The Formula for Curvature

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Suppose we have a curve in the plane given by the vector equation

$$\mathbf{r}(t) = x(t)\ \mathbf{i} + y(t)\ \mathbf{j}, \quad a \le t \le b,$$

where $x(t),\ y(t)$ are defined and continuously differentiable between $t = a$ and $t = b$. You can think of t as time. so that we have a particle located at the point $(x(t), y(t))$ at time t and it traces out a trajectory as t goes from a to b. Let's also assume that the particle never stops, i.e. that its speed

$$\frac{ds}{dt} = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} > 0$$

for all times between a and b. The instantaneous velocity vector (or tangent vector) to the curve is

$$\dot{\mathbf{r}}(t) = \dot{x}(t)\ \mathbf{i} + \dot{y}(t)\ \mathbf{j}.$$

We can also parameterize the curve by using arc length. Thus the arc length of the curve from the point $(x(a), y(a))$ to the point $x(t), y(t))$ is

$$s(t) = \int_a^t \sqrt{\dot{x}^2(\tau) + \dot{y}^2(\tau)}\ d\tau = \int_a^t ||\dot{\mathbf{r}}(\tau)||\ d\tau.$$

By the fundamental theorem of calculus we have

$$ds = ||\dot{\mathbf{r}}(t)||\ dt = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\ dt, \tag{1}$$

so we can either parameterize the curve by t or by the arc length s, and the equation (1) relates the two variables.

For any value of t the tangent vector $\dot{\mathbf{r}}(t)$ makes an angle $\phi(t)$ with the positive x axis. Thus we can write $\dot{\mathbf{r}}(t)$ in polar coordinates as

$$\dot{\mathbf{r}}(t) = ||\dot{\mathbf{r}}(t)||(\cos\phi(t)\ \mathbf{i} + \sin\phi(t)\ \mathbf{j}).$$

As the tangent vector moves along the curve it rotates in a counterclockwise or clockwise direction, depending on whether ϕ is increasing or decreasing. It should be clear from this that the derivative

$$\frac{d\phi}{dt}$$

gives information about how fast the curve is turning, and whether it is turning in a clockwise or counterclockwise direction. This information is, essentially, what we mean by the curvature of the curve at the point $(x(t), y(t))$.

However, the same curve can be parameterized in many different ways and the value of $\frac{d\phi}{dt}$ will depnd on the parameterization. To get a measure of how fast the curve is turning that depends on the curve alone, and not the specific parameterization, we fix on arc length s as a standard parameterization for the curve. Thus the curvature k at a point (x, y) on the curve is defined as the derivative

$$k = \frac{d\phi}{ds} = \frac{d\phi}{dt}\frac{dt}{ds},$$

where we have used the chain rule in the last equality. To compute the curvature from $(x(t), y(t))$ we note that

$$\tan\phi(t) = \frac{\dot{y}(t)}{\dot{x}(t)}.$$

Differentiating both sides of this equation implicitly with respect to t we find

$$\sec^2\phi\ \frac{d\phi}{dt} = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{(\dot{x})^2}.$$

Now

$$\sec^2\phi = \tan^2\phi + 1 = (\frac{\dot{y}}{\dot{x}})^2 + 1 = \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}^2},$$

so we can solve for $\frac{d\phi}{dt}$ to get

$$\frac{d\phi}{dt} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}.$$

Finally from (1) we get

$$k = \frac{d\phi}{ds} = \frac{d\phi}{dt}\frac{dt}{ds} = (\frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2})(\frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}}) = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}. \tag{2}$$

Thus

$$k = \frac{d\phi}{ds} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}},$$

which is the expression for curvature that appears in the course booklet.

Note that if the curve is a straight line $x = x_0 + at,\ y = y_0 + bt$ then $k = 0$ for all points on the line, i.e., the curvature is zero. If the curve is a circle with radius R, i.e.

$$x = R\cos t, \quad y = R\sin t,$$

then $k = 1/R$, i.e., the (constant) reciprocal of the radius. In this case the curvature is positive because the tangent to the curve is rotating in a counterclockwise direction.

In general the curvature will vary as one moves along the curve. For example, consider the parabola $y = x^2$. We can express this curve parametrically in the form

$$x = t, \quad y = t^2,$$

so that we identify the parameter t with x. Then $\dot{x} = 1,\ \ddot{x} = 0, \dot{y} = 2t,\ \ddot{y} = 2$, so

$$k = \frac{2}{(1 + 4t^2)^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$$

at the point $(x, y) = (x, x^2)$ on the curve.