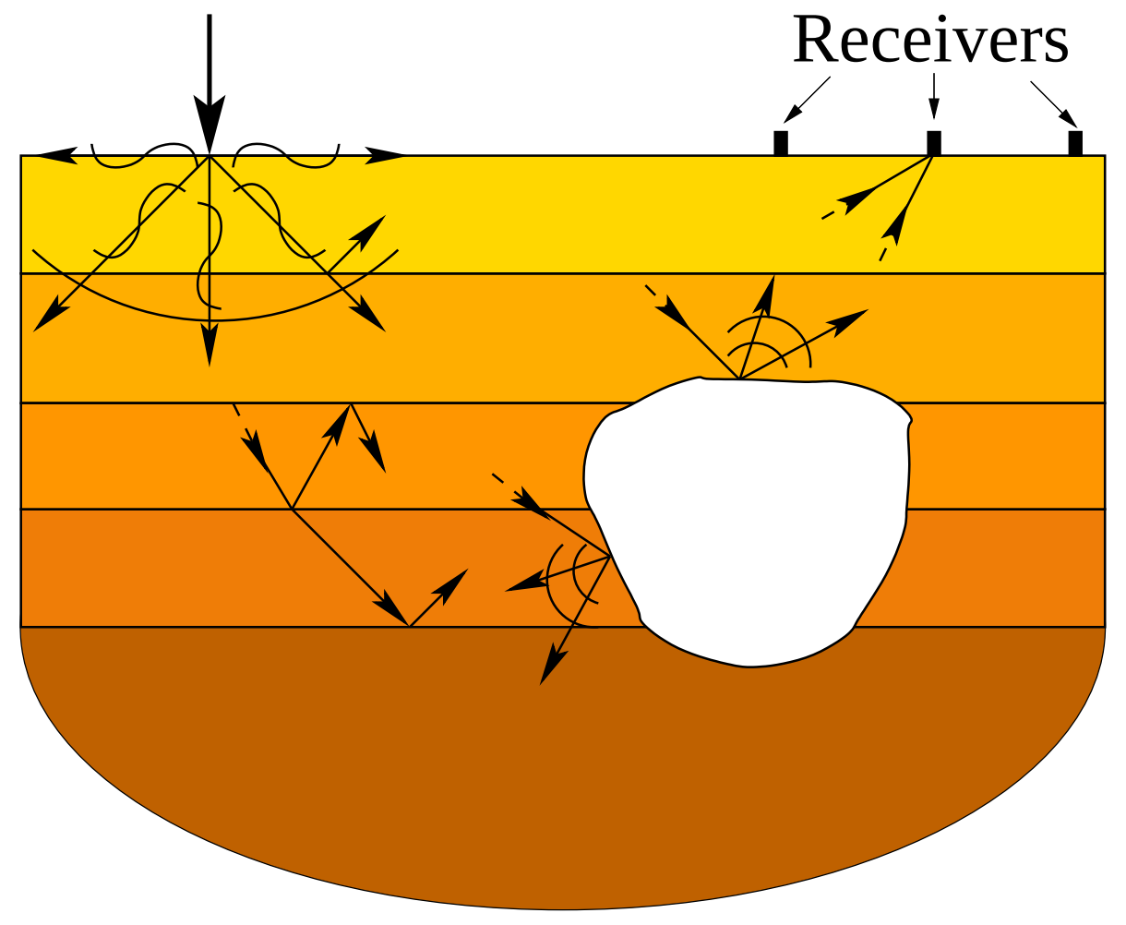
A Linear Sampling Method for Near-Field Inverse Problems in Elastodynamics

Bojan Guzina, Sylvain Nintcheu Fata, and Andrew Madyarov | **Civil Engineering, University of Minnesota**

`guzina@wave.ce.umn.edu`
`snintcheu@fata.ce.umn.edu`
`mady0007@umn.edu`

Introduction

The problem of rapid obstacle reconstruction in a (visco-) elastic **half-space** from near-field, surface seismic measurements is investigated within the framework of a **linear sampling method**.



✫ Applications

- ☞ Homeland security
- ☞ Environmental remediation
- ☞ Medical imaging
- ☞ Hydrocarbon exploration
- ☞ Land-mine detection

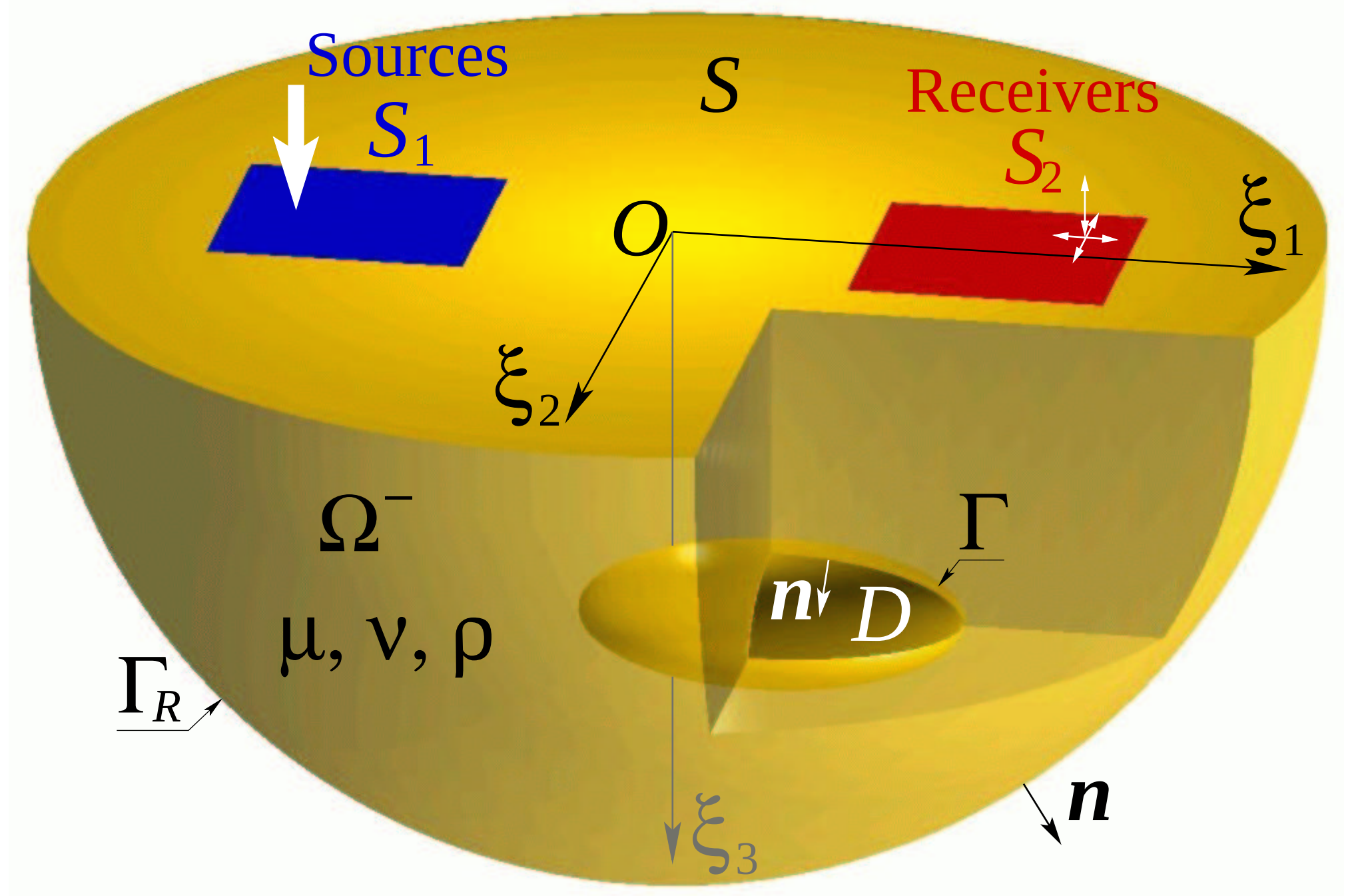
✫ Inverse scattering problem (nonlinear and ill-posed)

- ☞ Nonlinear optimization
  - Time consuming
  - Initial geometry and location
- ☞ Linear sampling method
  - Fast
  - No initial guess

✫ Linear sampling method in literature

Acoustics & Electromagnetism (full space)	**Elastodynamics**
☞ Far field waveforms ✍ Colton & Kirsch (1996) ✍ Colton et al. (2000)	☞ Full space, far field ✍ Gintides (1999) ✍ Pelekanos & Servoglou (2003)
☞ Near field waveforms ✍ Coyle (2000)	☞ Half-space, near field ✍ Nintcheu & Guzina (2004)

Direct Scattering Problem



✫ Total displacement: free (incident) and scattered field

$$\boldsymbol{u}(\boldsymbol{\xi}) = \boldsymbol{u}^{\mathrm{F}}(\boldsymbol{\xi}) + \boldsymbol{u}^{\mathrm{S}}(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \Omega^-$$

✫ Forward problem

☞ Navier equation

$$\mathcal{L}\boldsymbol{u}^{\mathrm{S}}(\boldsymbol{\xi}) + \rho\omega^2 \boldsymbol{u}^{\mathrm{S}}(\boldsymbol{\xi}) = \boldsymbol{0}, \;\; \boldsymbol{\xi} \in \Omega^-; \qquad \mathcal{L} \equiv \nabla \cdot \mathbf{C} : \nabla$$

☞ Neumann boundary conditions

$$\boldsymbol{t}^{\mathrm{S}}(\boldsymbol{\xi};\boldsymbol{n}) = -\boldsymbol{t}^{\mathrm{F}}(\boldsymbol{\xi};\boldsymbol{n}), \;\; \boldsymbol{\xi} \in \Gamma; \qquad \boldsymbol{t}^{\mathrm{S}}(\boldsymbol{\xi};\boldsymbol{n}) = \boldsymbol{0}, \;\; \boldsymbol{\xi} \in S$$

$\boldsymbol{t}^{\mathrm{F}}$ and $\boldsymbol{t}^{\mathrm{S}}$ are tractions corresponding to $\boldsymbol{u}^{\mathrm{F}}$ and $\boldsymbol{u}^{\mathrm{S}}$, $\boldsymbol{n}$ is normal to $\Gamma \cup S$

☞ Generalized radiation condition

$$\lim_{R\to\infty} \int_{\Gamma_R} \Big\{ \hat{\boldsymbol{u}}^j(\boldsymbol{\xi},\boldsymbol{x}) \cdot \boldsymbol{t}^{\mathrm{S}}(\boldsymbol{\xi};\boldsymbol{n}) - \hat{\boldsymbol{t}}^j(\boldsymbol{\xi},\boldsymbol{x};\boldsymbol{n}) \cdot \boldsymbol{u}^{\mathrm{S}}(\boldsymbol{\xi}) \Big\} \, \mathrm{d}s_{\boldsymbol{\xi}} = 0, \quad \forall \boldsymbol{x} \in \Omega^-$$

$\hat{\boldsymbol{u}}^j$ and $\hat{\boldsymbol{t}}^j$ are displacement and traction Green's functions for the elastic **half-space** due to a unit point force acting at $\boldsymbol{x}$ in direction j

Linear Sampling Method

✫ Green's and scattered displacement tensors

$$\widehat{\boldsymbol{U}}(\boldsymbol{\xi},\boldsymbol{\zeta}) \equiv \big(\, \hat{\boldsymbol{u}}^1(\boldsymbol{\xi},\boldsymbol{\zeta}), \;\; \hat{\boldsymbol{u}}^2(\boldsymbol{\xi},\boldsymbol{\zeta}), \;\; \hat{\boldsymbol{u}}^3(\boldsymbol{\xi},\boldsymbol{\zeta})\big) = \begin{pmatrix} \hat{u}_1^1 & \hat{u}_1^2 & \hat{u}_1^3 \\ \hat{u}_2^1 & \hat{u}_2^2 & \hat{u}_2^3 \\ \hat{u}_3^1 & \hat{u}_3^2 & \hat{u}_3^3 \end{pmatrix}$$

$$\boldsymbol{U}^{\mathrm{S}}(\boldsymbol{\xi},\boldsymbol{\zeta}) \equiv \big(\boldsymbol{u}^{\mathrm{S}^1}(\boldsymbol{\xi},\boldsymbol{\zeta}), \boldsymbol{u}^{\mathrm{S}^2}(\boldsymbol{\xi},\boldsymbol{\zeta}), \boldsymbol{u}^{\mathrm{S}^3}(\boldsymbol{\xi},\boldsymbol{\zeta})\big), \quad \boldsymbol{\xi} \in S_2, \;\; \boldsymbol{\zeta} \in S_1$$

Inverse scattering problem: Find D from the knowledge of $\boldsymbol{U}^{\mathrm{S}}$ over the observation surface S_2 for point sources over the source surface S_1.

✫ Free field: $$\boldsymbol{u}^{\mathrm{F}}(\boldsymbol{\xi}) = (\mathcal{F}\boldsymbol{g})(\boldsymbol{\xi}) \equiv \int_{S_1} \widehat{\boldsymbol{U}}(\boldsymbol{\xi},\boldsymbol{\zeta}) \cdot \boldsymbol{g}(\boldsymbol{\zeta}) \, \mathrm{d}s_{\boldsymbol{\zeta}}, \quad \boldsymbol{g} \in L_2(S_1)$$

✫ Scattered field: $$\boldsymbol{u}^{\mathrm{S}}(\boldsymbol{\xi}) = (\mathcal{S}\boldsymbol{g})(\boldsymbol{\xi}) \equiv \int_{S_1} \boldsymbol{U}^{\mathrm{S}}(\boldsymbol{\xi},\boldsymbol{\zeta}) \cdot \boldsymbol{g}(\boldsymbol{\zeta}) \, \mathrm{d}s_{\boldsymbol{\zeta}}$$

✫ Prescribed radiating field: $\hat{\boldsymbol{u}}_{\boldsymbol{z},\boldsymbol{d}}(\boldsymbol{\xi}) \equiv \widehat{\boldsymbol{U}}(\boldsymbol{\xi},\boldsymbol{z}) \cdot \boldsymbol{d}, \quad \boldsymbol{z} \in \Omega, \;\; |\boldsymbol{d}| = 1$

✫ Near-field integral equation:

$$\boxed{(\mathcal{S}\boldsymbol{g}_{\boldsymbol{z},\boldsymbol{d}})(\boldsymbol{\xi}) = \hat{\boldsymbol{u}}_{\boldsymbol{z},\boldsymbol{d}}(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in S_2}$$

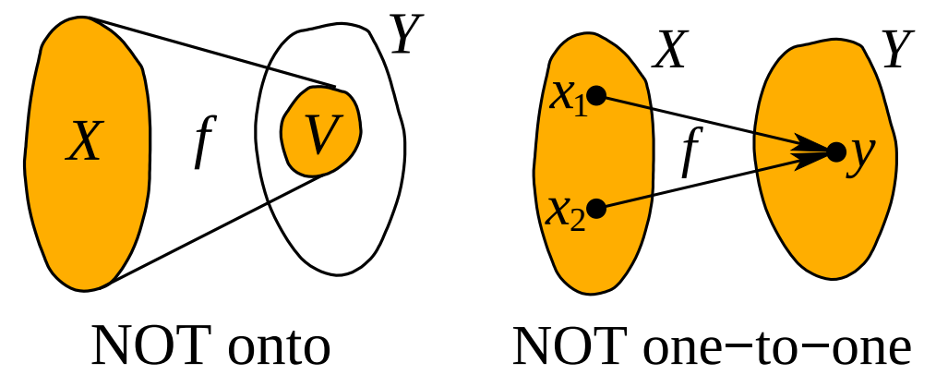
$$\|\boldsymbol{g}_{\boldsymbol{z},\boldsymbol{d}}\|_{L_2(S_1)} < \infty \iff \boldsymbol{z} \in D, \quad \text{i.e.} \quad \mathrm{supp}\left(\frac{1}{\|\boldsymbol{g}_{\boldsymbol{z},\boldsymbol{d}}\|_{L_2(S_1)}}\right) = \overline{D}$$

The formulation does not require any prior knowledge of obstacle geometry and boundary conditions. Issues: uniqueness and denseness.

Improperly Posed Problem

✫ Well-posed problem (Hadamard, 1923)

- ☞ Existence. $\mathcal{S}$ is onto
- ☞ Uniqueness. $\mathcal{S}$ is one-to-one
- ☞ Stability. $\mathcal{S}^{-1}$ is continuous



✫ Ill-posed problem – otherwise

$$\mathcal{S} : L_2(S_1) \to L_2(S_2)$$

- ☞ $\mathcal{S} \sim$ onto: $\overline{\mathcal{S}\{L_2(S_1)\}} = L_2(S_2)$
- ☞ $\mathcal{S}$ is one-to-one
- ☞ $\mathcal{S}$ – compact $\Rightarrow \nexists$ continuous $\mathcal{S}^{-1}$

Singular values of $\mathcal{S}$

Mathematical Foundation

✫ Denseness. Let $\boldsymbol{z} \in D$. $\forall\, \varepsilon > 0 \;\; \exists\, \boldsymbol{g}^{\varepsilon}_{\boldsymbol{z},\boldsymbol{d}} \in L_2(S_1)$:

$$\|\mathcal{S}\boldsymbol{g}^{\varepsilon}_{\boldsymbol{z},\boldsymbol{d}} - \hat{\boldsymbol{u}}_{\boldsymbol{z},\boldsymbol{d}}\|_{L_2(S_2)} < \varepsilon, \qquad \lim_{\boldsymbol{z}\to\boldsymbol{\xi}\in\Gamma} \|\boldsymbol{g}^{\varepsilon}_{\boldsymbol{z},\boldsymbol{d}}\|_{L_2(S_1)} = \infty$$

✫ Uniqueness $\iff \nexists\, \boldsymbol{g} \in L_2(S_1) : \boldsymbol{v} \equiv \mathcal{F}\boldsymbol{g} \neq \boldsymbol{0}$ solves

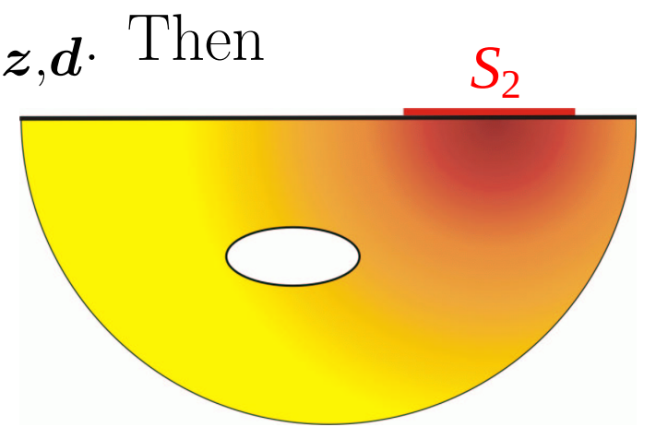
$$\mathcal{L}\boldsymbol{v} + \rho\omega^2\boldsymbol{v} = \boldsymbol{0} \;\; \text{in} \;\; D; \qquad \boldsymbol{t}_{\boldsymbol{v}} = \boldsymbol{0} \;\; \text{on} \;\; \Gamma$$

☞ Interior Neumann problem. $\mathcal{S}\boldsymbol{g}_{\boldsymbol{z},\boldsymbol{d}} = \hat{\boldsymbol{u}}_{\boldsymbol{z},\boldsymbol{d}}$ on $S_2 \iff \boldsymbol{u}^{\mathrm{F}} \equiv \mathcal{F}\boldsymbol{g}_{\boldsymbol{z},\boldsymbol{d}}$:

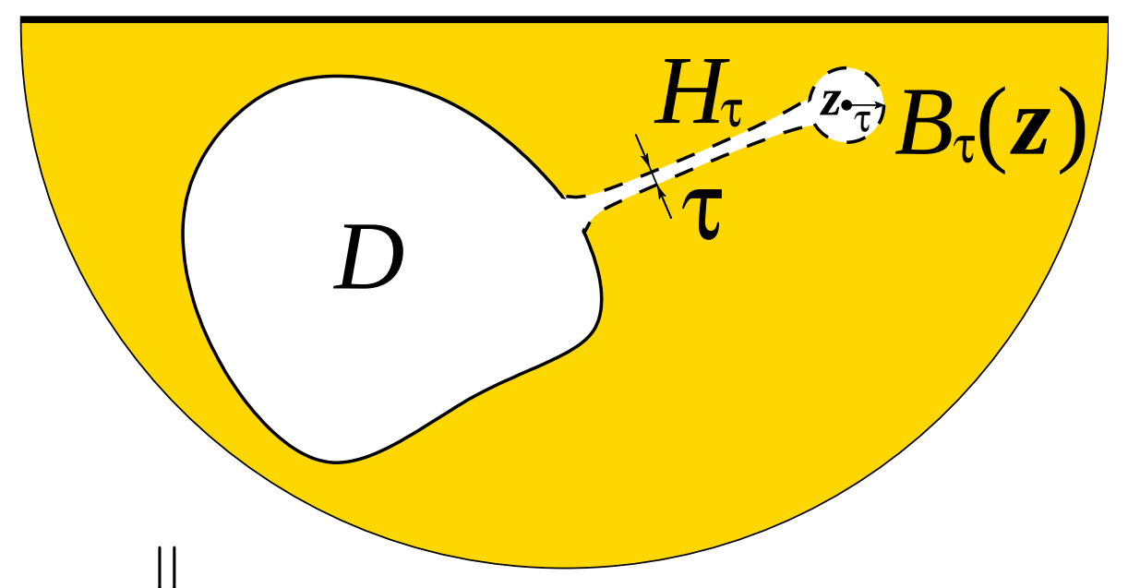
$$\mathcal{L}\boldsymbol{u}^{\mathrm{F}}(\boldsymbol{\xi}) + \rho\omega^2\boldsymbol{u}^{\mathrm{F}}(\boldsymbol{\xi}) = \boldsymbol{0}, \;\; \boldsymbol{\xi} \in D; \qquad \boldsymbol{t}^{\mathrm{F}}(\boldsymbol{\xi}) + \widehat{\boldsymbol{T}}(\boldsymbol{\xi},\boldsymbol{z}) \cdot \boldsymbol{d} = \boldsymbol{0}, \;\; \boldsymbol{\xi} \in \Gamma$$

☞ Holmgren's uniqueness theorem. Let $\boldsymbol{u}^{\mathrm{S}} \equiv \mathcal{S}\boldsymbol{g}_{\boldsymbol{z},\boldsymbol{d}}$. Then

$$\boldsymbol{u}^{\mathrm{S}} = \hat{\boldsymbol{u}}_{\boldsymbol{z},\boldsymbol{d}}, \;\; \boldsymbol{t}^{\mathrm{S}} = \hat{\boldsymbol{t}}_{\boldsymbol{z},\boldsymbol{d}} = \boldsymbol{0} \;\; \text{on} \;\; S_2$$
$$\Downarrow$$
$$\boldsymbol{u}^{\mathrm{S}} = \hat{\boldsymbol{u}}_{\boldsymbol{z},\boldsymbol{d}} \;\; \text{in} \;\; \overline{\Omega^-}$$



✫ Exterior Domain ($\boldsymbol{z} \in \Omega^-$)



Geometric perturbation of D

☞ $\tilde{D} \equiv D \cup H_\tau \cup B_\tau(\boldsymbol{z})$ – connected, $\boldsymbol{z} \in \tilde{D}$

☞ $\forall\, \varepsilon, \tau > 0 \quad \exists\, \tilde{\boldsymbol{g}}^{\varepsilon,\tau}_{\boldsymbol{z},\boldsymbol{d}} \in L_2(S_1)$:

$$\left\| \int_{S_1} \widetilde{\boldsymbol{U}}^{\mathrm{S}}(\cdot,\boldsymbol{\zeta}) \cdot \tilde{\boldsymbol{g}}^{\varepsilon,\tau}_{\boldsymbol{z},\boldsymbol{d}}(\boldsymbol{\zeta}) \, \mathrm{d}s_{\boldsymbol{\zeta}} - \hat{\boldsymbol{u}}_{\boldsymbol{z},\boldsymbol{d}}(\cdot) \right\|_{L_2(S_2)} < \varepsilon$$

$$\lim_{\tau\to0} \widetilde{\boldsymbol{U}}^{\mathrm{S}}(\boldsymbol{\xi},\boldsymbol{\zeta}) = \boldsymbol{U}^{\mathrm{S}}(\boldsymbol{\xi},\boldsymbol{\zeta}), \;\; \forall\, \boldsymbol{\xi} \in S_2, \; \boldsymbol{\zeta} \in S_1; \qquad \lim_{\tau\to0} \|\tilde{\boldsymbol{g}}^{\varepsilon,\tau}_{\boldsymbol{z},\boldsymbol{d}}\|_{L_2(S_1)} = \infty$$

Regularization

✫ Errors

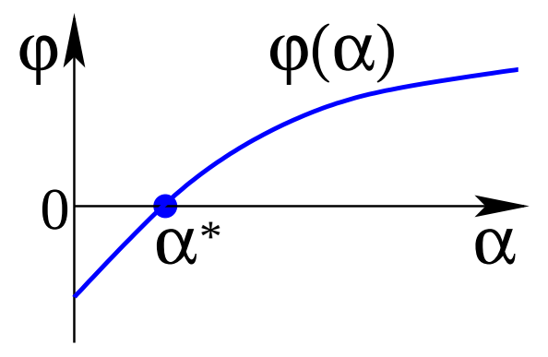
☞ Measurement (LHS) $$\|\mathcal{S} - \mathcal{S}^{\varepsilon}\| \equiv \sup_{\|\boldsymbol{g}\|_{L_2(S_1)}=1} \|(\mathcal{S} - \mathcal{S}^{\varepsilon})\boldsymbol{g}\|_{L_2(S_2)} \leq \varepsilon$$

☞ Computation (RHS) $$\|\hat{\boldsymbol{u}}_{\boldsymbol{z},\boldsymbol{d}} - \hat{\boldsymbol{u}}^{\delta}_{\boldsymbol{z},\boldsymbol{d}}\|_{L_2(S_2)} \leq \delta$$

✫ Tikhonov functional, $\alpha > 0$

$$J_\alpha(\boldsymbol{g}) = \|\mathcal{S}^{\varepsilon}\boldsymbol{g} - \hat{\boldsymbol{u}}^{\delta}_{\boldsymbol{z},\boldsymbol{d}}\|^2_{L_2(S_2)} + \alpha\|\boldsymbol{g}\|^2_{L_2(S_1)} \;\to\; \min \quad \Longrightarrow \quad \boldsymbol{g}^{\alpha,\varepsilon,\delta}_{\boldsymbol{z},\boldsymbol{d}}$$

✫ Discrepancy principle of Morozov



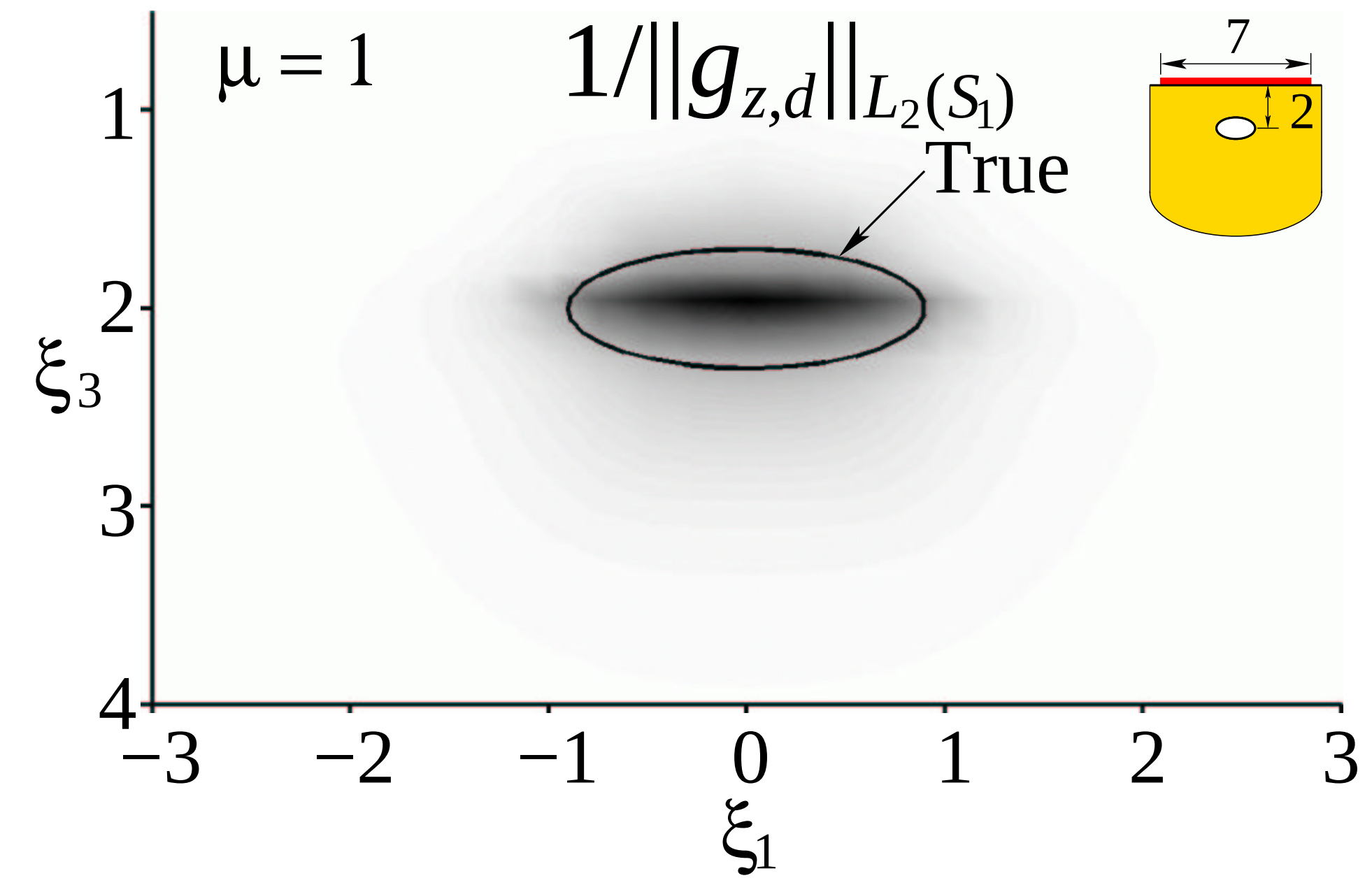
☞ **A posteriori** determination of α

$$\|\mathcal{S}^{\varepsilon}\boldsymbol{g}^{\alpha,\varepsilon,\delta}_{\boldsymbol{z},\boldsymbol{d}} - \hat{\boldsymbol{u}}^{\delta}_{\boldsymbol{z},\boldsymbol{d}}\|_{L_2(S_2)} = \delta + \varepsilon\|\boldsymbol{g}^{\alpha,\varepsilon,\delta}_{\boldsymbol{z},\boldsymbol{d}}\|_{L_2(S_1)}$$

☞ $\delta \ll \varepsilon \;\Rightarrow$ optimal value α^* of α:

$$\varphi(\alpha^*) \equiv \|\mathcal{S}^{\varepsilon}\boldsymbol{g}^{\alpha^*,\varepsilon,\delta}_{\boldsymbol{z},\boldsymbol{d}} - \hat{\boldsymbol{u}}^{\delta}_{\boldsymbol{z},\boldsymbol{d}}\|^2_{L_2(S_2)} - \varepsilon^2\|\boldsymbol{g}^{\alpha^*,\varepsilon,\delta}_{\boldsymbol{z},\boldsymbol{d}}\|^2_{L_2(S_1)} = 0$$

✫ Example

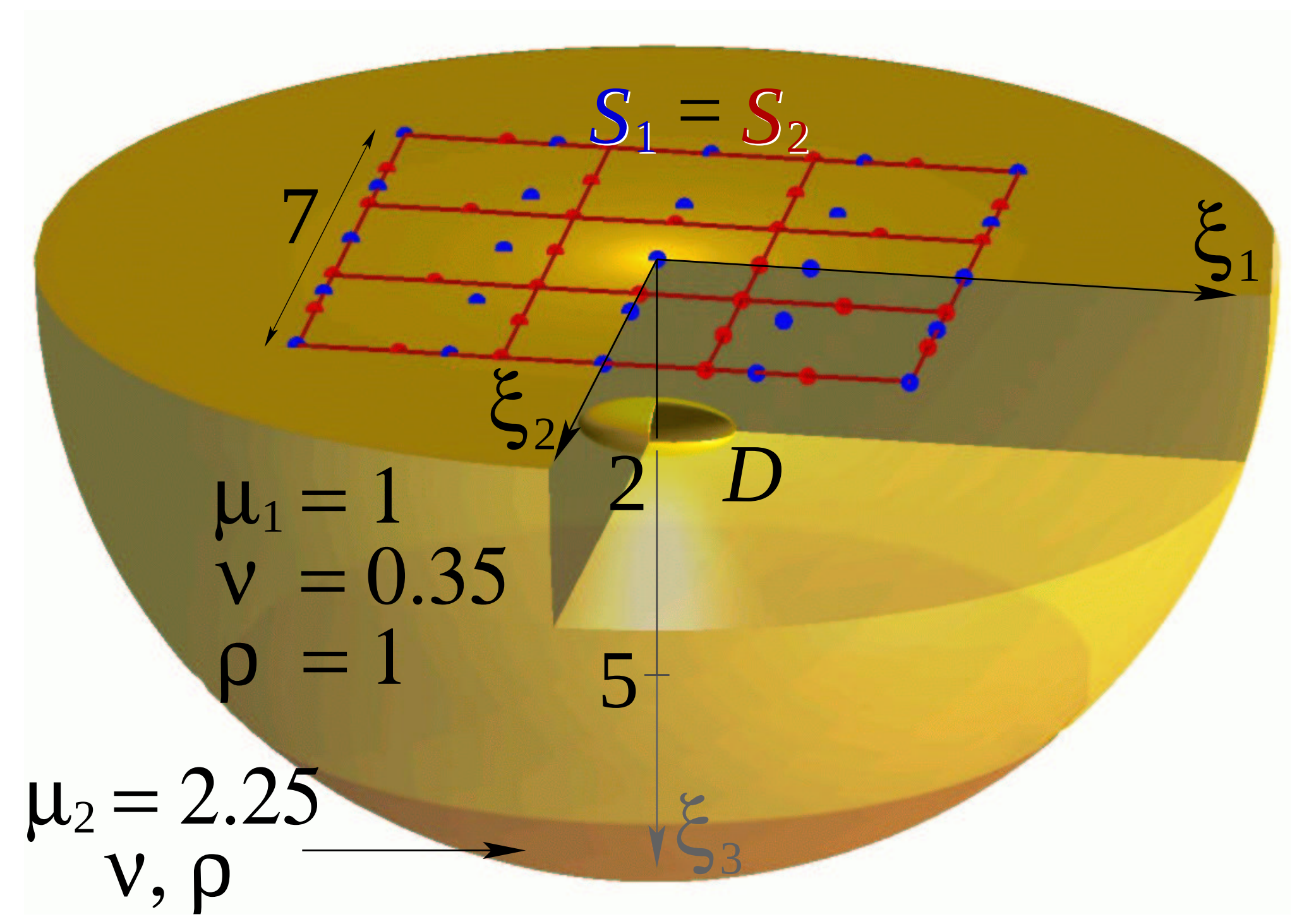


Layered Half-Space

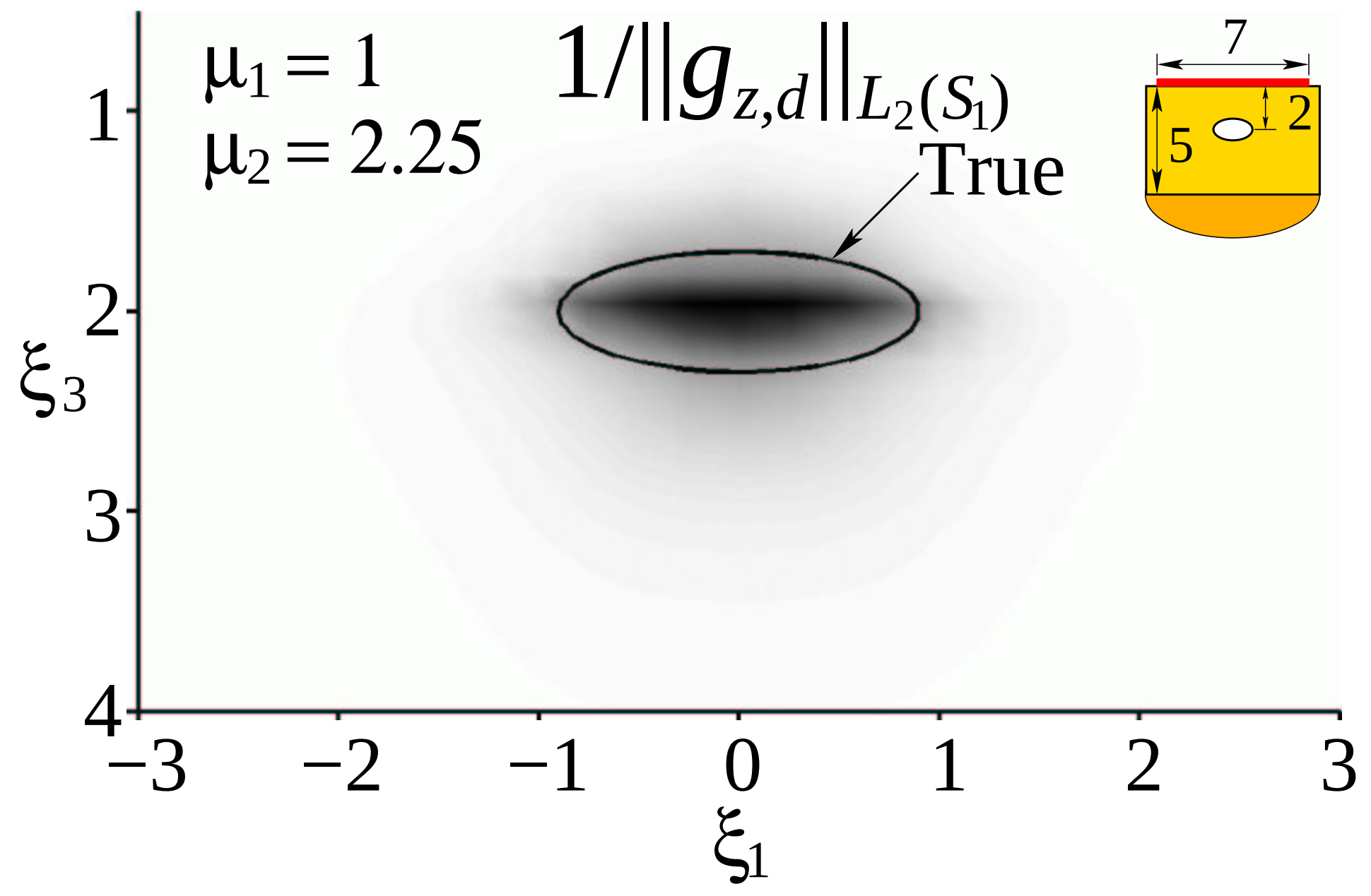
✫ Multi-layered Green's tensor $\widehat{\boldsymbol{U}}$ (Guzina & Pak, 2001)

$$\boldsymbol{u}^{\mathrm{S}} \equiv \boxed{\mathcal{S}\boldsymbol{g}_{\boldsymbol{z},\boldsymbol{d}} = \hat{\boldsymbol{u}}_{\boldsymbol{z},\boldsymbol{d}}} \equiv \widehat{\boldsymbol{U}}(\cdot,\boldsymbol{z}) \cdot \boldsymbol{d} \quad \text{on} \;\; S_2$$

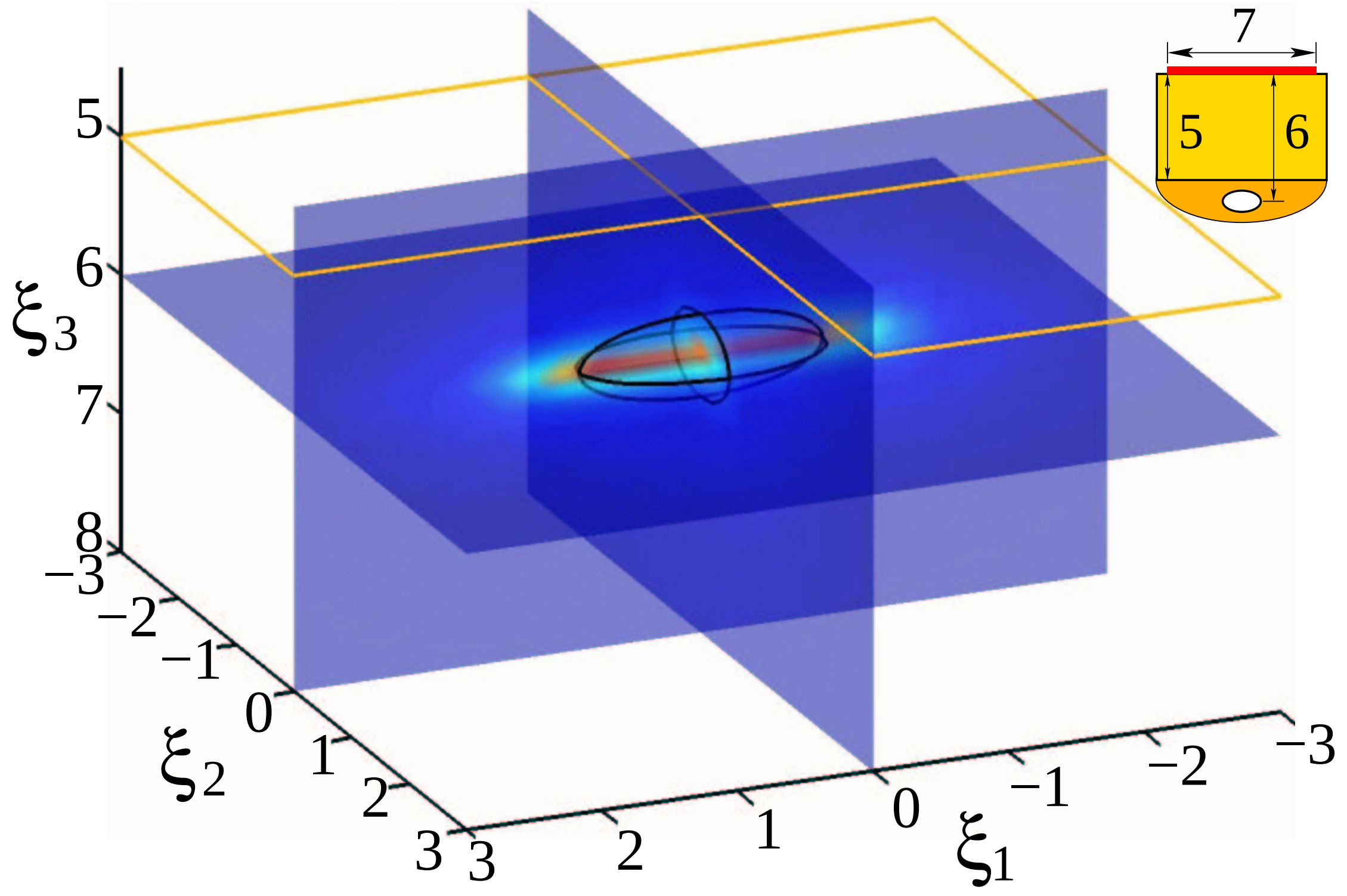
✫ Application of LSM



✫ Obstacle above layer interface



✫ Object below layer interface



✫ Theoretical foundation

Denseness & Uniqueness ⟸ Holmgren's uniqueness theorem

$$\nabla\boldsymbol{u}^{\mathrm{S}} \cdot \mathbf{C} : \nabla\boldsymbol{u}^{\mathrm{S}} + \rho\omega^2\boldsymbol{u}^{\mathrm{S}} = \boldsymbol{0}$$

Layered media ⟹ Piecewise constant $\mathbf{C}$, ρ

Summary and Conclusions

✫ Linear Sampling Method

- ☞ Near-field formulation, homogeneous/layered half-space
- ☞ Minimization free, computationally efficient
- ☞ Regularization, image resolution
- ☞ Experimental setup, measurement errors

References

COLTON, D. AND KIRSCH, A. (1996). *Inverse Problems*, **12**:383–393.

COLTON, D., COYLE, J. AND MONK, P. (2000). *SIAM Review*, **42**:369–414.

COYLE, J. (2000). *Inverse Problems*, **16**:275–292.

GINTIDES, D. (1999). *Z. Angew. Math. Mech.*, **79**:675–684.

GUZINA, B. B. AND PAK, R. Y. S. (2001). *Quart. J. Mech. Appl. Math.*, **54**:13–37.

HADAMARD, J. (1923). *Lectures on the Cauchy Problem in Linear Partial Differential Equations*, Yale University Press.

NINTCHEU FATA, S. AND GUZINA, B. B. (2004). *Inverse Problems*, **20**:713–736.

PELEKANOS, G. AND SEVROGLOU, V. (2003). *J. Comp. and Appl. Math.*, **151**:129–140.