

Bivariate Uniqueness and Endogeny for Recursive Distributional Equations : Two Examples

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Abstract

In this work we prove the *bivariate uniqueness* property for two “max-type” *recursive distributional equations* which then lead to the proof of *endogeny* for the associated *recursive tree processes*. Thus providing two concrete instances of the general theory developed by Aldous and Bandyopadhyay [3]. The first example discussed here deals with the construction of a frozen percolation process on a infinite regular binary tree. For this we prove that the construction do not involve any external randomness. It is also shown that same is true for any r -regular tree and more interestingly for any infinite regular Galton-Watson branching process trees with mild moment condition on the progeny distribution. The second example is proving the endogeny for the *Logistic* recursive distributional equation which appears for studying the asymptotic limit of the random assignment problem using local-weak convergence method. The two examples are quite unrelated and hence illustrate a broad range of applicability of the general methods of [3].

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1 Introduction and Main Results

In recent years distributional identities have become an intense topic of discussion. Recently the general framework developed by Aldous and Bandyopadhyay [3] covers a variety of settings, for example, the study of Galton-Watson branching process; characterization of probability distributions [13, 24]; probabilistic analysis of random algorithms [11, 20, 21, 12, 10]; study of branching random walks [15, 2, 8, 7, 3]; various statistical physics problems, like the frozen percolation on regular trees [5]; combinatorial optimization problems and the local-weak convergence method [1, 6, 4]. In the general setting (see [3] for a detailed definition) a distributional identity looks like

$$X \stackrel{d}{=} g(\xi; (X_j : 1 \leq j \leq^* N)),$$

where $(X_j)_{j \geq 1}$ are i.i.d with same distribution as X and are independent of the pair (ξ, N) , where N is a positive integer valued random variable which may take the value ∞ , and g is a given function. (In the above equation by “ $\leq^* N$ ” we mean that left hand side is “ $\leq N$ ” if $N < \infty$ and “ $< N$ ” if $N = \infty$). Following [3] we will call such distributional identities *recursive distributional equations* (RDE). Typically the unknown quantity for a RDE is the distribution of X , while the joint distribution of the pair (ξ, N) is known and so is the function g . As shown in [3] exploiting the natural recursive structure one can formalize a solution of a RDE in terms of a tree-indexed random process, called *recursive tree process* (RTP). In many applications the variables of the RTP are used as auxiliary variables to define or construct some useful random structure, in this article we will see two such applications. In [3] Aldous and Bandyopadhyay emphasize on *max-type* RDEs, where the function g is a “maximum” or “minimum” function. One of the main interest of [3] is the question of *endogeny* : whether the associated RTP is measurable with respect to the *data*, namely i.i.d copies of the pair (ξ, N) . In this article we provide two examples falling under this general framework [3] and prove that the RTPs are endogenous.

Our first example is related to the *frozen percolation process* on a infinite regular binary tree which was first studied by Aldous [5]. The RDE involved here is given by

$$Y \stackrel{d}{=} \Phi(Y_1 \wedge Y_2; U) \quad \text{on } I := [\frac{1}{2}, 1] \cup \{\infty\}, \quad (1)$$

where (Y_1, Y_2) are i.i.d with same distribution as Y and are independent of $U \sim \text{Uniform}[0, 1]$, and $\Phi : I \times [0, 1] \rightarrow I$ is a function defined as

$$\Phi(x; u) = \begin{cases} x & \text{if } x > u \\ \infty & \text{otherwise} \end{cases} . \quad (2)$$

We will call this the *frozen percolation RDE*. The following theorem is our main result for this example.

Theorem 1 *The invariant recursive tree process associated with the RDE (1) with marginal ν is endogenous, where ν is given by*

$$\nu(dy) = \frac{dy}{2y^2}, \quad \frac{1}{2} < y < 1, \quad \nu(\{\infty\}) = \frac{1}{2}. \quad (3)$$

The second example arise from the study of the asymptotic limit of *random assignment problem* using local-weak convergence method [6]. For this example the RDE is given by

$$X \stackrel{d}{=} \min_{j \geq 1} (\xi_j - X_j) \quad \text{on } \mathbb{R}, \quad (4)$$

where $(X_j)_{j \geq 1}$ are i.i.d with same law as X and are independent of $(\xi_j)_{j \geq 1}$ which are points of a Poisson point process of rate 1 on $(0, \infty)$. This we call the *Logistic RDE* for the natural reason that it has a unique solution given by the *Logistic* distribution, which is defined as

$$\mathbf{P}(X \leq x) = \frac{1}{1 + e^{-x}}, \quad x \in \mathbb{R}. \quad (5)$$

The following is our second main result.

Theorem 2 *The invariant recursive tree process associated with the RDE (4) is endogenous.*

The two examples we consider here have different backgrounds and hence motivations for studying them are different too. However the proofs of endogeny for the associated RTPs in both cases use the general technique of proving the *bivariate uniqueness* for the RDEs (Theorem 11 of [3]). Thus provide nice illustration of the general methods developed in [3]. Having said that we would also like to point out that the details of the proofs are very different and they use intricate analytic techniques. Thus also demonstrating the need of analytic methods for studying these kind of max-type RDEs.

The following two sections, Sections 2 and 3, provide the backgrounds for the two examples. Section 4 proves the bivariate uniqueness property for the frozen percolation RDE, and leads to a proof of Theorem 1 given in Section 5. Section 6 gives the proof of the bivariate uniqueness for the Logistic RDE, and the proof of Theorem 2 is given in Section 7. Finally Section 8 provides some more results related to frozen percolation process on other regular trees and discussion of some open problems.

2 Background and Motivation for Frozen Percolation RDE

Frozen percolation process was first studied by Aldous [5] where he constructed the process on a infinite regular binary tree. Let $\mathcal{T}_3 = (\mathcal{V}, \mathcal{E})$ be the infinite binary tree, where each vertex has degree three; \mathcal{V} is the set of vertices and \mathcal{E} is the set of undirected edges. Let $(U_e)_{e \in \mathcal{E}}$ be i.i.d Uniform[0, 1] edge weights. Consider a collection of random subsets $\mathcal{A}_t \subseteq \mathcal{E}$ for $0 \leq t \leq 1$, whose evolution is described informally by :

\mathcal{A}_0 is empty; for each $e \in \mathcal{E}$, at time $t = U_e$ set $\mathcal{A}_t = \mathcal{A}_{t-} \cup \{e\}$ if each end-vertex of e is in a finite cluster of \mathcal{A}_{t-} ; otherwise set $\mathcal{A}_t = \mathcal{A}_{t-}$. (*)

(A *cluster* is formally a connected component of edges, but we also consider it as the induced set of vertices). Qualitatively, in the process (\mathcal{A}_t) the clusters may grow to infinite size but, at the instant of becoming infinite they are “frozen”, in the sense that no extra edge may be connected to an infinite cluster. The final set \mathcal{A}_1 will be a forest on \mathcal{T}_3 with both infinite and finite clusters, such that no two finite clusters are separated by a single edge. Aldous [5] defines this process (\mathcal{A}_t) as the *frozen percolation process*.

Although this process is intuitively quite natural, rigorously speaking it is not clear that it exists or that (*) does specify a unique process. In fact Itai Benjamini and Oded Schramm have an argument that such a process does not exist on the \mathbb{Z}^2 -lattice with its natural invariance property (see the remarks in Section 5.1 of [5]). But for the infinite binary tree case [5] gives a rigorous construction of a process satisfying (*) which is automorphism invariant. This construction uses the frozen percolation RDE (1).

It is easy to see [5] that the non-atomic solutions of the RDE (1) are given by

$$\nu_a(dx) = \frac{1}{2x^2} dx, \quad \frac{1}{2} < x \leq a; \quad \nu_a(\infty) = \frac{1}{2a}, \quad (6)$$

where $a \in [\frac{1}{2}, 1]$. So $\nu = \nu_1$ is the unique non-atomic solution with full support.

Using the natural recursive structure and Kolmogorov’s consistency theorem we can now construct a tree indexed process $(Y_{\mathbf{i}})_{\mathbf{i} \in \tilde{\mathcal{V}}}$ such that for all $\mathbf{i} \in \tilde{\mathcal{V}}$

- $Y_{\mathbf{i}} \sim \nu$,
- $Y_{\mathbf{i}} = \Phi(Y_{\mathbf{i}1} \wedge Y_{\mathbf{i}2}; U_{\mathbf{i}})$, and
- $Y_{\mathbf{i}}$ is independent of $\{U_{\mathbf{i}'} \mid \text{length}(\mathbf{i}') < \text{length}(\mathbf{i})\}$;

where $\tilde{\mathcal{V}} := \cup_{m \geq 1} \{1, 2\}^m \cup \{\emptyset\}$, which is endowed with the natural rooted binary tree structure with \emptyset as the root, and $(U_{\mathbf{i}})_{\mathbf{i} \in \tilde{\mathcal{V}}}$ are i.i.d Uniform[0, 1] random variables. Here it is worth to note that the infinite binary tree structure on $\tilde{\mathcal{V}}$ is not regular since the degree of the root \emptyset is just two. The process $(Y_{\mathbf{i}})_{\mathbf{i} \in \tilde{\mathcal{V}}}$ is the invariant RTP associated with the frozen percolation RDE (1) with marginal ν .

Aldous [5] constructed automorphism invariant version of the frozen percolation process on a infinite regular binary tree using the random variables $(Y_{\mathbf{i}})_{\mathbf{i} \in \tilde{\mathcal{V}}}$. Without going into the details of the technicalities of this construction here we only mention briefly what is the significance of the RTP $(Y_{\mathbf{i}})_{\mathbf{i} \in \tilde{\mathcal{V}}}$. Let $e = (u, v)$ be an edge of the infinite regular binary tree \mathcal{T}_3 and let $\vec{e} = \frac{1}{(u, v)}$ be a direction of it which is from the vertex u to vertex v . Naturally \vec{e} has two children which are the two neighbors of v not including u . These can be represented by two directional edges coming out of v and so on. Thus \vec{e} represent a rooted infinite

binary tree structure exactly like $\tilde{\mathcal{V}}$ with the weights defined appropriately using the i.i.d Uniform edge weights (U_e). If the frozen percolation process exists then the time for the edge e to join to infinite along the subtree defined by \vec{e} is then given by the variable Y_\emptyset , more precisely, such time should satisfy the distributional recursion (1). However to prove the existence of the process such times are then *externally* constructed using the RTP. Naturally it make sense to ask whether these variables can only be defined using the i.i.d Uniform[0, 1] edge weights (see Remark 5.7 in [5]), which is same as asking whether the RTP is endogenous. This is one of our main motivation for proving Theorem 1.

3 Background and Motivation for Logistic RDE

For a given $n \times n$ matrix of costs (C_{ij}), consider the problem of assigning n jobs to n machines in the most “cost effective” way. Thus the task is to find a permutation π of $\{1, 2, \dots, n\}$, which solves the following minimization problem

$$A_n := \min_{\pi} \sum_{i=1}^n C_{i, \pi(i)}. \quad (7)$$

This problem has been extensively studied in literature for a fixed cost matrix, and there are various algorithms to find the optimal permutation π . A probabilistic model for the assignment problem can be obtained by assuming that the costs are independent random variables each with Uniform[0, 1] distribution. Although this model appears to be quite simple, careful investigations of it in the last few decades have shown that it has enormous richness in its structure. See [23, 4] for survey and other related works.

Our interest in this problem is from another perspective. In 2001 Aldous [6] showed

$$\lim_{n \rightarrow \infty} \mathbf{E}[A_n] = \zeta(2) = \frac{\pi^2}{6}, \quad (8)$$

confirming the earlier work of Mézard and Parisi [16], where they computed the same limit using some non-rigorous arguments based on the *replica method* [17]. In an earlier work Aldous [1] showed that the limit of limit of $\mathbf{E}[A_n]$ as $n \rightarrow \infty$ exists for any cost distribution, and does not depend on the specifics of it, except only on the value of its density at 0, provided it exists and is strictly positive. So for calculation of the limiting constant one can assume that C_{ij} 's are independent and each has Exponential distribution with mean n , and re-write the objective function A_n in the normalized form,

$$A_n = \min_{\pi} \frac{1}{n} \sum_{i=1}^n C_{i, \pi(i)}. \quad (9)$$

From historical perspective it is worth mentioning that in 1998 Parisi [19] conjectured the following fascinating exact formula

$$\mathbf{E}[A_n] = 1 + \frac{1}{4} + \dots + \frac{1}{n^2}, \quad \forall n \geq 1.$$

when the costs are i.i.d Exponential(1). Recently two separate groups Linusson and Wästlund [14] and C. Nair, B. Prabhakar and M. Sharma [18] have independently proved this conjecture using combinatorial techniques. Thus also proving the limit.

However Aldous [6] used local-weak convergence techniques to identify the limit constant $\zeta(2)$ in terms of an optimal matching problem on a limit infinite tree with random edge weights. This structure is called *Poisson Weighted Infinite Tree*, or, *PWIT*, it is described as follows (see [4] for a more friendly account).

Let $\mathcal{T} := (\mathcal{V}, \mathcal{E})$ be the canonical infinite rooted labelled tree with vertex set $\mathcal{V} := \cup_{m=0}^{\infty} \mathbb{N}^m$ (where $\mathbb{N}^0 := \{\emptyset\}$), and edge set $\mathcal{E} := \{e = (\mathbf{i}, \mathbf{ij}) \mid \mathbf{i} \in \mathcal{V}, j \in \mathbb{N}\}$. We consider \emptyset as the root of the tree, and will write $\emptyset j = j, \forall j \in \mathbb{N}$. For every vertex $\mathbf{i} \in \mathcal{V}$, let $(\xi_{\mathbf{ij}})_{j \geq 1}$ be points of independent Poisson point process of rate 1 on $(0, \infty)$. Define the weight of the edge $e = (\mathbf{i}, \mathbf{ij})$ as $\xi_{\mathbf{ij}}$.

One of the key ingredient for solving the optimal matching problem on PWIT is the Logistic RDE given by (4). It is easy to prove [6] that the (4) has unique solution as the Logistic distribution as defined by the equation (5). Thus by a similar application of Kolmogorov's consistency theorem one can construct the invariant RTP associated with this RDE, which is a tree indexed process $(X_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ such that for every $\mathbf{i} \in \mathcal{V}$

- $X_{\mathbf{i}}$ has Logistic distribution,
- $X_{\mathbf{i}} = \min_{j \geq 1} (\xi_{\mathbf{ij}} - X_{\mathbf{ij}})$, and
- $X_{\mathbf{i}}$ is independent of $\{(\xi_{\mathbf{i}'j})_{j \geq 1} \mid \text{length}(\mathbf{i}') < \text{length}(\mathbf{i})\}$.

The heuristic interpretation used in [6] for the random variables $X_{\mathbf{i}}$ is as follows

$$X_{\mathbf{i}} = \begin{aligned} & \text{Total cost of a maximal matching on the subtree } \mathcal{T}^{\mathbf{i}} \\ & - \text{Total cost of a maximal matching on the forest } \mathcal{T}^{\mathbf{i}} \setminus \{\mathbf{i}\}, \end{aligned}$$

where $\mathcal{T}^{\mathbf{i}}$ is the subtree rooted at the vertex \mathbf{i} . Here by "total cost" we mean the sum total of all the edge weights in the matching. Naturally this is not well defined since both the "total costs" are ∞ almost surely. On the other hand one can define *externally* the random variables $(X_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ through the RTP construction and then use them to construct the optimal matching. This is done in [6] (see [4] for a more friendly account).

Once again a natural question is to figure out whether the random variables $X_{\mathbf{i}}$'s are truly external or not, in other words to see whether the RTP is endogenous or not (see remarks (4.2.d) and (4.2.e) in [6]). This is one of our main motivation for proving Theorem 2. Other significance of this result has been pointed out in Section 7.5 of [3]. Without going into the technical details we

would like to comment that the endogeny of the Logistic RTP helps to define approximately feasible solution for the finite n -matching problem using the optimal solution of the matching problem on PWIT. That in turn helps in proving the $\zeta(2)$ -limit for the random assignment problem.

4 Bivariate Uniqueness for the Frozen Percolation RDE

In this section we prove the *bivariate uniqueness* property for the frozen percolation RDE (1) which is the first step to prove endogeny.

Theorem 3 *Consider the following bivariate RDE,*

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \Phi(X_1 \wedge X_2; U) \\ \Phi(Y_1 \wedge Y_2; U) \end{pmatrix}, \quad (10)$$

where $(X_j, Y_j)_{j=1,2}$ are i.i.d with same joint law as (X, Y) and have same marginal distribution ν given by (3), and are independent of $U \sim \text{Uniform}[0, 1]$; and Φ is given by (2). Then the unique solution of this bivariate RDE (10) is given by the diagonal measure $\nu^\nearrow := \text{dist}((X, X))$ where $X \sim \nu$.

Proof. Observe that the diagonal measure ν^\nearrow is a solution of the bivariate RDE (10), so all we need to show is it is the unique solution. For a solution of (10), let $F(x, y) := \mathbf{P}(X \leq x, Y \leq y)$, for $x, y \in [\frac{1}{2}, 1]$, then from definition we get

$$\begin{aligned} F(x, y) &= \mathbf{P}(\Phi(X_1 \wedge X_2; U) \leq x, \Phi(Y_1 \wedge Y_2; U) \leq y) \\ &= \mathbf{P}(U < X_1 \wedge X_2 \leq x, U < Y_1 \wedge Y_2 \leq y) \\ &= \mathbf{E}[(\mathbf{1}_{(X_1 \wedge X_2 > U)} - \mathbf{1}_{(X_1 \wedge X_2 > x)}) (\mathbf{1}_{(Y_1 \wedge Y_2 > U)} - \mathbf{1}_{(Y_1 \wedge Y_2 > y)})] \\ &= \int_0^{x \wedge y} (G^2(x, y) - G^2(x, u) - G^2(u, y) + G^2(u, u)) du \quad (11) \end{aligned}$$

where $G(x, y) := \mathbf{P}(X > x, Y > y)$, which can be written as

$$\begin{aligned} G(x, y) &= F(x, y) - \mathbf{P}(X \leq x) - \mathbf{P}(Y \leq y) + 1 \\ &= F(x, y) + \frac{1}{2x} + \frac{1}{2y} - 1 \quad (12) \end{aligned}$$

for $x, y \in [\frac{1}{2}, 1]$. Further notice that $G(x, y) = 1$ if $x, y < \frac{1}{2}$; $G(x, y) = \frac{1}{2x}$ if $x \in [\frac{1}{2}, 1]$ and $y < \frac{1}{2}$; and finally $G(x, y) = \frac{1}{2y}$ if $y \in [\frac{1}{2}, 1]$ and $x < \frac{1}{2}$.

We know that $G_0(x, y) := \frac{1}{2(x \vee y)}$ on $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ is a solution of the equation (11) which represent the diagonal solution. Note that for this solution $F_0(x, y) = 1 - \frac{1}{2(x \wedge y)}$.

Let $D := [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ be the domain of the integral equation. Put $H(x, y) = 1 - G(x, y)/G_0(x, y)$. Notice that $H \equiv 0$ if $(x, y) \in [0, 1] \times [0, 1] \setminus D$.

Moreover for $(x, y) \in D$,

$$\begin{aligned} G(x, y) &= \mathbf{P}(X > x, Y > y) \\ &\leq \min(\mathbf{P}(X > x), \mathbf{P}(Y > y)) \\ &= \frac{1}{2(x \vee y)} = G_0(x, y). \end{aligned}$$

Thus $0 \leq H(x, y) \leq 1$ for all $(x, y) \in D$. To prove bivariate uniqueness all we need to show is $H \equiv 0$ on D .

Since G_0 is a solution, so by (11)

$$F_0(x, y) = \int_0^{x \wedge y} (G_0^2(x, y) - G_0^2(x, u) - G_0^2(u, y) + G_0^2(u, u)) du. \quad (13)$$

Further by (12) and definition of H we have

$$F_0(x, y) - F(x, y) = G_0(x, y) - G(x, y) = G_0(x, y)H(x, y).$$

Thus using (11) it follows

$$\begin{aligned} G_0(x, y)H(x, y) &= F_0(x, y) - F(x, y) \\ &= F_0(x, y) \\ &\quad - \int_0^{x \wedge y} (G^2(x, y) - G^2(x, u) - G^2(u, y) + G^2(u, u)) du \\ &= F_0(x, y) \\ &\quad - \int_0^{x \wedge y} (G_0^2(x, y) - G_0^2(x, u) - G_0^2(u, y) + G_0^2(u, u)) du \\ &\quad + \int_0^{x \wedge y} \Lambda(x, y, u) du \\ &= \int_0^{x \wedge y} \Lambda(x, y, u) du \quad [\text{using (13)}] \end{aligned} \quad (14)$$

where Λ is given by

$$\begin{aligned} \Lambda(x, y, u) &:= G_0^2(x, y) (2H(x, y) - H^2(x, y)) - G_0^2(x, u) (2H(x, u) - H^2(x, u)) \\ &\quad - G_0^2(u, y) (2H(u, y) - H^2(u, y)) + G_0^2(u, u) (2H(u, u) - H^2(u, u)) \end{aligned}$$

because $G_0^2 - G^2 = G_0^2 - G_0^2(1 - H)^2 = G_0^2(2H - H^2)$. Notice that $\Lambda \equiv 0$ outside D .

Find $\frac{1}{2} = \alpha_0 < \alpha_1 < \dots < \alpha_{k-1} < \alpha_k = 1$ such that

$$\int_{\alpha_{i-1}}^{\alpha_i} G_0^2(u, u) du < \frac{1}{48}, \quad (15)$$

we can do this since $G_0^2(u, u) = \frac{1}{4u^2}$ is a continuous function. Define $L_i := \{(x, y) \mid \alpha_{i-1} \leq x \wedge y < \alpha_i\}$, for $1 \leq i \leq k$. Certainly L_i 's are disjoint and

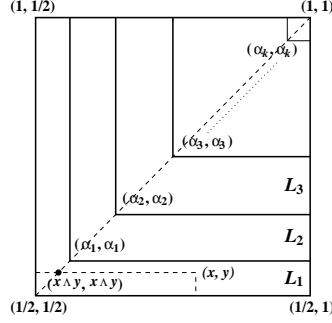


Figure 1: Partition of D in L -shape parts

their union is D . See figure 1 for a picture of the partition. On L_i define $\|H\|_i := \sup_{x,y \in L_i} |H(x,y)|$.

Start with $i = 1$ and let $(x,y) \in L_i$, observe that from (14) and definition of Λ we get the estimate

$$\begin{aligned}
|H(x,y)| &\leq \frac{2 \|H\|_i + \|H\|_i^2}{G_0(x,y)} \\
&\quad \times \int_{\alpha_{i-1}}^{x \wedge y} G_0^2(u,u) \left(\frac{G_0^2(x,y)}{G_0^2(u,u)} + \frac{G_0^2(x,u)}{G_0^2(u,u)} + \frac{G_0^2(u,y)}{G_0^2(u,u)} + 1 \right) du \\
&\leq 6 (x \wedge y) \|H\|_i \int_{\alpha_{i-1}}^{x \wedge y} 4 G_0^2(u,u) du \\
&\quad \text{[since } \|H\|_i \leq 1 \text{ and } G_0(x,y) \leq G_0(u,u) \text{ for } u \leq x \wedge y] \\
&\leq 24 \|H\|_i \int_{\alpha_{i-1}}^{\alpha_i} G_0^2(u,u) du \quad \text{[since } x \wedge y < \alpha_i] \\
&\leq \frac{1}{2} \|H\|_i \quad \text{[using (15)]}
\end{aligned}$$

Thus we get $\|H\|_i \leq \frac{1}{2} \|H\|_i$ which implies $\|H\|_i = 0$, that is $H \equiv 0$ on L_i . We can now proceed exactly in the same way by taking $i = 2, 3, \dots, k$ recursively to get $H \equiv 0$ on whole on D . This completes the proof. \blacksquare

5 Proof of Theorem 1

To prove the Theorem 1 we will apply the general result of Aldous and Bandyopadhyay [3], namely their Theorem 11(b). Since Theorem 3 proves the bivariate uniqueness property for the frozen percolation RDE (1) thus all remains is to check the technical condition of Theorem 11(b) of [3]. The following proposition shows that the technical condition is valid.

Proposition 4 Let \mathcal{Q} be the set of all probabilities on I^2 where $I = [\frac{1}{2}, 1] \cup \{\infty\}$ and let $\Xi : \mathcal{Q} \rightarrow \mathcal{Q}$ be the operator associated with the RDE (10), that is,

$$\Xi(\nu^{(2)}) \stackrel{d}{=} \begin{pmatrix} \Phi(X_1 \wedge X_2; U) \\ \Phi(Y_1 \wedge Y_2; U) \end{pmatrix}, \quad (16)$$

where $(X_j, Y_j)_{j=1,2}$ are i.i.d with joint law $\nu^{(2)} \in \mathcal{Q}$ and are independent of $U \sim \text{Uniform}[0, 1]$. Then Ξ is continuous with respect to the weak topology when restricted to the subspace \mathcal{Q}^* defined as

$$\mathcal{Q}^* := \left\{ \nu^{(2)} \mid \text{both the marginals of } \nu^{(2)} \text{ are } \nu \right\}. \quad (17)$$

Proof. Suppose $\nu_n^{(2)} \xrightarrow{d} \nu^{(2)}$ on \mathcal{Q}^* , and let F_n be the distribution function for $\nu_n^{(2)}$ and F be that for $\nu^{(2)}$. We define G_n and G in similar manner as done in equation (12). Following argument similar of derivation of the equation (11) we get that

$$\Xi(F_n)(x, y) = \int_0^{x \wedge y} (G_n^2(x, y) - G_n^2(x, u) - G_n^2(u, y) + G_n^2(u, u)) du.$$

The rest follows by the *dominated convergence theorem*. ■

6 Bivariate Uniqueness for the Logistic RDE

This section provides a proof of the the bivariate uniqueness property for the Logistic RDE (4) which is the main step for proving the endogeny for the Logistic RTP.

Theorem 5 Consider the following bivariate RDE

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \min_{j \geq 1} (\xi_j - X_j) \\ \min_{j \geq 1} (\xi_j - Y_j) \end{pmatrix}, \quad (18)$$

where $(X_j, Y_j)_{j \geq 1}$ are i.i.d. pairs with same joint distribution as (X, Y) and are independent of $(\xi_j)_{j \geq 1}$ which are points of a Poisson process of rate 1 on $(0, \infty)$. Then the unique solution of this RDE is given by the diagonal measure $\mu^\nearrow := \text{dist}((X, X))$ where $X \sim \text{Logistic distribution}$.

6.1 Proof of Theorem 5

First observe that if the equation (18) has a solution then, the marginal distributions of X and Y solve the Logistic RDE (4), and hence they are both Logistic. Further by inspection μ^\nearrow is a solution of (18). So it is enough to prove that μ^\nearrow is the only solution of (18).

Let $\mu^{(2)}$ be a solution of (18). Notice that the points $\{(\xi_j; (X_j, Y_j)) \mid j \geq 1\}$ form a Poisson point process, say \mathcal{P} , on $(0, \infty) \times \mathbb{R}^2$, with mean intensity $\rho(t; (x, y)) dt d(x, y) := dt \mu^{(2)}(d(x, y))$. Thus if $G(x, y) := \mathbf{P}(X > x, Y > y)$, for $x, y \in \mathbb{R}$, then

$$\begin{aligned}
G(x, y) &= \mathbf{P}\left(\min_{j \geq 1} (\xi_j - X_j) > x, \text{ and, } \min_{j \geq 1} (\xi_j - Y_j) > y\right) \\
&= \mathbf{P}\left(\text{No points of } \mathcal{P} \text{ are in } \left\{(t; (u, v)) \mid t - u \leq x, \text{ or, } t - v \leq y\right\}\right) \\
&= \exp\left(-\int_{t-u \leq x, \text{ OR, } t-v \leq y} \int \int \rho(t; (u, v)) dt d(u, v)\right) \\
&= \exp\left(-\int_0^\infty [\overline{H}(t-x) + \overline{H}(t-y) - G(t-x, t-y)] dt\right) \\
&= \overline{H}(x) \overline{H}(y) \exp\left(\int_0^\infty G(t-x, t-y) dt\right), \tag{19}
\end{aligned}$$

where \overline{H} is the right tail of Logistic distribution, defined as $\overline{H}(x) = e^{-x} / (1 + e^{-x})$ for $x \in \mathbb{R}$. The last equality follows from properties of the Logistic distribution (see Fact 1 of appendix). For notational convenience in this paper we will write $\overline{F}(\cdot) := 1 - F(\cdot)$, for any distribution function F .

The following simple lemma reduces the bivariate problem to a univariate problem.

Lemma 6 *For any two random variables U and V , $U = V$ a.s. if and only if $U \stackrel{d}{=} V \stackrel{d}{=} U \wedge V$.*

Proof. First of all if $U = V$ a.s. then $U \wedge V = U$ a.s.

Conversely suppose that $U \stackrel{d}{=} V \stackrel{d}{=} U \wedge V$. Fix a rational q , then under our assumption,

$$\begin{aligned}
\mathbf{P}(U \leq q < V) &= \mathbf{P}(V > q) - \mathbf{P}(U > q, V > q) \\
&= \mathbf{P}(V > q) - \mathbf{P}(U \wedge V > q) \\
&= 0
\end{aligned}$$

A similar calculation will show that $\mathbf{P}(V \leq q < U) = 0$. These are true for any rational q , thus $\mathbf{P}(U \neq V) = 0$. \blacksquare

Thus if we can show that $X \wedge Y$ also has Logistic distribution, then from the lemma above we will be able to conclude that $X = Y$ a.s., and hence the proof will be complete. Put $g(\cdot) := \mathbf{P}(X \wedge Y > \cdot)$, we will show $g = \overline{H}$. Now, for every fixed $x \in \mathbb{R}$, $g(x) = G(x, x)$ by definition. So using (19) we get

$$g(x) = \overline{H}^2(x) \exp\left(\int_{-x}^\infty g(s) ds\right), \quad x \in \mathbb{R}. \tag{20}$$

Notice that from (A1) (see Fact 3 of appendix) $g = \overline{H}$ is a solution of this non-linear integral equation (20), which corresponds to the solution $\mu^{(2)} = \mu^{\nearrow}$ of

the original equation (18). To complete the proof of Theorem 5 we need to show that this is the only solution. For that we will prove that the operator associated with (20) (defined on an appropriate space) is monotone and has unique fixed-point as \overline{H} . The techniques we will use here are similar to Eulerian recursion [22], and are heavily based on analytic arguments.

Let \mathfrak{F} be the set of all functions $f : \mathbb{R} \rightarrow [0, 1]$ such that

- $\overline{H}^2(x) \leq f(x) \leq \overline{H}(x)$, $\forall x \in \mathbb{R}$,
- f is a tail of a distribution, that is, \exists random variable say W such that $f(x) = \mathbf{P}(W > x)$, $x \in \mathbb{R}$.

Observe that by definition $\overline{H} \in \mathfrak{F}$. Further from (20) it follows that $g(x) \geq \overline{H}^2(x)$, as well as, $g(x) = \mathbf{P}(X \wedge Y > x) \leq \mathbf{P}(X > x) = \overline{H}(x)$, $\forall x \in \mathbb{R}$. So it is appropriate to search for solutions of (20) in \mathfrak{F} .

Let $T : \mathfrak{F} \rightarrow \mathfrak{F}$ be defined as

$$T(f)(x) := \overline{H}^2(x) \exp\left(\int_{-x}^{\infty} f(s) ds\right), \quad x \in \mathbb{R}. \quad (21)$$

Proposition 11 of Section 6.2 shows that T does indeed map \mathfrak{F} into itself. Observe that the equation (20) is nothing but the fixed-point equation associated with the operator T , that is,

$$g = T(g) \text{ on } \mathfrak{F}. \quad (22)$$

We here note that using (A1) (see Fact 3 of appendix) T can also be written as

$$T(f)(x) := \overline{H}(x) \exp\left(-\int_{-x}^{\infty} (\overline{H}(s) - f(s)) ds\right), \quad x \in \mathbb{R}, \quad (23)$$

which will be used in the subsequent discussion.

Define a partial order \preceq on \mathfrak{F} as, $f_1 \preceq f_2$ in \mathfrak{F} if $f_1(x) \leq f_2(x)$, $\forall x \in \mathbb{R}$, then the following result holds.

Lemma 7 T is a monotone operator on the partially ordered set (\mathfrak{F}, \preceq) .

Proof. Let $f_1 \preceq f_2$ be two elements of \mathfrak{F} , so from definition $f_1(x) \leq f_2(x)$, $\forall x \in \mathbb{R}$. Hence

$$\begin{aligned} \int_{-x}^{\infty} f_1(s) ds &\leq \int_{-x}^{\infty} f_2(s) ds, \quad \forall x \in \mathbb{R} \\ \Rightarrow T(f_1)(x) &\leq T(f_2)(x), \quad \forall x \in \mathbb{R} \\ \Rightarrow T(f_1) &\preceq T(f_2). \end{aligned}$$

Put $f_0 = \overline{H}^2$, and for $n \in \mathbb{N}$, define $f_n \in \mathfrak{F}$ recursively as, $f_n = T(f_{n-1})$. Now from Lemma 7 we get that if g is a fixed-point of T in \mathfrak{F} then, ■

$$f_n \preceq g, \quad \forall n \geq 0. \quad (24)$$

If we can show $f_n \rightarrow \overline{H}$ pointwise, then using (24) we will get $\overline{H} \preceq g$, so from definition of \mathfrak{F} it will follow that $g = \overline{H}$, and our proof will be complete. For that, the following lemma gives an explicit recursion for the functions $\{f_n\}_{n \geq 0}$.

Lemma 8 *Let $\beta_0(s) = 1 - s$, $0 \leq s \leq 1$. Define recursively*

$$\beta_n(s) := \int_s^1 \frac{1}{w} \left(1 - e^{-\beta_{n-1}(1-w)}\right) dw, \quad 0 < s \leq 1. \quad (25)$$

Then for $n \geq 1$,

$$f_n(x) = \overline{H}(x) \exp(-\beta_{n-1}(\overline{H}(x))), \quad x \in \mathbb{R}. \quad (26)$$

Proof. We will prove this by induction on n . Fix $x \in \mathbb{R}$, for $n = 1$ we get

$$\begin{aligned} f_1(x) &= T(f_0)(x) \\ &= \overline{H}(x) \exp\left(-\int_{-x}^{\infty} (\overline{H}(s) - \overline{H}^2(s)) ds\right) \quad [\text{using (23)}] \\ &= \overline{H}(x) \exp\left(-\int_{-x}^{\infty} \overline{H}(s) (1 - \overline{H}(s)) ds\right) \\ &= \overline{H}(x) \exp\left(-\int_{-x}^{\infty} \overline{H}(s) H(s) ds\right) \\ &= \overline{H}(x) \exp\left(-\int_{-x}^{\infty} H'(s) ds\right) \quad [\text{using Fact 1 of appendix}] \\ &= \overline{H}(x) \exp(-H(x)) \\ &= \overline{H}(x) \exp(-\beta_0(\overline{H}(x))) \end{aligned}$$

Now, assume that the assertion of the Lemma is true for $n \in \{1, 2, \dots, k\}$, for some $k \geq 1$, then from definition we have

$$\begin{aligned} f_{k+1}(x) &= T(f_k)(x) \\ &= \overline{H}(x) \exp\left(-\int_{-x}^{\infty} (\overline{H}(s) - f_k(s)) ds\right) \quad [\text{using (23)}] \\ &= \overline{H}(x) \exp\left(-\int_{-x}^{\infty} \overline{H}(s) (1 - e^{-\beta_{k-1}(\overline{H}(s))}) ds\right) \\ &= \overline{H}(x) \exp\left(-\int_{\overline{H}(x)}^1 \frac{1}{w} (1 - e^{-\beta_{k-1}(1-w)}) dw\right) \quad (27) \end{aligned}$$

The last equality follows by substituting $w = H(s)$ and thus from Fact 1 and Fact 2 of the appendix we get that $\frac{dw}{w} = \overline{H}(s) ds$ and $H(-x) = \overline{H}(x)$. Finally by definition of β_n 's and using (27) we get $f_{k+1} = T(f_k)$. \blacksquare

To complete the proof it is now enough to show that $\beta_n \rightarrow 0$ pointwise, which will imply by Lemma 8 that $f_n \rightarrow \overline{H}$ pointwise, as $n \rightarrow \infty$. Using Proposition 12 (see Section 6.2) we get the following characterization of the pointwise limit of these β_n 's.

Lemma 9 *There exists a function $L : [0, 1] \rightarrow [0, 1]$ with $L(1) = 0$, such that*

$$L(s) = \int_s^1 \frac{1}{w} \left(1 - e^{-L(1-w)}\right) dw, \quad \forall s \in [0, 1], \quad (28)$$

and $L(s) = \lim_{n \rightarrow \infty} \beta_n(s)$, $\forall 0 \leq s \leq 1$.

Proof. From part (b) of Proposition 12 we know that for any $s \in [0, 1]$ the sequence $\{\beta_n(s)\}$ is decreasing, and hence \exists a function $L : [0, 1] \rightarrow [0, 1]$ such that $L(s) = \lim_{n \rightarrow \infty} \beta_n(s)$. Now observe that $\beta_n(1-w) \leq \beta_0(1-w) = w$, $\forall 0 \leq w \leq 1$, and hence

$$0 \leq \frac{1}{w} \left(1 - e^{-\beta_n(1-w)}\right) \leq \frac{\beta_n(1-w)}{w} \leq 1, \quad \forall 0 \leq w \leq 1.$$

Thus by taking limit as $n \rightarrow \infty$ in (25) and using the *dominated convergence theorem* along with part (a) of Proposition 12 we get that

$$L(s) = \int_s^1 \frac{1}{w} \left(1 - e^{-L(1-w)}\right) dw, \quad \forall 0 \leq s < 1. \quad \blacksquare$$

The above lemma basically translates the non-linear integral equation (20) to the non-linear integral equation (28), where the solution $g = \overline{H}$ of (20) is given by the solution $L \equiv 0$ of (28). So at first sight this may not lead us to the conclusion. But fortunately, something nice happens for equation (28), and we have the following result which is enough to complete the proof of Theorem 5.

Lemma 10 *If $L : [0, 1] \rightarrow [0, 1]$ is a function which satisfies the non-linear integral equation (28), namely,*

$$L(s) = \int_s^1 \frac{1}{w} \left(1 - e^{-L(1-w)}\right) dw, \quad \forall 0 \leq s < 1,$$

and if $L(1) = 0$, then $L \equiv 0$.

Proof. First note that $L \equiv 0$ is a solution. Now let L be any solution of (28), then L is infinitely differentiable on the open interval $(0, 1)$, by repetitive application of Fundamental Theorem of Calculus.

Consider,

$$\eta(w) := (1-w)e^{L(1-w)} + we^{-L(w)} - 1, \quad w \in [0, 1]. \quad (29)$$

Observe that $\eta(0) = \eta(1) = 0$ as $L(1) = 0$. Now, from (28) we get that

$$L'(w) = -\frac{1}{w} \left(1 - e^{-L(1-w)}\right), \quad w \in (0, 1). \quad (30)$$

Thus differentiating the function η we get

$$\eta'(w) = e^{-L(w)} \left[2 - \left(e^{L(1-w)} + e^{-L(1-w)}\right)\right] \leq 0, \quad \forall w \in (0, 1). \quad (31)$$

So the function η is decreasing in $(0, 1)$ and is continuous in $[0, 1]$ with boundary values as 0, hence $\eta \equiv 0 \Leftrightarrow L \equiv 0$. \blacksquare

6.2 Some Technical Details

This section provides some of the technical results which were needed in the previous section.

Proposition 11 *The operator T maps \mathfrak{F} into \mathfrak{F} .*

Proof. First note that if $f \in \mathfrak{F}$, then by definition $T(f)(x) \geq \overline{H}^2(x)$, $\forall x \in \mathbb{R}$. Next by definition of \mathfrak{F} we get that $f \in \mathfrak{F} \Rightarrow f \preceq \overline{H}$, thus

$$\begin{aligned} \int_{-x}^{\infty} f(s) ds &\leq \int_{-x}^{\infty} \overline{H}(s) ds, \quad \forall x \in \mathbb{R} \\ \Rightarrow T(f)(x) &\leq \overline{H}^2(x) \exp\left(\int_{-x}^{\infty} \overline{H}(s) ds\right) = \overline{H}(x), \quad \forall x \in \mathbb{R} \end{aligned}$$

The last equality follows from (A1) (see Fact 3 of appendix). So,

$$\overline{H}^2(x) \leq T(f)(x) \leq \overline{H}(x), \quad \forall x \in \mathbb{R}. \quad (32)$$

Now we need to show that for $f \in \mathfrak{F}$, $T(f)$ is a tail of a distribution. From the definition $T(f)$ is continuous (in fact, infinitely differentiable). Further using (32) and the fact that \overline{H} is a tail of a distribution we get that

$$\lim_{x \rightarrow \infty} T(f)(x) = 0, \quad \text{and} \quad \lim_{x \rightarrow -\infty} T(f)(x) = 1. \quad (33)$$

Finally let $x \leq y$ be two real numbers, then

$$\int_{-x}^{\infty} (\overline{H}(s) - f(s)) ds \leq \int_{-y}^{\infty} (\overline{H}(s) - f(s)) ds,$$

because $f \preceq \overline{H}$. Also $\overline{H}(x) \geq \overline{H}(y)$, thus using (23) we get

$$T(f)(x) \geq T(f)(y) \quad (34)$$

So using (32), (33), (34) we conclude that $T(f) \in \mathfrak{F}$ if $f \in \mathfrak{F}$. ■

Proposition 12 *The following are true for the sequence of functions $\{\beta_n\}_{n \geq 0}$ as defined in (25).*

(a) *For every $n \geq 1$, $\lim_{s \rightarrow 0^+} \beta_n(s)$ exists, and is given by*

$$\int_0^1 \frac{1}{w} \left(1 - e^{-\beta_{n-1}(1-w)}\right) dw,$$

we will write this as $\beta_n(0)$.

(b) *For every fixed $s \in [0, 1]$, the sequence $\{\beta_n(s)\}$ is decreasing.*

Proof. (a) Note that for $n = 1$,

$$\beta_1(s) = \int_s^1 \frac{1}{w} (1 - e^w) dw, \quad \forall s \in (0, 1],$$

Thus $\lim_{s \rightarrow 0^+} \beta_1(s)$ exists and is given by

$$\int_0^1 \frac{1}{w} (1 - e^{-\beta_0(1-w)}) dw.$$

Now we assume that the assertion is true for $n \in \{1, 2, \dots, k\}$ for some $k \geq 1$, we will show that it is true for $n = k + 1$. For that note

$$\beta_{k+1}(s) = \int_s^1 \frac{1}{w} (1 - e^{-\beta_k(1-w)}) dw, \quad \forall s \in (0, 1].$$

But,

$$\begin{aligned} & \lim_{w \rightarrow 0^+} \frac{1}{w} (1 - e^{-\beta_k(1-w)}) \\ &= \lim_{w \rightarrow 0^+} \frac{1 - e^{-\beta_k(1-w)}}{\beta_k(1-w)} \times \frac{\beta_k(1-w)}{w} \\ &= \lim_{w \rightarrow 0^+} \frac{1}{w} \int_{1-w}^1 \frac{1}{v} (1 - e^{-\beta_{k-1}(1-v)}) dv \\ &= 1 - e^{-\beta_{k-1}(0)} \end{aligned}$$

The last equality follows from *mean-value theorem* and the induction hypothesis. The rest follows from the definition.

(b) Notice that $\beta_0(s) = 1 - s$ for $s \in [0, 1]$, thus

$$\beta_1(s) = \int_s^1 \frac{1 - e^{-w}}{w} dw < 1 - s = \beta_0(s), \quad \forall s \in [0, 1].$$

Now assume that for some $n \geq 1$ we have $\beta_n(s) < \beta_{n-1}(s) < \dots < \beta_0(s)$, $\forall s \in [0, 1]$, if we show that $\beta_{n+1}(s) < \beta_n(s)$, $\forall s \in [0, 1]$ then by induction the proof will be complete. For that, fix $s \in [0, 1]$ then

$$\begin{aligned} \beta_{n+1}(s) &= \int_s^1 \frac{1}{w} (1 - e^{-\beta_n(1-w)}) dw \\ &< \int_s^1 \frac{1}{w} (1 - e^{-\beta_{n-1}(1-w)}) dw \\ &= \beta_n(s) \end{aligned}$$

Hence the proof of the proposition. ■

7 Proof of Theorem 2

Once again we will use the general result Theorem 11(b) of [3] to prove Theorem 2. We note that by Theorem 5 the Logistic RDE (4) has bivariate uniqueness property and hence all remains is to check the technical condition of Theorem 11(b) of [3].

Proposition 13 *Let \mathfrak{S} be the set of all probabilities on \mathbb{R}^2 and let $\Gamma : \mathfrak{S} \rightarrow \mathfrak{S}$ be the operator associated with the RDE (18), that is,*

$$\Gamma(\mu^{(2)}) \stackrel{d}{=} \left(\begin{array}{c} \min_{j \geq 1} (\xi_j - X_j) \\ \min_{j \geq 1} (\xi_j - Y_j) \end{array} \right), \quad (35)$$

where $(X_j, Y_j)_{j \geq 1}$ are i.i.d with joint law $\mu^{(2)} \in \mathfrak{S}$ and are independent of $(\xi_j)_{j \geq 1}$ which are points of a Poisson point process of rate 1 on $(0, \infty)$. Then Γ is continuous with respect to the weak topology when restricted to the subspace \mathfrak{S}^* defined as

$$\mathfrak{S}^* := \left\{ \mu^{(2)} \mid \text{both the marginals of } \mu^{(2)} \text{ are Logistic distribution} \right\}. \quad (36)$$

Proof. Let $\left\{ \mu_n^{(2)} \right\}_{n=1}^{\infty} \subseteq \mathfrak{S}^*$ and suppose that $\mu_n^{(2)} \xrightarrow{d} \mu^{(2)} \in \mathfrak{S}^*$. We will show that $\Gamma(\mu_n^{(2)}) \xrightarrow{d} \Gamma(\mu^{(2)})$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space such that, $\exists \{(X_n, Y_n)\}_{n=1}^{\infty}$ and (X, Y) random vectors taking values in \mathbb{R}^2 , with $(X_n, Y_n) \sim \mu_n^{(2)}$, $n \geq 1$, and $(X, Y) \sim \mu^{(2)}$. Notice that by definition $X_n \stackrel{d}{=} Y_n \stackrel{d}{=} X \stackrel{d}{=} Y$, and each has Logistic distribution.

Fix $x, y \in \mathbb{R}$, then using similar calculations as in (19) we get

$$\begin{aligned} G_n(x, y) &:= \Gamma(\mu_n^{(2)})((x, \infty) \times (y, \infty)) \\ &= \overline{H}(x) \overline{H}(y) \exp \left(- \int_0^{\infty} \mathbf{P}(X_n > t - x, Y_n > t - y) dt \right) \\ &= \overline{H}(x) \overline{H}(y) \exp \left(- \int_0^{\infty} \mathbf{P}((X_n + x) \wedge (Y_n + y) > t) dt \right) \\ &= \overline{H}(x) \overline{H}(y) \exp(-\mathbf{E}[(X_n + x)^+ \wedge (Y_n + y)^+]), \end{aligned} \quad (37)$$

and a similar calculation will also give that

$$\begin{aligned} G(x, y) &:= \Gamma(\mu^{(2)})((x, \infty) \times (y, \infty)) \\ &= \overline{H}(x) \overline{H}(y) \exp(-\mathbf{E}[(X + x)^+ \wedge (Y + y)^+]). \end{aligned} \quad (38)$$

Now to complete the proof all we need is to show

$$\mathbf{E}[(X_n + x)^+ \wedge (Y_n + y)^+] \longrightarrow \mathbf{E}[(X + x)^+ \wedge (Y + y)^+].$$

Since we assumed that $(X_n, Y_n) \xrightarrow{d} (X, Y)$ thus

$$(X_n + x)^+ \wedge (Y_n + y)^+ \xrightarrow{d} (X + x)^+ \wedge (Y + y)^+, \quad \forall x, y \in \mathbb{R}. \quad (39)$$

Fix $x, y \in \mathbb{R}$, define $Z_n^{x,y} := (X_n + x)^+ \wedge (Y_n + y)^+$, and $Z^{x,y} := (X + x)^+ \wedge (Y + y)^+$. Observe that

$$0 \leq Z_n^{x,y} \leq (X_n + x)^+ \leq |X_n + x|, \quad \forall n \geq 1. \quad (40)$$

But, $|X_n + x| \stackrel{d}{=} |X + x|$, $\forall n \geq 1$. So clearly $\{Z_n^{x,y}\}_{n=1}^\infty$ is uniformly integrable. Hence we conclude (using Theorem 25.12 of Billingsley [9]) that

$$\mathbf{E}[Z_n^{x,y}] \longrightarrow \mathbf{E}[Z^{x,y}].$$

This completes the proof. \blacksquare

It is worth mentioning that the operator Γ is not continuous on the whole space \mathfrak{S} , in fact, it is every where discontinuous on \mathfrak{S} with respect to the weak convergence topology. But fortunately for applying Theorem 11(b) of [3] we only need the continuity of Γ when restricted to the subspace \mathfrak{S}^* .

8 Compliments

8.1 Frozen Percolation on r -regular Trees

Using exactly similar arguments as done in the case of infinite regular binary tree one can construct an automorphism invariant version of frozen percolation process on a infinite r -regular tree \mathcal{T}_r , in which each vertex has degree $r \geq 3$ (see [5] for details). In this setting the RDE is given by

$$Y^r \stackrel{d}{=} \Phi^r(Y_1^r \wedge Y_2^r \wedge \dots \wedge Y_{r-1}^r; U) \quad \text{on } I^r := \left[\frac{1}{r-1}, 1\right] \cup \{\infty\}, \quad (41)$$

where $(Y_j^r)_{1 \leq j \leq r-1}$ are i.i.d with same law as Y^r and are independent of $U \sim \text{Uniform}[0, 1]$; and $\Phi^r : I^r \times [0, 1] \rightarrow I^r$ is the function defined by equation (2). It is easy to check that the unique non-atomic solution of this RDE with full support is given by

$$\nu^r(dy) = \frac{dy}{(r-2)(r-1)^{\frac{1}{r-2}} y^{\frac{r-1}{r-2}}}, \quad \frac{1}{r-1} < y < 1, \quad \nu^r(\{\infty\}) = \frac{1}{(r-1)^{\frac{1}{r-2}}}. \quad (42)$$

Naturally the case $r = 3$ gives back the RDE (1) and its fundamental solution ν . Interesting enough our argument to prove the bivariate uniqueness for the frozen percolation RDE (1) extend essentially unchanged (needs only some changes of the constants) in this setting. Thus proving the endogeny for the invariant RTP associated with the RDE (41) with marginal ν^r .

8.2 Frozen Percolation on Infinite Regular Galton-Watson Trees

We can go a step further, and can construct a frozen percolation process (using essentially similar argument as in [5]) on a *infinite regular Galton-Watson branching process tree* with progeny distribution N , which satisfy

$$\mathbf{P}(N \geq 1) = 1 \quad \text{and} \quad \mathbf{P}(N = 1) < 1. \quad (43)$$

By *regular* we mean that the degree of each vertex has same distribution and are independent. The standard definition of a Galton-Watson branching process provides a *non-regular* tree since the degree of the root is N while the degree of all other vertices have distribution same as $(1 + N)$. There are several ways of defining a infinite regular Galton-Watson tree, for our particular case where N satisfies (43), we consider the following easy construction. Let \mathfrak{J}_1 and \mathfrak{J}_2 be two independent and identical realizations of infinite Galton-Watson trees with roots \emptyset_1 and \emptyset_2 say. The tree \mathfrak{J} obtained by joining \emptyset_1 and \emptyset_2 by an edge will be called an infinite regular Galton-Watson tree. On such a tree we can put i.i.d Uniform $[0, 1]$ edge weights independent of the tree and then one can construct a version of frozen percolation process. In this case the RDE involved is given by

$$Y^* \stackrel{d}{=} \Phi^*(Y_1^* \wedge Y_2^* \wedge \cdots \wedge Y_N^*; U) \quad \text{on} \quad I^* := \left[\frac{1}{m}, 1\right] \cup \{\infty\}, \quad (44)$$

where $(Y_j^*)_{j \geq 1}$ are i.i.d and have the same law as Y^* and are independent of the pair $(U; N)$ where N is the progeny distribution and $m := \mathbf{E}[N] \leq \infty$, and $\Phi^* : I^* \times [0, 1] \rightarrow I^*$ is the function defined by the equation (2). The following proposition gives the fundamental solution needed for construction of the frozen percolation process. The proof is essentially a rewriting of the argument of Aldous (see Lemma 3 of [5]) with the necessary changes required for this set up.

Proposition 14 *Suppose N satisfies the conditions given by (43) then the RDE (44) has a unique non-atomic solution with full support which is given by*

$$\nu^*((y, 1] \cup \{\infty\}) = \psi\left(\frac{1}{y}\right), \quad y \in \left[\frac{1}{m}, 1\right], \quad (45)$$

where $\psi = (\phi^!)^{-1}$ and $\phi(t) := \mathbf{E}[t^N]$, $0 \leq t \leq 1$ is the probability generating function of N .

Proof. Let $F(y) = \mathbf{P}(Y^* \leq y)$ for $\frac{1}{m} \leq y \leq 1$ be distribution function of Y^* , a solution of (44). Then from definition we have

$$F(y) = \sum_{n=1}^{\infty} p_n \mathbf{P}(U < Y_1^* \wedge Y_2^* \wedge \cdots \wedge Y_n^* \leq y), \quad \frac{1}{m} \leq y \leq 1, \quad (46)$$

where we write $p_n = \mathbf{P}(N = n)$ for $n \geq 1$. In the non-atomic case (46) is equivalent to

$$dF(y) = y \phi'(\bar{F}(y)) dF(y), \quad \frac{1}{m} < y < 1, \quad (47)$$

which in turn is equivalent to

$$F(y) = 1 - \psi\left(\frac{1}{y}\right) \text{ on } \left[\frac{1}{m}, 1\right] \cap \text{support}(F). \quad (48)$$

Now $\phi'(t) = \sum_{n=1}^{\infty} np_n t^{n-1}$ is strictly increasing and hence $\psi := (\phi')^{-1} : [p_1, m] \rightarrow [0, 1]$ is also strictly increasing. Thus the function $y \mapsto 1 - \psi\left(\frac{1}{y}\right)$ is strictly increasing but then (48) can only happen when the $\text{support}(F) = \left[\frac{1}{m}, a\right]$ for some $\frac{1}{m} < a \leq 1$. Thus the only non-atomic solution with full support is given by (45). \blacksquare

The construction of the frozen percolation process uses the invariant RTP associated with the RDE (44) with marginal ν^* . To be more specific, we can construct a joint law for $\left(\mathfrak{J}, (U_e)_{e \in \mathfrak{E}}, \left(Y_{\vec{e}}^*\right)_{\vec{e} \in \vec{\mathfrak{E}}}\right)$ such that

- $\mathfrak{J} = (\mathfrak{V}, \mathfrak{E})$ is a infinite regular Galton-Watson tree with progeny distribution given by N ,
- Given \mathfrak{J} , $(U_e)_{e \in \mathfrak{E}}$ are i.i.d Uniform $[0, 1]$,
- $Y_{\vec{e}}^*$ has law ν^* for each directed edge $\vec{e} \in \vec{\mathfrak{E}}$, and
- $Y_{\vec{e}}^* = \Phi^* \left(Y_{\vec{e}_1}^* \wedge Y_{\vec{e}_2}^* \wedge \dots \wedge Y_{\vec{e}_{N_{\vec{e}}}}^* ; U_e \right)$ a.s. where $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{N_{\vec{e}}}$ are $N_{\vec{e}}$ children of the directed edge $\vec{e} \in \vec{\mathfrak{E}}$.

Note for a realization of \mathfrak{J} every edge of it can be given two directions. $\vec{\mathfrak{E}}$ denotes the set of all the directed edges of a particular realization of \mathfrak{J} . If $e = (u, v)$ be an edge of \mathfrak{J} and $\vec{e} = \overrightarrow{(u, v)}$ be its one direction from vertex u to vertex v then $N_{\vec{e}} := \text{degree of } v - 1$, thus has same distribution as N . Finally similar to the construction of the frozen percolation process on infinite regular binary tree [5], given a realization of \mathfrak{J} we can now construct the set process $(\mathcal{A}_t)_{0 \leq t \leq 1}$ as

$$\begin{aligned} \mathcal{A}_1 &:= \left\{ e \in \mathfrak{E} \mid U_e < \min \left(Y_{\vec{e}'}^*, : \vec{e}' \in \partial(\{e\}) \right) \right\}, \text{ and} \\ \mathcal{A}_t &:= \left\{ e \in \mathcal{A}_1 \mid U_e \leq t \right\}, \quad 0 \leq t < 1, \end{aligned} \quad (49)$$

where for an edge $e \in \mathfrak{E}$, $\partial(\{e\})$ is the set of all children (as directed edges) of the two directed edges \vec{e} and \overleftarrow{e} . Though proving that this construction satisfies (*) needs some further arguments but they are similar to what is done in [5] and hence we do not repeat them here. The *externally* defined random variables, namely, $Y_{\vec{e}}^*$'s have natural interpretation as the time an edge e takes to join to infinity along a particular direction, namely, \vec{e} . These of course form invariant RTP associated with the RDE (44) with marginal ν^* .

Interesting enough under some mild moment condition on N , we now prove that the invariant RTP with marginal ν^* is endogenous. Thus the frozen percolation process on infinite regular Galton-Watson branching process tree does not exhibit any “spatial chaos” property. Naturally the endogeny for r -regular trees is a special case of this general result.

Theorem 15 *Suppose $\mathbf{E}[2^N] < \infty$ then the invariant RTP associated with the RDE (44) with marginal ν^* is endogenous.*

Proof. Naturally we will again show that the RDE (44) has bivariate uniqueness property and then use Theorem 11(b) of [3]. The proof is essentially similar to the proof of Theorem 3 but needs some changes which are given below.

As in the other cases first we consider the following bivariate RDE

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \Phi^*(X_1^* \wedge X_2^* \wedge \cdots \wedge X_N^*; U) \\ \Phi^*(Y_1^* \wedge Y_2^* \wedge \cdots \wedge Y_N^*; U) \end{pmatrix}, \quad (50)$$

where $(X_j, Y_j)_{j \geq 1}$ are i.i.d pairs with same joint distribution as (X, Y) and have same marginal distribution ν^* and are independent of $U \sim \text{Uniform}[0, 1]$; We note that the bivariate RDE (50) has a solution given by the diagonal measure $\nu^{*\nearrow} := \text{dist}((X^*, X^*))$ where $X^* \sim \nu^*$, so all we need to show is it is the unique solution. For any solution of (50) let $F(x, y) := \mathbf{P}(X^* \leq x, Y^* \leq y)$ and $G(x, y) := \mathbf{P}(X^* > x, Y^* > y)$ for $x, y \in [\frac{1}{m}, 1]$, then a similar derivation like equation (11) will give

$$\begin{aligned} F(x, y) &= \mathbf{E} \left[\left(\mathbf{1}_{(X_1^* \wedge X_2^* \wedge \cdots \wedge X_N^* > U)} - \mathbf{1}_{(X_1^* \wedge X_2^* \wedge \cdots \wedge X_N^* > x)} \right) \right. \\ &\quad \left. \times \left(\mathbf{1}_{(Y_1^* \wedge Y_2^* \wedge \cdots \wedge Y_N^* > U)} - \mathbf{1}_{(Y_1^* \wedge Y_2^* \wedge \cdots \wedge Y_N^* > y)} \right) \right] \\ &= \sum_{n=1}^{\infty} p_n \int_0^{x \wedge y} (G^n(x, y) - G^n(x, u) - G^n(u, y) + G^n(u, u)) du \quad (51) \end{aligned}$$

Let G_0 be the tail of the distribution function for the solution $\nu^{*\nearrow}$. Put $D^* := [\frac{1}{m}, 1] \times [\frac{1}{m}, 1]$, the domain of the integral equation. Define $H(x, y) = 1 - G(x, y)/G_0(x, y)$. Notice that $H \equiv 0$ if $(x, y) \in [0, 1] \times [0, 1] \setminus D^*$, since both G and G_0 have same marginal. Moreover for $(x, y) \in D^*$,

$$\begin{aligned} G(x, y) &= \mathbf{P}(X^* > x, Y^* > y) \\ &\leq \min(\mathbf{P}(X^* > x), \mathbf{P}(Y^* > y)) \\ &= \min(\mathbf{P}(X^* > x), \mathbf{P}(X^* > y)) \\ &= \mathbf{P}(X^* > x \vee y) = G_0(x, y). \end{aligned}$$

Thus $0 \leq H(x, y) \leq 1$ for all $(x, y) \in D^*$. To prove bivariate uniqueness all we need to show is $H \equiv 0$ on D^* .

Now by similar calculation like equation (14) we get

$$\begin{aligned}
G_0(x, y)H(x, y) &= G_0(x, y) - G(x, y) \\
&= F_0(x, y) - F(x, y) \\
&= \int_0^{x \wedge y} \Lambda^*(x, y, u) du
\end{aligned} \tag{52}$$

where Λ^* is given by $\Lambda^*(x, y, u) = \sum_{n=1}^{\infty} p_n \Lambda_n^*(x, y, u)$, and for $n \geq 1$,

$$\begin{aligned}
\Lambda_n^*(x, y, u) &:= G_0^n(x, y) (1 - (1 - H(x, y))^n) - G_0^n(x, u) (1 - (1 - H(x, u))^n) \\
&\quad - G_0^n(u, y) (1 - (1 - H(u, y))^n) + G_0^n(u, u) (1 - (1 - H(u, u))^n).
\end{aligned}$$

The last expression follows from (51) and since $G_0^n - G^n = G_0^n (1 - (1 - H)^n)$, for all $n \geq 1$. Note that $\Lambda^* \equiv 0$ outside D^* .

Now we observe $G_0(x, y) = \mathbf{P}(X^* > x \vee y) = \psi\left(\frac{1}{x \vee y}\right)$ where $x, y \in [\frac{1}{m}, 1]$. Moreover ψ being an increasing function, the minimum of G_0 on D^* is obtained at $x = y = 1$. So the minimum is $\psi(1) > 0$, since ψ is the inverse function of ϕ' and $\phi'(0) = p_1 < 1$. Thus G_0 remains bounded away from 0 on D^* , so there is $M > 0$ a constant such that $\frac{1}{G_0(x, y)} < M$ for all $(x, y) \in D^*$.

Further consider the function $u \mapsto \sum_{n=1}^{\infty} p_n 2^n G_0^n(u, u)$ defined on $[\frac{1}{m}, 1]$. It is well defined by our assumption, $\mathbf{E}[2^N] < \infty$. Moreover it is also continuous. Thus given $0 < \varepsilon < \frac{1}{8M}$ there exists a partition $\frac{1}{m} = \alpha_0 < \alpha_1 < \dots < \alpha_k = 1$ such that

$$\int_{\alpha_{i-1}}^{\alpha_i} \sum_{n=1}^{\infty} p_n 2^n G_0^n(u, u) du < \varepsilon, \tag{53}$$

Similar to the proof of Theorem 3 define $L_i := \{(x, y) \mid \alpha_{i-1} \leq x \wedge y < \alpha_i\}$, for $1 \leq i \leq k$. Certainly L_i 's form a partition of D^* into L -shape parts. Finally on L_i we define $\|H\|_i := \sup_{x, y \in L_i} |H(x, y)|$.

Similar to the calculations in the proof of Theorem 3 we start with $i = 1$ and let $(x, y) \in L_i$. Then from (52) and definition of Λ^* we get the following estimate

$$\begin{aligned}
|H(x, y)| &\leq \frac{\|H\|_i}{G_0(x, y)} \\
&\quad \times \int_{\alpha_{i-1}}^{x \wedge y} \sum_{n=1}^{\infty} p_n 2^n G_0^n(u, u) \left(\frac{G_0^n(x, y)}{G_0^n(u, u)} + \frac{G_0^n(x, u)}{G_0^n(u, u)} + \frac{G_0^n(u, y)}{G_0^n(u, u)} + 1 \right) du \\
&\leq 4M \|H\|_i \int_{\alpha_{i-1}}^{x \wedge y} \sum_{n=1}^{\infty} p_n 2^n G_0^n(u, u) du \\
&\quad \text{[since } G_0(x, y) \leq G_0(u, u) \text{ for } u \leq x \wedge y] \\
&\leq 4M \|H\|_i \int_{\alpha_{i-1}}^{\alpha_i} \sum_{n=1}^{\infty} p_n 2^n G_0^n(u, u) du \quad \text{[since } x \wedge y < \alpha_i] \\
&\leq 4M\varepsilon \|H\|_i \quad \text{[using (53)]}
\end{aligned}$$

So we get $\|H\|_{i \leq \frac{1}{2}} \leq \frac{1}{2} \|H\|_i$ which implies $\|H\|_i = 0$, that is $H \equiv 0$ on L_i . We can now proceed exactly in the same way by taking $i = 2, 3, \dots, k$ recursively to get $H \equiv 0$ on whole of D^* . This proves the bivariate uniqueness property for the invariant recursive tree process associated with the RDE (44) with marginal ν^* .

Finally to complete the proof of endogeny we notice that under our assumption $\mathbf{E}[2^N] < \infty$, the technical condition of Theorem 11(b) of [3] follows by using the dominated convergence theorem. \blacksquare

8.3 Uniqueness of frozen percolation process

Theorem 1 proves that Aldous' construction of the frozen percolation process on a infinite regular binary tree [5] do not exhibit any "spatial chaos" property, in the sense that the only randomness is through the i.i.d Uniform[0, 1] edge weights. On the other hand it does not exclude the possibility of having another version of the process which may have *external* randomness present in it. The question of uniqueness thus remain open. All we can conclude here is that if the process satisfying (*) is also measurable with respect to the edge weights then it is unique and has to be the one constructed by Aldous [5]. Similar conclusion holds for the frozen percolation process on other regular trees.

8.4 Comments on the proof of Theorem 5

(a) Intuitively, a natural approach to show that the fixed-point equation $\Gamma(\mu^{(2)}) = \mu^{(2)}$ on \mathfrak{S} has unique solution, would be to specify a metric ρ on \mathfrak{S} such that the operator Γ becomes a contraction with respect to it. Unfortunately, this approach seems rather hard or may even be impossible. Perhaps the reason being the Logistic RDE (4) itself does not have a contractive property, in fact, it does not have a full domain of attraction (see [3]). However its exact domain of attraction is not yet known (see open problem 62 of [3]). On the other hand from the proof of Theorem 5 it is clear that equation (20) has the whole of \mathfrak{F} within its domain of attraction. So it is possible to have a suitable metric of contraction for T but, we have been unable to find it.

(b) Although at first glance it seems that the operator T as defined in (21) is just an analytic tool to solve the equation (20) but, it has a nice interpretation through Logistic RDE (4). Suppose \mathfrak{A} is the operator associated with Logistic RDE, that is,

$$\mathfrak{A}(\mu) \stackrel{d}{=} \min_{j \geq 1} (\xi_j - X_j), \quad (54)$$

where $(\xi_j)_{j \geq 1}$ are points of a Poisson point process of mean intensity 1 on $(0, \infty)$, and are independent of $(X_j)_{j \geq 1}$, which are i.i.d with distribution μ on \mathbb{R} . It is easy to check that the domain of definition of \mathfrak{A} is the space

$$\mathcal{A} := \left\{ F \mid F \text{ is a distribution function on } \mathbb{R} \text{ and } \int_0^\infty \overline{F}(s) ds < \infty \right\}. \quad (55)$$

Note that the condition $\int_0^\infty \overline{F}(s) ds < \infty$ means $\mathbf{E}_F[X^+] < \infty$. From definition $\mathfrak{F} \subseteq \mathcal{A}$, and T can be naturally extended to the whole of \mathcal{A} . In that case the following identity holds

$$\frac{\overline{T(\mu)}(\cdot)}{\overline{H}(\cdot)} \times \frac{\overline{\mathfrak{A}(\mu)}(\cdot)}{\overline{H}(\cdot)} = 1, \quad \forall \mu \in \mathcal{A}. \quad (56)$$

This at least explains the monotonicity of T through anti-monotonicity property of the Logistic operator \mathfrak{A} (easy to check).

Appendix

Here we provide some known basic facts about the Logistic distribution which are used in the Sections 6 and 7.

First recall that we say a real valued random variable X has Logistic distribution if its distribution function is given by (5), namely,

$$H(x) = \mathbf{P}(X \leq x) = \frac{1}{1 + e^{-x}}, \quad x \in \mathbb{R}.$$

The following facts hold for the function H .

Fact 1 H is infinitely differentiable, and $H'(\cdot) = H(\cdot)\overline{H}(\cdot)$, where $\overline{H}(\cdot) = 1 - H(\cdot)$.

Proof. From the definition it follows that H is infinitely differentiable on \mathbb{R} . Further,

$$\begin{aligned} H'(x) &= \frac{1}{1 + e^{-x}} \times \frac{e^{-x}}{1 + e^{-x}} \\ &= H(x)\overline{H}(x) \quad \forall x \in \mathbb{R} \end{aligned}$$

■

Fact 2 H is symmetric around 0, that is, $H(-x) = \overline{H}(x) \quad \forall x \in \mathbb{R}$.

Proof. From the definition we get that for any $x \in \mathbb{R}$,

$$H(-x) = \frac{1}{1 + e^x} = \frac{e^{-x}}{1 + e^{-x}} = \overline{H}(x).$$

■

Fact 3 \overline{H} is the unique solution of the non-linear integral equation

$$\overline{H}(x) = \exp\left(-\int_{-x}^{\infty} \overline{H}(s) ds\right), \quad \forall x \in \mathbb{R}. \quad (A1)$$

Proof. Notice that the equation (A1) is nothing but Logistic RDE, since $\mathfrak{A}(H)(x) = \exp\left(-\int_{-x}^{\infty} \overline{H}(s) ds\right)$, $\forall x \in \mathbb{R}$ (see proof of Lemma 5 in Aldous [6]). Thus from the fact that \overline{H} is the unique solution of Logistic RDE (Lemma 5 of Aldous [6]) we conclude that \overline{H} is unique solution of equation (A1). ■

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