# RESEARCH STATEMENT

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My recent research interest has been to develop, analyze and implement effective numerical methods for multiscale phenomena arising from flows in heterogeneous porous media. The main purpose is to develop innovative numerical and analytical methods that can capture the effect of small scales on the large scales without resolving the small scale details on a coarse computational grid. This research activity is strongly motivated by many important practical applications arising in contaminant transport in heterogeneous porous media, oil reservoir simulations, subsurface characterization and etc. During my Ph.D study at Texas A&M University and postdoctoral research at University of Minnesota, I have made contributions in the following research areas: (1) Develop and analyze Galerkin *multiscale finite element methods* (MsFEM), mixed MsFEM and mixed *multiscale finite volume methods* (MsFVM) for flows in heterogeneous porous media that employ limited global information; (2) Develop and analyze mixed MsFEM using approximate global information based on partial upscaling and apply the methods to multiphase flow simulations in porous media, and extend the idea to develop stochastic mixed MsFEM for random porous media; (3) Design and analyze efficient a *partition of unity methods* (PUM) involving global multiscale fields for partial differential equations with continuum spatial scales; (4) Investigate individual homogenization theorems for nonlinear parabolic equations when the coefficients are almost periodic in the sense of Besikovitch; (5) Explore a unified analysis of 4D variational data assimilation in functional differential operator and theoretically investigate its numerical approximation .

## 1 MsFEM, mixed MsFEM and mixed MsFVM using limited global information

The main idea of MsFEM is to incorporate the small-scale information into finite element basis functions and capture their effect on the large scale via finite element computations. This method is first proposed in [6] for certain class of multiscale problems and later it is extended and generalized in [10]. These methods also share similarities with some other multiscale methods, e.g., [4, 7, 11]. The coupling of the small scales is done through a variational formulation of the global problem. The multiscale basis functions are constructed from the solution of the leading order homogeneous elliptic equation on each coarse element with some specified boundary conditions. Hou et al. [10] have demonstrated that a careful choice of boundary conditions would improve the accuracy of the method, and in particular, they have introduced oversampling technique, where the local problems in the domain larger than the target coarse block is used for constructing basis functions. Standard MsFEM produces resonance error between the grid scale and the characteristic length scales. Although over-sampling technique improves the convergence of standard MsFEM, it still produces cell resonance error caused by the mismatch between the mesh size and the perfect sample size. To overcome this difficulty, multiscale finite element methods that use limited global information are introduced in [8].

One of our contributions in MsFEM is the extensive studies of MsFEMs using limited global information and the implementation for flows in highly heterogeneous, channelized, porous media. The method uses some global field information (e.g., single-phase flow) to construct multiscale basis functions. We have proposed a mixed MsFEM using multiple global fields [2] and provided a rigorous analysis of the stability and convergence for the method, and performed numerical experiments for two-phase flow equations in highly heterogenous porous media. For construction of velocity basis function for the mixed MsFEM, some global fields are utilized. The analysis of the global multiscale methods (Galerkin MsFEM [1] and mixed MsFEM) uses the fact that the solution (or velocity for mixed MsFEM) smoothly depends on global fields. Under this assumption, the convergence rate for Galerkin MsFEM and mixed MsFEM are obtained. We have studied both separable scales and continuum scales. The analysis of the global MsFEM demonstrates that the convergence rate in $H^1$ norm is $O(h^\alpha)$ for the case with continuum scales and $O(h^\alpha + \sqrt{\varepsilon})$ for separable scales, where $h$ is the coarse mesh size, $\varepsilon$ is physical scale and $\alpha$ depends on the smoothness of the solution in terms of the utilized global fields. Analogously for the global mixed MsFEM, the convergence rate under the norm $\| \cdot \|_{0,\Omega} + \| \cdot \|_{H(div,\Omega)}$ is $O(h^\alpha)$ for the case with continuum scales and $O(h^\alpha + \sqrt{\varepsilon})$ for separable scales. Our results reveal that global MsFEM and global mixed MsFEM can improve accuracy substantially compared to standard local MsFEM and remove resonance error completely. Further, we have extended and generalized the global MsFEM and mixed MsFEM to parabolic equations and acoustic wave equations with continuum spatial scales and presented rigorous analysis and numerical results. The details of the results and analysis can be found in [14] and [15].

It is often necessary to use an unstructured coarse grid when highly heterogeneous reservoirs are discretized via irregular anisotropic fine grids. The latter often occurs in petroleum applications. Motivated by the development of coarse-scale models for coupled flow and transport equations in a multi-phase system, we have studied mixed MsFEM on unstructured coarse grids [3]. The unstructured coarse grid is used to upscale the transport equation as well. Solving the flow equation on the same coarse grid provides a general robust coarse-scale model for a multi-phase flow and transport.

Mixed MsFEM is one of the favorite methods for multiphase flow simulations in heterogeneous media and provides very good approximation, but the computational cost is relatively high for real-life problems and not easy to use for unstructured grids. MsFVM proposed by [12] is cheap and works well for general meshes, but it is not applicable for full tensor coefficients. Recently we have proposed a new mixed MsFVM [17], which combines the advantages of both mixed MsFEM and MsFVM and overcomes the drawbacks in the mixed MsFEM and the MsFVM. A new multiscale velocity basis function using global information is devised in the mixed MsFVM. We have analyzed the mixed MsFVM and applied it to two-phase flows in highly heterogenous porous media.

## 2 MsFEMs and stochastic MsFEMs using approximate global information based on partial upscaling

The use of limited global information in multiscale simulations is needed when there is no scale separation. Previous approaches entail fine-scale simulations in the computation of the global information. The computation of the global information is expensive. We propose the use of approximate global information based on partial upscaling. Partial homogenization upscales some of the smallest scales that can be captured through local computations of multiscale basis functions.

Such global information can be computed inexpensively and is less sensitive to the changes of the media properties at smallest scales. A requirement for partial homogenization is to capture long-range (non-local) effects present in the fine-scale solution, while homogenizing some of the smallest scales. The proposed approach allows us to avoid the computations at the scales that can be homogenized. This results to coarser problems for the computation of global fields. We consider a mathematical framework, where the media have both non-separable and separable scales. Separable scales are assumed to be much smaller and, thus, it is desirable to homogenize them. As for approximate global fields, we homogenize the media properties over separable scales and compute the global fields on the coarser grid. Rigorous analysis for various cases are made. We consider two scenarios for the scales that are homogenized. In the first case, the scales, which are homogenized, are periodic. For the second case, we use a general $G$-convergence framework. We show that the proposed MsFEM is stable and the convergence only depends on the small localizable scales, but it is independent of non-local scales. The proposed method is applied to simulate two-phase flows in various heterogeneous porous media. The use of approximate global information provides better accuracy than purely local multiscale methods. The details of the results and analysis can be found in [16].

We extend the idea to random porous media and develop a framework of stochastic MsFEMs, which can efficiently solve the stochastic partial equations with multiscale structures. We investigate two approaches of stochastic MsFEMs in the framework. The first approach entails no stochastic interpolation in constructing multiscale basis functions that capture both spatial scales information and stochastic information. This is accomplished by selecting a small number of representative samples that are sufficiently scattered in the uncertainty space (or in a part of the uncertainty space), and then computing multiscale basis functions for each of the samples. This approach avoids using interpolation in the stochastic space. The second approach combines MsFEMs with stochastic collocation methods. This approach is to construct multiscale basis functions at collocation points specially chosen from the stochastic space. For each arbitrary realization, we reconstruct multiscale basis functions that are interpolated by the functions at the collocation points. Each of stochastic MsFEM approaches has its own advantage. The non-collocation approach is applicable to random porous media with complicated statistical feature (e.g., hierarchical stochastic model), but may have less accuracy of the approximation for stochastic space if the representative samples are selected improperly. The collocation approach can provide good approximation for stochastic space when proper collocation points are used; however, it is difficult to apply it to the models with complicated stochastic features. We make convergence analysis of the stochastic MsFEMs and investigate their applications to incompressible two-phase flows in random porous media. The analysis and related numerical results can be found in [17].

## 3 Partition of unity methods for partial differential equations with continuum scales

Partition of unity approach is first introduced in [6] to obtain an accurate numerical solution of second order elliptic equations with rough coefficients. Based on this idea, the approach is further elaborated on in [19], where it is named by *partition of unity methods* (PUM). Later it is also referred as *generalized finite element methods*. The main idea of PUM is to find an accurate approximation in each "patch" and then use partition of unity functions to "paste" those patch approximations together. If one knows more information about solutions of partial differential equations and enrich this information into patch approximation, then an accurate approximation of numerical solution can be obtained. Motivated by this idea, we have designed a PUM to solve elliptic and evolution equations where the coefficients have continuum spatial scales. Standard localized multiscale methods or upscaling techniques are not very suitable for these problems because these problems do not possess scale separation and homogenization techniques are not applicable. For our analysis, we use the fact that the solutions of partial differential equations smoothly depend on certain global fields (defined over the entire region), which carry the information on the small scale structure of the solution. These global fields are used to construct shape functions in each patch. This is not an trivial extension of MsFEM that employs limited global information because the construction of basis function in every "patch" are different from that global MsFEM discussed in Section 1, i.e., basis functions in each "patch" are the span of multiplication of partial unity functions and global fields. Our mathematical analysis shows that the convergence rate of this PUM in $H^1$ for elliptic equations with continuous scales is $O(h^\alpha)$, which is free of resonance errors as in Section 1. Further, we construct PUM framework for parabolic equations and acoustic wave equations with continuum spatial scales. For these equations, time continuous Galerkin methods and time discrete Galerkin methods are analyzed and convergence rate is obtained. This method shares some similarities with the recent work in [20], where conforming Galerkin finite element method is constructed on a grid consisting of level sets of certain global fields. The results and analysis can be found in [14] and [15].

## 4 Homogenization of nonlinear parabolic operators

My another contribution is to investigate an individual homogenization theory for nonlinear pseudomonotone parabolic operators when the coefficients are almost periodic in the sense of Besikovitch. Homogenization of nonlinear parabolic equations comes from applications arisen in flow in porous media for both saturated and unsaturated media, though one encounter nonlinear parabolic equations in many different applications. The fluxes in applied problems may be discontinuous in space and time. For this purpose, we studied the case where the spatial and temporal heterogeneities are almost periodic coefficients in the sense of Besikovitch. This allows discontinuities and is suitable for many applications. It was shown that any almost periodic nonlinear pseudomonotone parabolic operators can be considered as a particular realization of certain random homogeneous operator. Although this realization is not generic, one can use almost periodicity to pass from a generic realization to every particular realization. For the purpose, we derive stability and comparison results for $G-$convergent sequence of parabolic operators and then estimate the difference between $G-$limits of two $G-$convergent sequence of the parabolic operators. Finally, using these estimates, the individual homogenization results are established. The results can be found in [9]. In future, I plan to study the numerical homogenization of nonlinear parabolic equations and obtain explicit convergence rates when the coefficients are random homogeneous with possibly additional assumptions.

## 5 A unified analysis of 4D variational data assimilation

Data assimilation is an analysis technique in which the observed information is accumulated into the model state by taking advantage of consistency constraints with laws of time evolution and physical properties. 4D Var is to handle observations that are distributed in 3D space and in time. We have given a general procedure and a unified analysis for 4D variation data assimilation, where cost functional is described by a nonlinear function in Hilbert spaces and model output is constrained by a nonlinear evolution equations with complicated structures in Hilbert spaces. In the nonlinear functional partial differential equation setting, we have explored adjoint equations, sensitivity analysis and Marchuk-Strang symmetrical multi-component operator splitting techniques. Error estimates based on the operator splitting method are discussed for the functional 4D-Var problem. Some quasi-Newton iterative techniques (e.g., modified BFGS) are investigated and applied to the optimization of the nonlinear cost functional. Convergence of the numerical methods is analyzed. The analysis and results can be found in [13].

## 6 Some ongoing research plans

I am currently studying finite element exterior calculus [5] proposed by Arnold et al. This approach is to the design and understanding of finite element methods for various partial differential equations (PDEs). The approach brings elegant connections among finite element methods, differential geometry and homological algebra. We are planing to apply this exterior calculus approach to multiscale PDEs and propose a unified description of MsFEMs and a framework of MsFE exterior calculus, and investigate its applications involving multiscale phenomena. We expect to use the framework of MsFE exterior calculus to develop some novel multiscale finite element methods to solve elasticity equations and Maxwell's equations with multiscale coefficients.

## A Additional research

In addition, we have studied some holomorphic function spaces on unit disc and the composition operators between the holomorphic function spaces. Some important properties of holomorphic Bers-type spaces are comprehensively explored, for example, the value distribution theory, the connection with Carleson measures and the factorization decomposition of holomorphic Bers-type functions. Necessary and sufficient conditions are established for composition operators to be bounded and compact on Bers-type spaces and on their generalized spaces referred to be normal function spaces. We have also studied the composition operators from Bers-type spaces to weighted Bergman spaces and Hardy spaces. The composition operators on general analytic function spaces (also referred as $F(p,q,s)$ spaces) are also extensively studied. From the latter, we have obtained a unified characterization for necessary and sufficient conditions of composition operators from Bloch-type spaces to other classic holomorphic function spaces, i.e., analytic function spaces of bounded mean oscillation (also referred as BMOA spaces) and holomorphic $Q$ spaces (also referred as $Q_p$ spaces).

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