An application of Large Deviation Principle to pricing multi asset derivatives

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Method of Steepest Descent

Large Deviation Principle The pdf of diffusion processes in $\mathbb{R}^d$ with $t \ll 1$ is

Brownian Motion $\Rightarrow p(t,x,y) = c_t \exp\left[-\frac{1}{2t}\|x-y\|^2\right]$

Diffusion $\Rightarrow p(t,x,y) = c_t \exp\left[-\frac{1}{2t}d^2(x,y)\right]$

$p(t,x,y) \to \delta_x(y)$

$\mathbb{E}[f(x(t))] \to f(x(0))$

where c_t is the normalizing constant, $\|\cdot\|$ is the Euclidean norm and $d(\cdot,\cdot)$ is the Riemannian metric from the vol matrix of the diffusion. $\delta_x(y)$ denotes Dirac delta function.

The conditional pdf on a Borel set A with $t \ll 1$ is

$p(t,x,y \mid A) \to \delta_{x^*}(y)$

$\mathbb{E}[f(x(t)) \mid A] \to f(x^*)$

$x^* = \text{argmin}_{y\in A} d^2(x,y)$

Asymptotics for standard error

Asymptotic density of a multivariate diffusion

$$dy(t) = \mu(t,y(t))\,dt + \sigma(t,y(t))\,dW(t)$$

$$\frac{y(t)-y(0)}{\sqrt{t}} \to N\left(\vec{0},\Sigma\right)$$

in distribution as $t \to 0+$.

The most likely configuration of a mutivariate diffusion with constant volatility matrix

$$dx(t) = b(t,x(t))\,dt + a\,dW(t)$$

is simply a solution to quadratic programming

$$x^* = \text{argmin}_A (x - x(0))^T [a^{-1}](x - x(0)).$$

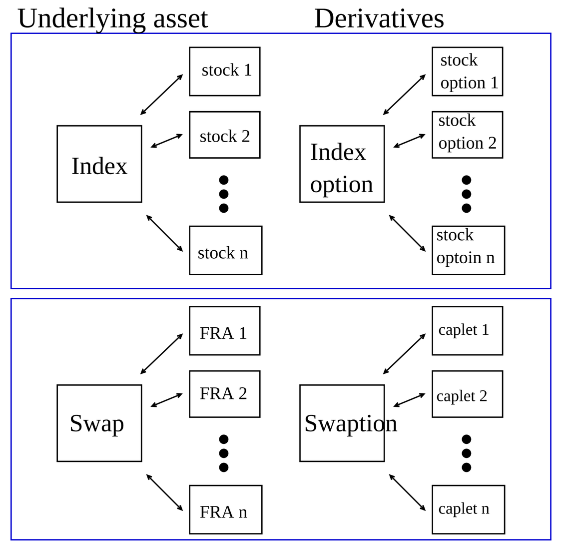
For reducible diffusions like $y(t) = \gamma(x(t))$ it is easy to convert to problems in $x(t)$. Otherwise more involved technique is necessary to find the most likely configuraiton.

Asymptotic conditional density of $x(t)$ is

$$\frac{x(t) - \mathbb{E}[x(t) \mid A]}{\sqrt{t}} \to N\left(\vec{0},[a]\right) \quad \text{on a plane}$$

for $x(t) \in A$. Easy to convert to this for $y(t) = \gamma(x(t))$.

Application to Relative pricing



Local volatility for general index

- Assets: $d\vec{s}(t) = \alpha(t,\vec{s}(t))\,dt + \beta(t,\vec{s}(t))dW(t)$
- General index: $I(t) = f(\vec{s}(t))$
- The index has stochastic volatility in $(s_1,\ldots,s_n)$
- Effective drift : $\alpha_L(t,I) = \mathbb{E}[\alpha(\vec{s},t)\cdot\nabla f(\vec{s}) \mid f(\vec{s}(t)) = I]$
- Effective variance:

$$\Gamma_L(t,I) = \sum_{i=1}^{d} \mathbb{E}^i\left[\frac{\partial f}{\partial s_i}(\vec{s}(t))\,|f(\vec{s}(t)) > I\right]\mathbb{E}\left[\sum_{j,k=1}^{d}\frac{\partial f}{\partial s_j}\beta_{ik}\beta_{kj}|f(\vec{s}(t)) = I\right]$$

For a linear index $I(t) = w_1 s_1(t) + \ldots + w_d s_d(t)$ it is simply

$$\sigma_L^2(t,I) = \mathbb{E}\left[\sum_{i,j=1}^{d} p_i(s)p_j(s)\sigma_i(s)\sigma_j(s)\rho_{ij}\Big|\sum_{i=1}^{d} w_i s_i(t) = I\right]$$

$$p_i(s) = \frac{w_i s_i}{\sum_{i=1}^d w_i s_i} \quad \text{for} i = 1\ldots d.$$

Combine LDP and Relative Pricing to estimate

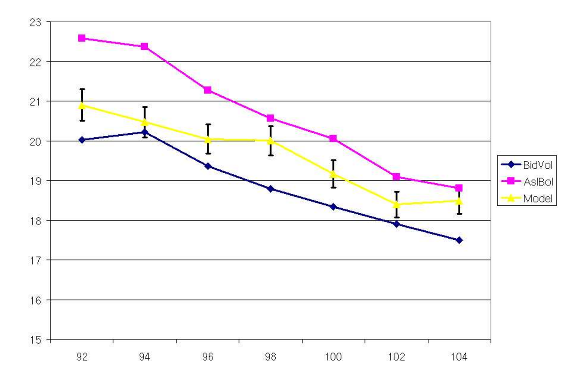
- the effective volatility surface (skew, termstructure) of index option relative to stock options and
- confidence band surrounding the skew.

$$\sigma_L^2(t,I) \to \sigma_L^2(\vec{s}^*) = \sum_{i,j=1}^{d} p_i(s^*)p_j(s^*)\sigma_i(s_i^*(t))\sigma_j(s_j^*(t))\rho_{ij}$$

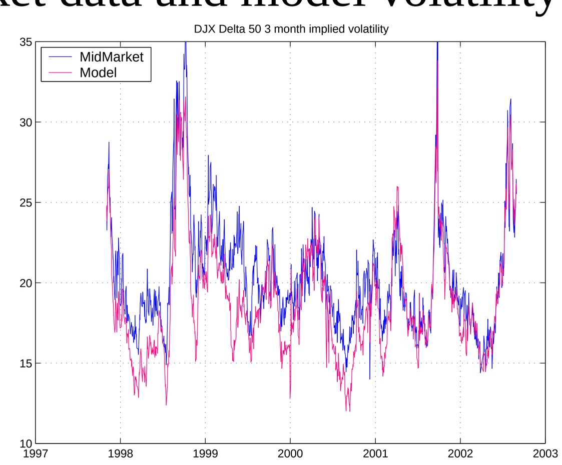
$$\frac{\sigma_L(\vec{s}(\vec{x}^*)) - \sqrt{\mathbb{E}[\sigma_L^2(\vec{s}(\vec{x})) \mid \vec{w}\cdot\vec{s}(\vec{x}) = I]}}{\sqrt{t}} \to N(0,C)$$

Empirical Evidence

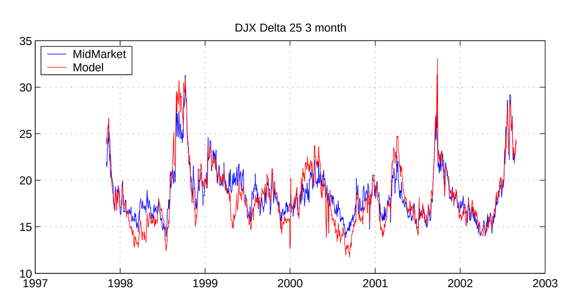
- Data: Dow Jones Industrial Average 11/1/1997-8/31/2002
- Reconstructed index volatility with confidence band

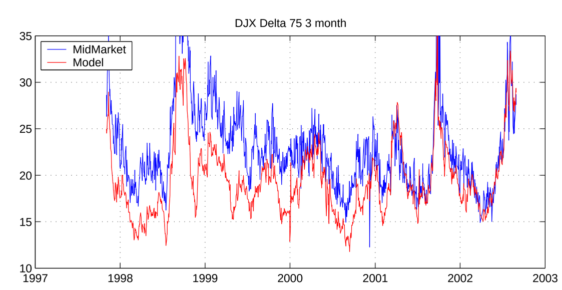


- DJX 3 month at the money implied volatility from market data and model volatility

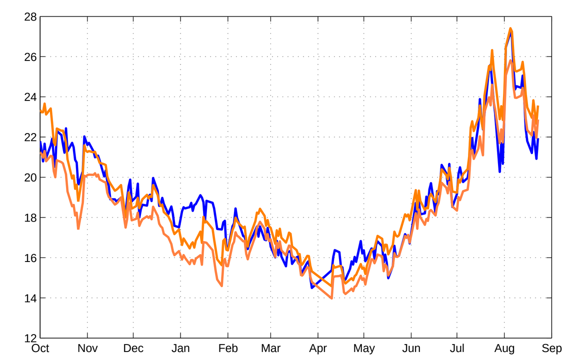


- DJX 3 month implied volatility from market data and model volatility.Top one is with Delta 25 and bottom one is with Delta 75





- The DJX 6 month at the money volatility overlapped with 99% confidence interval of model volatility 10/1/2001-8/24/2002



- Suspects for mispricing – left for future research
  - Correlation matrix is constant
    $\Rightarrow$ Fully state dependent correlaiton matrix is to be studied
  - Unexpected movement of correlation matrix
    $\Rightarrow$ Stochastic (or conditional heteroskedascic) correlation matrix

Conclusions

- LDP offers an effective way to estimate (conditional) expectations of multidimensional diffusion process without simulation.
- Assuming a short time horizon, we get consistency and asymptotic normality like in large sample theory.
- We can calibrate models with many variables relative to each other using marginalization and LDP asymptotics.
- More study is needed to enrich the model for practical purposes.