES

$Z \in$ TOP

$$Glob^{top}(Z) = \left(Z \times [0,1] \Big/ \begin{matrix} (z,0)=(z',0), & \{(*,0), \\ (z,1)=(z',1) & (*,1)\} \end{matrix} \right)$$

MULTIPOINTED TOPOLOGICAL SPACE

COMPARE WITH x n-morphism

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "s_{n-1}x"] \arrow[r, bend right=60, "t_{n-1}x"'] \arrow[r, phantom, "\Downarrow x"] & \bullet \end{tikzcd}$$

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MULTIPOINTED TOPOLOGICAL SPACE (X, X^0)

X^0 discrete $\subset X$

MORPHISM BETWEEN MULTIPOINTED SPACES

$$(X, X^0) \xrightarrow{f} (Y, Y^0)$$

$$f(X^0) \subset Y^0$$

COCOMPLETE CATEGORY

$\to \lambda$ - SEQUENCE

$$X : \lambda \longrightarrow \{\text{MULTIPOINTED SPACES}\}$$

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GLOBULAR COMPLEX

$X : \lambda \longrightarrow \{\text{MULTIPOINTED SPACE}\}$

$\forall \alpha < \lambda$

$$\begin{tikzcd}
\mathrm{Glob}^{top}(S^{n_\alpha - 1}) \arrow[r, "\text{NON DECREASING}"] \arrow[d] & X_\alpha \arrow[d] \\
\mathrm{Glob}^{top}(D^{n_\alpha}) \arrow[r] & X_{\alpha+1} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

MORPHISM $f : X \longrightarrow Y$

C^{tie} MAP $f : \varinjlim X \longrightarrow \varinjlim Y$

$$f(X^0) \subset Y^0$$

f NON DECREASING

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$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

TWO MORPHISMS OF $glTop$

S-HOMOTOPY EQUIVALENT IF

$\exists\ H : [0,1] \times X \longrightarrow Y$

$H_u = H(u, -)$

$H_u : X \to Y \in glTop$

$H_0 = f \quad H_1 = g$

$f \sim_S g$

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y \qquad \begin{array}{l} f \circ g \sim_S Y \\ g \circ f \sim_S X \end{array}$$

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$f : X \longrightarrow Y$ T-homotopy equivalence if

f induces homeomorphism.

GLOBULAR COMPLEXES UP TO S-HOMOTOPY

T-HOMOTOPY

Pb: . glTop NOT COCOMPLETE

. NOT ENOUGH T-HOMOTOPY

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THERE EXISTS A COMPLETE AND COCOMPLETE CATEGORY FLOW, A FUNCTOR $CAT : glTop \longrightarrow Flow$, A CLASS OF MORPHISMS OF FLOWS CALLED WEAK S-HOMOTOPY EQUIVALENCE:

$$\begin{tikzcd} glTop \arrow[r] \arrow[d] & Flow \arrow[d] \\ glTop[\mathcal{SH}^{-1}] \arrow[r, "\simeq"] & Flow[\mathcal{S}^{-1}] \end{tikzcd}$$

$\mathcal{SH}$ = S-homotopy

$\mathcal{S}$ = weak S-homotopy

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FLOW

$$\begin{tikzcd} \mathbb{P}X \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ X^0 \end{tikzcd}$$

$\mathbb{P}X$ = COMPACTLY GENERATED TOPOLOGICAL SPACE

X^0 = DISCRETE SPACE

$$\mathbb{P}_{\alpha\beta}X \times \mathbb{P}_{\beta\gamma}X \xrightarrow{*} \mathbb{P}_{\alpha\gamma}X$$

$*$ ASSOCIATIVE

$$\alpha \xrightarrow{u} \beta \qquad u \mapsto v * w \qquad \alpha \xrightarrow{v} \bullet \xrightarrow{w} \beta$$

$$R : \{0,1\} \longrightarrow \{0\}$$

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ω-CATEGORY $\longrightarrow$ FLOW

Z = COMPACTLY GENERATED TOPOLOGICAL SPACE

$$Glob(Z) = \begin{cases} \mathbb{P}Glob(Z) = Z \\ Glob(Z)^0 = \{0, 1\} \end{cases}$$

$s = 0 \quad t = 1$

NO COMPOSITION LAW

COMPARE WITH x n-morphism

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "s_{n-1}x"] \arrow[r, bend right=60, "t_{n-1}x"'] \arrow[r, phantom, "\Downarrow x"] & \bullet \end{tikzcd}$$

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GLOBULAR COMPLEXES ARE INTERESTING BECAUSE THEY ARE "TRUE" TOPOLOGICAL SPACES

MOREOVER, THERE EXISTS

$$F : glTop \longrightarrow LpTop$$

WITH THE ORDERING

$$(z,t) \leq (z',t') \iff \begin{cases} z = z' \\ t \leq t' \end{cases}$$

FLOWS ARE INTERESTING BECAUSE THEY HAVE "NICE" CATEGORICAL PROPERTIES

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$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \quad \in \mathrm{Flow}$$

$f \sim_S g$ IF

THERE EXISTS

$$[0,1] \xrightarrow{H} \mathrm{FLOW}(X,Y)$$

SUCH THAT

$$H(0)=f \quad H(1)=g$$

$X \xrightarrow{f} Y$ WEAK S-HOMOTOPY

IF $X^0 \xrightarrow{f} Y^0$ BIJECTION

$\mathbb{P}X \xrightarrow{f} \mathbb{P}Y$ WEAK HOMOTOPY EQUIVALENCE

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Thm: X, Y TWO GLOBULAR COMPLEXES.

$f : X \to Y$ T-HOMOTOPY EQUIVALENCE IFF

$cat(f) : cat(X) \longrightarrow cat(Y)$

TRANSFINITE COMPOSITION OF PUSHOUT OF $\varphi : \vec{I} \longrightarrow \vec{I} * \vec{I}$

WITH $\varphi([0,1]) = [0,1] * [0,1]$

WHERE $\vec{I}$ FLOW DEFINED BY

$\vec{I} = Glob(\{[0,1]\})$

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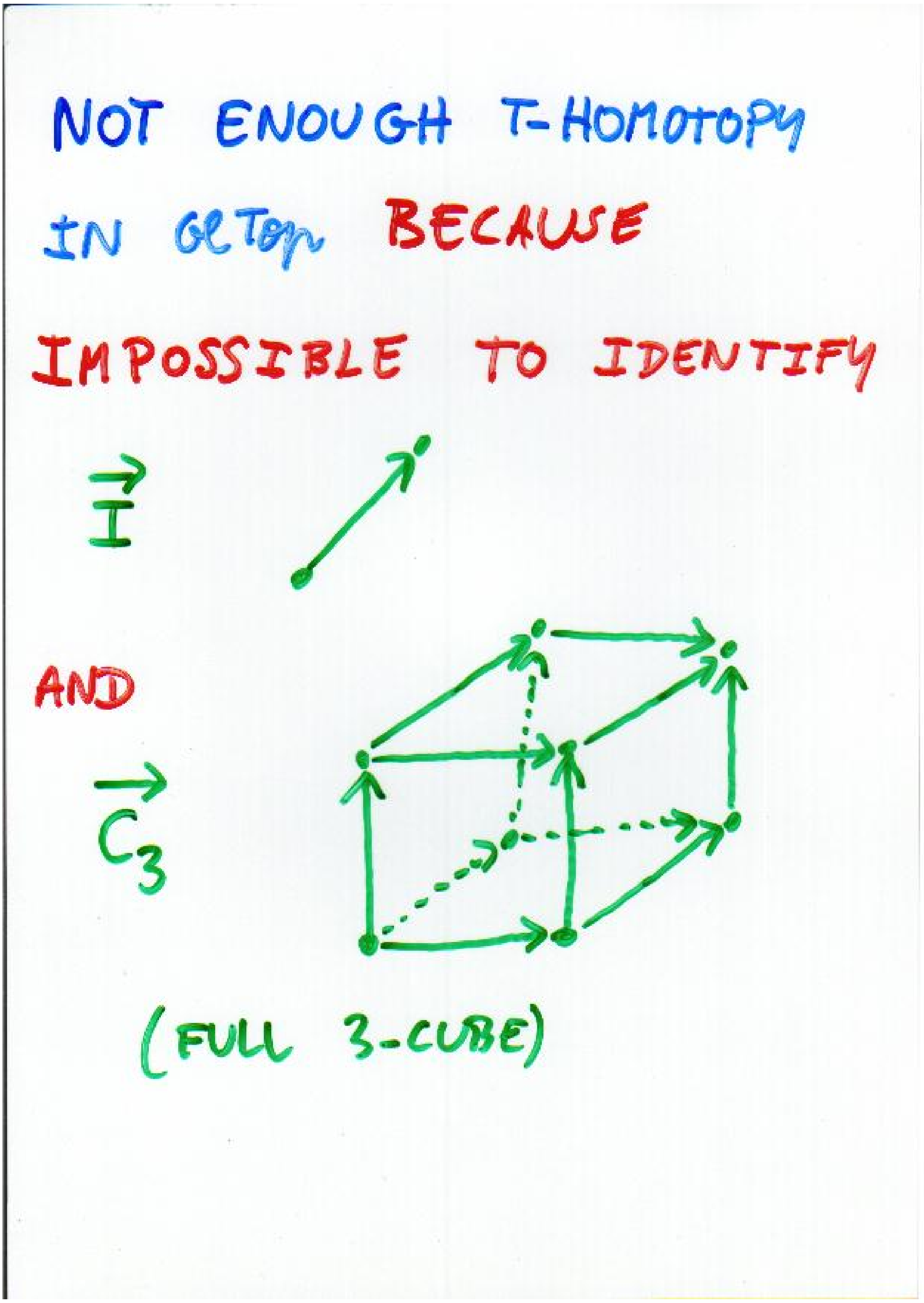

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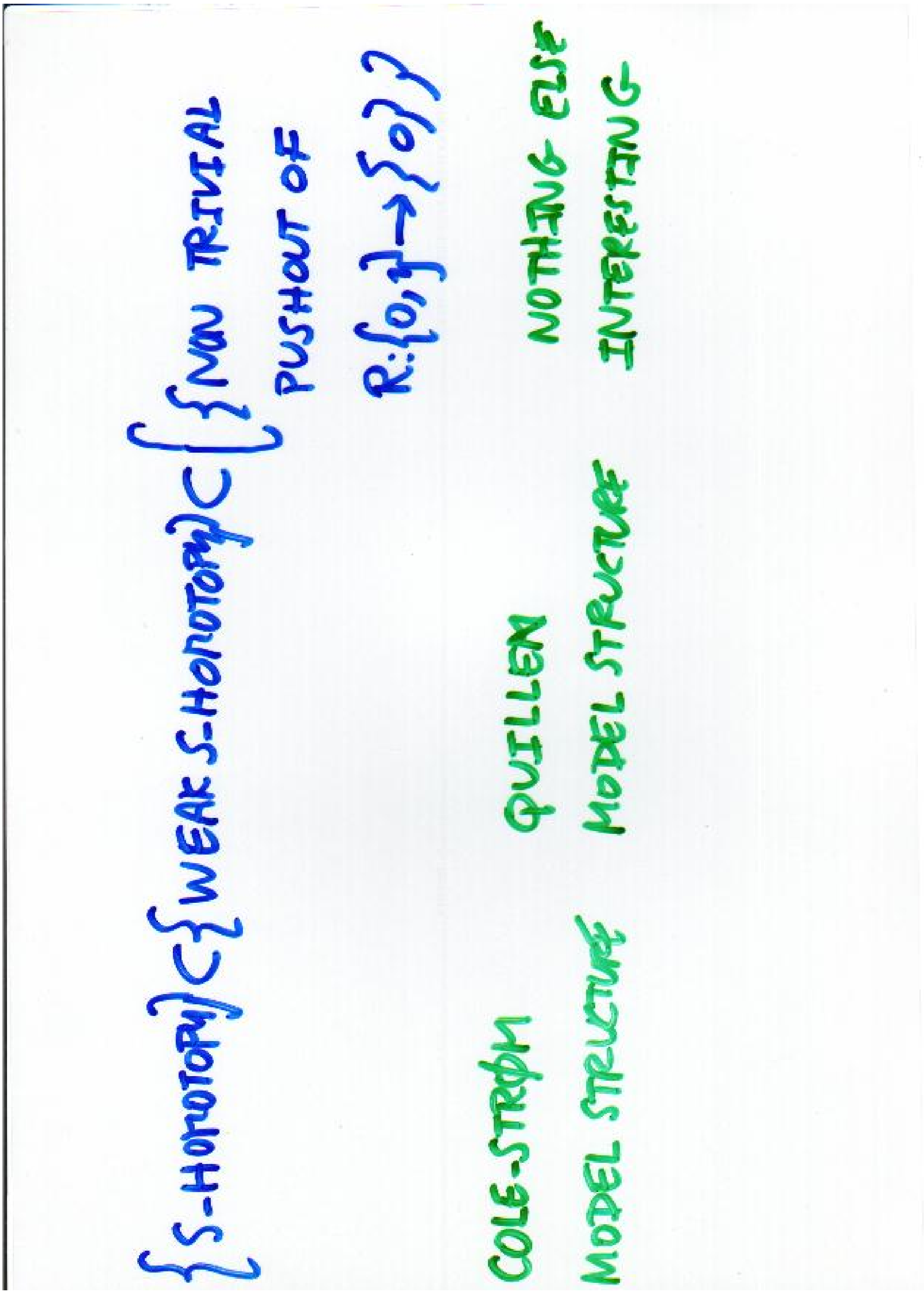

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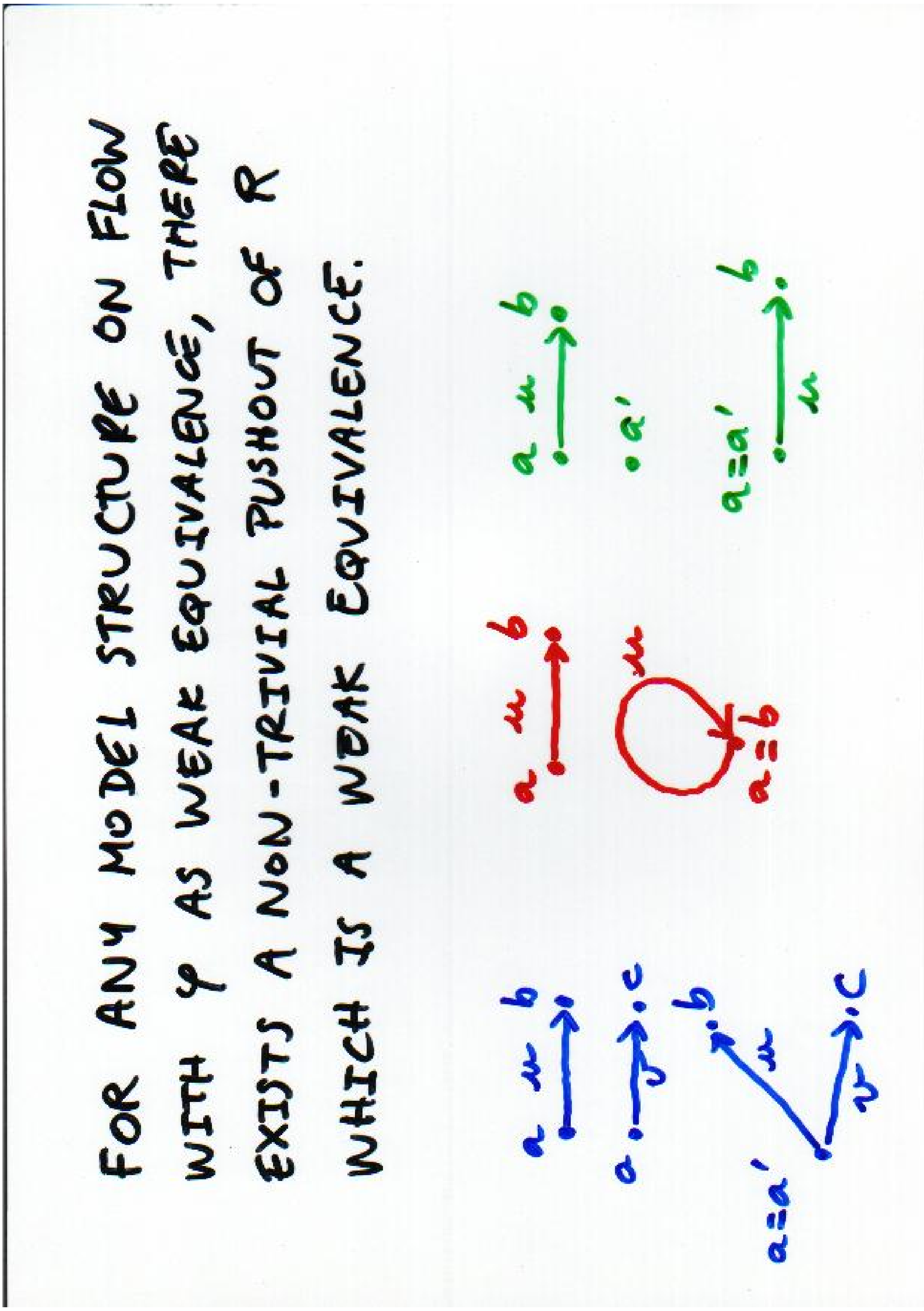

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QUILLEN MODEL STRUCTURE OF FLOW

COFIBRANTLY GENERATED

$$I = \{Glob(S^{n-1}) \to Glob(D^n)\}$$

$$\cup \{C : \emptyset \to \{0\}\} \cup \{R : \{0,1\} \to \{0\}\}$$

$$J = \{Glob(D^n \times \{0\}) \to Glob(D^n \times [0,1])\}$$

$\mathcal{W} = \{$ WEAK S-HOMOTOPY $\}$

Fib $= \{ f : X \to Y$ s.t.

$f : \mathbb{P}X \to \mathbb{P}Y$ SERRE FIBRATION $\}$

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COLE-STRØM MODEL STRUCTURE

PROBABLY NOT COFIBRANTLY GENERATED

$$\mathrm{Flow}([0,1]\boxtimes X, Y) \cong \mathrm{Top}([0,1], \mathrm{FLOW}(X,Y))$$

$$\mathrm{Fib} = \left\{ f \text{ RLP} \,/\, \begin{array}{c} \{0\}\boxtimes M \\ \downarrow \\ [0,1]\boxtimes M \end{array} \quad \forall M \right\}$$

$$\mathcal{W} = \{\text{S-HOMOTOPY}\}$$

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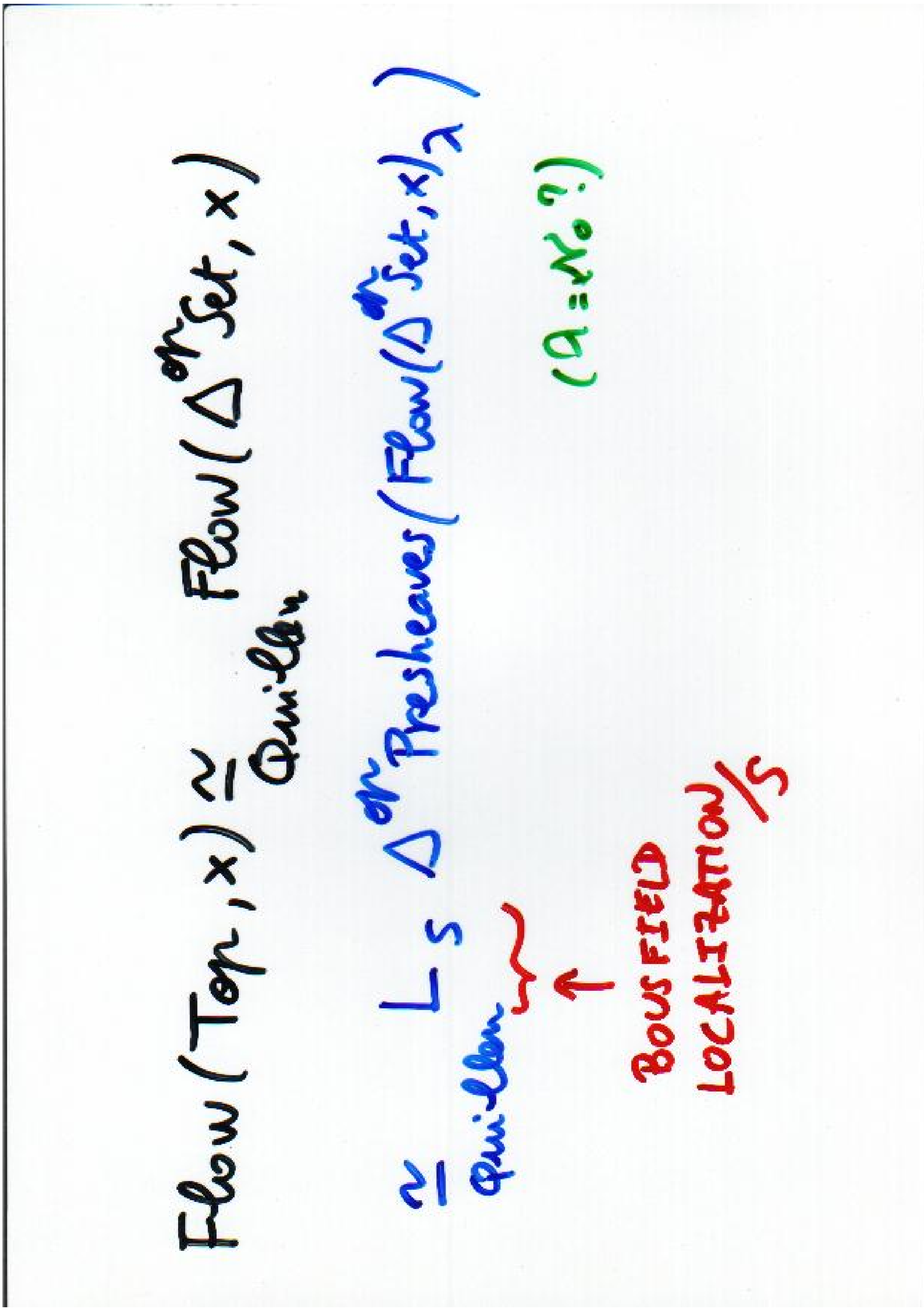

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THM: X, Y TWO GLOBULAR COMPLEXES $f : X \to Y$ T-HOMOTOPY EQUIVALENCE IFF

$cat(f) : cat(X) \to cat(Y)$

SATISFIES

1) $cat(X) \xrightarrow[\cong]{cat(f)} cat(Y)\restriction_{f(X^0)}$

2) $\alpha \in Y^0 \setminus f(X^0)$ NEITHER AN INITIAL NOR A FINAL STATE

3) $\alpha \in Y^0 \setminus f(X^0)$

$\mathbb{P}^-_\alpha Y$ and $\mathbb{P}^+_\alpha Y$ singletons

$\mathbb{P}^- Y = \mathbb{P}Y / _{x = x * y}$ $\quad \mathbb{P}^+ Y = \mathbb{P}Y / _{y = x * y}$

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FIRST IDEA OF GENERALIZATION FOR T-HOMOTOPY

$f : X \longrightarrow Y$

1) $X \xrightarrow{\cong} Y \restriction f(X^0)$

2) $\alpha \in Y^0 \setminus f(X^0)$ NEITHER AN INITIAL NOR A FINAL STATE

3) $\alpha \in Y^0 \setminus f(X^0)$

$\mathrm{ho}\mathbb{P}^-_\alpha Y$ and $\mathrm{ho}\mathbb{P}^+_\alpha Y$ (weakly) CONTRACTIBLE

Pb: IS THE UNDERLYING HOMOTOPY TYPE PRESERVED?

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ANOTHER (RECENT) IDEA OF GENERALIZATION FOR T-HOMOTOPY

$f : X \longrightarrow Y$ MORPHISM OF GLOBULAR COMPLEXES.

1) $\exists\, r : Y \longrightarrow X$ CONTINUOUS MAP
$r \circ f = Id_X$

2) $\forall \gamma$ EXECUTION PATH OF Y FROM $f(\alpha)$ TO $f(\beta)$, $r \circ \gamma$ EXECUTION PATH OF X

3) $\alpha \in Y^0 \setminus f(X^0)$ NEITHER INITIAL NOR FINAL

4) $f \circ r \underset{H}{\sim} Id_Y$

H SATISFYING 2) AS WELL

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PB: WHAT IS THE ANALOG OF THE LAST NOTION FOR FLOW

PB: COMPARING THE TWO GENERALIZATIONS

CONCLUSION:

FLOW ALLOWS THE STUDY OF HDA UP TO DIHOMOTOPY.

A LOT OF PROBLEMS REMAIN TO BE SOLVED BEFORE ANY APPLICATION IN COMPUTER SCIENCE

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References

[AHRT02] J. Adámek, H. Herrlich, J. Rosický, and W. Tholen. On a generalized small-object argument for the injective subcategory problem. *Cah. Topol. Géom. Différ. Catég.*, 43(2):83–106, 2002.

[AR94] J. Adámek and J. Rosický. *Locally presentable and accessible categories.* Cambridge University Press, Cambridge, 1994.

[BK72] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations.* Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 304.

[Bor94a] F. Borceux. *Handbook of categorical algebra. 1.* Cambridge University Press, Cambridge, 1994. Basic category theory.

[Bor94b] F. Borceux. *Handbook of categorical algebra. 2.* Cambridge University Press, Cambridge, 1994. Categories and structures.

[Bro88] R. Brown. *Topology.* Ellis Horwood Ltd., Chichester, second edition, 1988. A geometric account of general topology, homotopy types and the fundamental groupoid.

[Dug01a] D. Dugger. Combinatorial model categories have presentations. *Adv. Math.*, 164(1):177–201, 2001.

[Dug01b] D. Dugger. Universal homotopy theories. *Adv. Math.*, 164(1):144–176, 2001.

[Dug03] D. Dugger. Notes on Delta-generated spaces, 2003.

[Gau03a] P. Gaucher. Comparing globular CW-complex and flow, 2003. arXiv:math.AT/0308063.

[Gau03b] P. Gaucher. Homotopy branching space and weak dihomotopy, 2003. arXiv:math.AT/0304112.

[Gau03c] P. Gaucher. A model category for the homotopy theory of concurrency. *Homology, Homotopy and Applications*, 5(1):p.549–599, 2003.

[Gau04] P. Gaucher. S-homotopy as an analogue of homotopy of spaces, 2004. arXiv:math.AT/0401033.

[Gau04b] P. Gaucher. Flow does not model flows up to weak dihomotopy, 2004. arXiv:math.AT/0404328.

[GJ99] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory.* Birkhäuser Verlag, Basel, 1999.

[Hir03] P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2003.

[Hov99] M. Hovey. *Model categories.* American Mathematical Society, Providence, RI, 1999.

[Lew78] L. G. Lewis. *The stable category and generalized Thom spectra.* PhD thesis, University of Chicago, 1978.

[May99] J. P. May. *A concise course in algebraic topology.* University of Chicago Press, Chicago, IL, 1999.

[Qui67] D. G. Quillen. *Homotopical algebra.* Springer-Verlag, Berlin, 1967.

[Smi] J. Smith. A really convenient category of topological spaces. unpublished.

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