CHAPTER 3

The Three-Variable Helmholtz and Laplace Equations

3.1 The Helmholtz equation $(\Delta_3+\omega^2)\Psi=0$

The Helmholtz or reduced wave equation in three variables

$$(\Delta_3+\omega^2)\Psi(x_1,x_2,x_3)=0, \qquad \Delta_3=\partial_{x_1x_1}+\partial_{x_2x_2}+\partial_{x_3x_3}, \qquad \omega>0, \tag{1.1}$$

has been widely studied from the point of view of separation of variables, and the possible separable coordinate systems for this equation are well known [97, 98]. The connection between the separable systems and the Euclidean symmetry group $E(3)$ of (1.1) was first pointed out in [76]. However, it is only recently that this connection with group theory has been employed systematically to derive properties of the separable solutions of the Helmholtz equation.

Applying our usual methods, we find that (apart from the trivial symmetry E) the symmetry algebra of (1.1) is six dimensional with basis

$$\begin{aligned} &P_j=\partial_j=\partial_{x_j}, \qquad j=1,2,3;\\ &J_1=x_3\partial_2-x_2\partial_3, \qquad J_2=x_1\partial_3-x_3\partial_1, \qquad J_3=x_2\partial_1-x_1\partial_2, \end{aligned} \tag{1.2}$$

and commutation relations

$$[J_l,J_m]=\sum_n \varepsilon_{lmn}J_n, \quad [J_l,P_m]=\sum_n \varepsilon_{lmn}P_n, \quad [P_l,P_m]=0, \qquad l,m,n=1,2,3, \tag{1.3}$$

where ε_{lmn} is the tensor such that $\varepsilon_{123}=\varepsilon_{312}=\varepsilon_{231}=1$, $\varepsilon_{132}=\varepsilon_{213}=\varepsilon_{321}=-1$, with all other components zero. We take the real Lie algebra $\mathcal{E}(3)$ with basis (1.2) as the symmetry algebra of (1.1). In terms of the P operators,

ENCYCLOPEDIA OF MATHEMATICS and Its Applications, Gian-Carlo Rota (ed.).
Vol. 4: Willard Miller, Jr., Symmetry and Separation of Variables

ISBN-0-201-13503-5

the Helmholtz equation reads

$$\left(P_1^2+P_2^2+P_3^2\right)\Psi=-\omega^2\Psi. \tag{1.4}$$

Here $\mathcal{E}(3)$ is isomorphic to the Lie algebra of the Euclidean group in three-space $E(3)$ and the subalgebra $so(3)$ with basis $\{J_1,J_2,J_3\}$ is isomorphic to the Lie algebra of the proper rotation group $SO(3)$. To show this explicitly we first consider the well-known realization of $SO(3)$ as the group of real 3×3 matrices A such that $A'A=E_3$ and $\det A=1$ (see, e.g., [45, 85]). Here E_3 is the 3×3 identity matrix $(E_3)_{jl}=\delta_{jl}$ and $(A')_{jl}=A_{lj}$, $j,l=1,2,3$. The Lie algebra of $SO(3)$ in this realization is the space of 3×3 skew-symmetric matrices $\mathcal{A}$ ($\mathcal{A}'=-\mathcal{A}$). A basis for this Lie algebra is provided by the matrices

$$\mathcal{J}_1'=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix},\quad \mathcal{J}_2'=\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix},\quad \mathcal{J}_3'=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}, \tag{1.5}$$

with commutation relations $[\mathcal{J}_l',\mathcal{J}_m']=\sum_n \varepsilon_{lmn}\mathcal{J}_n'$, in agreement with (1.3). A convenient parametrization of $SO(3)$ is that in terms of *Euler angles* (φ,θ,ψ):

$$A(\varphi,\theta,\psi)=\exp(\varphi\mathcal{J}_3')\exp(\theta\mathcal{J}_1')\exp(\psi\mathcal{J}_3'), \tag{1.6}$$

$$0\leqslant\varphi<2\pi,\quad 0\leqslant\theta\leqslant\pi,\quad 0\leqslant\psi<2\pi.$$

As the Euler angles run over their full domain of values, $A(\varphi,\theta,\psi)$ runs over all elements of $SO(3)$. The coordinates are one to one on the group manifold except for those elements for which $\theta=0,\pi$, in which cases only the sum $\varphi+\psi$ is uniquely determined. More detailed discussions of these coordinates can be found in many references (e.g., [45, 85, 124]).

The Euclidean group in three-space $E(3)$ can be realized as a group of 4×4 real matrices. The elements of $E(3)$ are

$$g(A,\mathbf{a})=\begin{bmatrix} & & & 0\\ & A & & 0\\ & & & 0\\ a_1 & a_2 & a_3 & 1\end{bmatrix},\qquad A\in SO(3),\quad \mathbf{a}=(a_1,a_2,a_3)\in R^3, \tag{1.7}$$

and the group product is given by matrix multiplication

$$g(A,\mathbf{a})\,g(A',\mathbf{a}')=g(AA',\mathbf{a}A'+\mathbf{a}'). \tag{1.8}$$

$E(3)$ acts as a transformation group in three-space R^3. The group element $g(A,\mathbf{a})$ maps the point $\mathbf{x}\in R^3$ to the point

$$\mathbf{x}g=\mathbf{x}A+\mathbf{a}\in R^3. \tag{1.9}$$

ISBN-0-201-13503-5

It follows easily from this definition that $\mathbf{x}(gg')=(\mathbf{x}g)g'$ for all $\mathbf{x}\in R^3$, $g,g'\in E(3)$, and that $\mathbf{x}g(E_3,\mathbf{0})=\mathbf{x}$ where $g(E_3,\mathbf{0})$ is the identity element of $E(3)$. Geometrically, g corresponds to a rotation A about the origin $(0,0,0)\in R^3$ followed by a translation $\mathbf{a}$ [85].

A basis for the Lie algebra of the matrix group $E(3)$ is provided by the matrices

$$\mathcal{J}_l=\left[\begin{array}{ccc|c} & & & 0\\ & \mathcal{J}_l' & & 0\\ & & & 0\\ 0 & 0 & 0 & 0\end{array}\right],\quad l=1,2,3;\qquad \mathcal{P}_1=\left[\begin{array}{ccc|c} & & & 0\\ & 0 & & 0\\ & & & 0\\ 1 & 0 & 0 & 0\end{array}\right],$$

$$\mathcal{P}_2=\left[\begin{array}{ccc|c} & & & 0\\ & 0 & & 0\\ & & & 0\\ 0 & 1 & 0 & 0\end{array}\right],\qquad \mathcal{P}_3=\left[\begin{array}{ccc|c} & & & 0\\ & 0 & & 0\\ & & & 0\\ 0 & 0 & 1 & 0\end{array}\right],\tag{1.10}$$

with commutation relations identical to (1.3). This shows that the Lie algebra $\mathcal{E}(3)$ with basis (1.2) is isomorphic to the Lie algebra of $E(3)$. The explicit relation between the Lie algebra generators (1.10) and the group elements (1.7) is

$$\begin{aligned} g(\varphi,\theta,\psi,\mathbf{a})&\equiv g(A(\varphi,\theta,\psi),\mathbf{a})\\ &=\exp(\varphi\mathcal{J}_3)\exp(\theta\mathcal{J}_1)\exp(\psi\mathcal{J}_3)\exp(a_1\mathcal{P}_1+a_2\mathcal{P}_2+a_3\mathcal{P}_3). \end{aligned}\tag{1.11}$$

Using standard Lie theory, we can extend the action of $\mathcal{E}(3)$ by Lie derivatives (1.2) on the space $\mathcal{F}$ of analytic functions defined on some open connected set $\mathcal{D}\subseteq R^3$ to a local representation $\mathbf{T}$ of $E(3)$ on $\mathcal{F}$. We find

$$\begin{aligned} \mathbf{T}(g)\Phi(\mathbf{x})=\{&\exp(\varphi J_3)\exp(\theta J_1)\exp(\psi J_3)\\ &\times\exp(a_1P_1+a_2P_2+a_3P_3)\}\Phi(\mathbf{x})=\Phi(\mathbf{x}g) \end{aligned}\tag{1.12}$$

where $\mathbf{x}g$ is given by (1.9). Thus the action (1.9) of $E(3)$, as a transformation group is exactly that induced by the Lie derivatives (1.2). As usual,

$$\mathbf{T}(gg')=\mathbf{T}(g)\mathbf{T}(g'),\qquad g,g'\in E(3),\tag{1.13}$$

and the operators $\mathbf{T}(g)$ map solutions of the Helmholtz equation into solutions.

Computing the space $\mathcal{S}$ of second-order symmetries of (1.1), we find that this equation is class I. Indeed, factoring out the space $\mathfrak{q}$ of trivial symmetries RQ, $R\in\mathcal{F}$, $Q=P_1^2+P_2^2+P_3^2+\omega^2$ (recall that RQ is the zero operator on the solution space of (1.1)), we find that the factor space $\mathcal{S}/\mathfrak{q}$ is 41 dimensional, with a basis consisting of the identity operator E, the 6

ISBN-0-201-13503-5

first-order operators J_l, P_l, and 34 purely second-order symmetrized operators. The space $\mathcal{S}(3)^2$ of second-order symmetrized operators is spanned by the elements $\{J_l,J_m\}$, $\{J_l,P_m\}$, $\{P_l,P_m\}\Rightarrow 2P_lP_m$, and these elements are subject only to the relations $\mathbf{J}\cdot\mathbf{P}=J_1P_1+J_2P_2+J_3P_3\equiv 0$ and $\mathbf{P}\cdot\mathbf{P}=P_1^2+P_2^2+P_3^2=-\omega^2$, the latter relation holding on the solution space of (1.1) (see [76]).

The group $E(3)$ acts on $\mathcal{S}(3)$ via the adjoint representation and decomposes $\mathcal{S}(3)$ into three orbit types with representatives

$$P_3, \quad J_3, \quad J_3+aP_3, \qquad a\neq 0. \tag{1.14}$$

Note that $\exp(aP_3)$ is a translation along the three-axis, $\exp(\varphi J_3)$ is a rotation about this axis, and $\exp(\varphi J_3+\varphi aP_3)=\exp(\varphi J_3)\exp(\varphi aP_3)$ is a rotation about the three-axis followed by a translation along the axis (a *screw-displacement*). Thus we have the Lie algebra version of the theorem that every Euclidean transformation is a translation, a rotation, or a screw-displacement (see [85]).

Since (1.1) is an equation in three variables, two separation constants are associated with each separable coordinate system. Thus we expect the separated solutions to be characterized as common eigenfunctions of a pair of commuting symmetry operators in the enveloping algebra of $\mathcal{S}(3)$. This turns out to be the case. Just as for the two-variable Helmholtz equation in Section 1.2, we find a number of rather trivial nonorthogonal coordinate systems which correspond to the diagonalization of first-order operators. In addition to these, there are eleven types of orthogonal separable coordinate systems, each of which corresponds to a pair of independent commuting operators S_1, S_2 in $\mathcal{S}(3)^2$. The associated separable solutions $\Psi=U(u)V(v)W(w)$ are characterized by the eigenvalue equations

$$(\Delta_3+\omega^2)\Psi=0, \qquad S_1\Psi=\omega_1^2\Psi, \qquad S_2\Psi=\omega_2^2\Psi \tag{1.15}$$

where ω_1^2, ω_2^2 are the separation constants [121, 76]. (It can be shown that there are no nontrivial R-separable solutions.)

Put another way, a separable coordinate system is associated with a two-dimensional subspace of commuting operators in $\mathcal{S}(3)^2$ and S_1, S_2 is a basis (nonunique) for this subspace. The group $E(3)$ acts on the set of all two-dimensional subspaces of commuting operators in $\mathcal{S}(3)^2$ via the adjoint representation and decomposes this set into orbits of equivalent subspaces. As usual, one regards separable coordinates associated with equivalent subspaces as equivalent, since one can obtain any such system from any other by a Euclidean transformation. As proved in [76], there are eleven types of distinct (nontrivial) orbits, and they match exactly the eleven types of orthogonal separable coordinates. Representative operators from each orbit and the associated coordinate systems are listed in Table 14.

ISBN-0-201-13503-5

Table 14 Operators and Separable Coordinates for $(\Delta_3+\omega^2)\Psi=0$ $((x_1,x_2,x_3)=(x,y,z))$

	Commuting operators S_1, S_2	Separable coordinates
1	P_2^2, P_3^2	Cartesian x, y, z
2	J_3^2, P_3^2	Cylindrical $x=r\cos\varphi$, $y=r\sin\varphi$, $z=z$
3	$\{J_3,P_2\}, P_3^2$	Parabolic cylindrical $x=(\xi^2-\eta^2)/2$, $y=\xi\eta$, $z=z$
4	$J_3^2+d^2P_1^2, P_3^2$, $d>0$	Elliptic cylindrical $x=d\cosh\alpha\cos\beta$, $y=d\sinh\alpha\sin\beta$, $z=z$
5	$\mathbf{J}\cdot\mathbf{J}, J_3^2$	Spherical $x=\rho\sin\theta\cos\varphi$, $y=\rho\sin\theta\sin\varphi$, $z=\rho\cos\theta$
6	$\mathbf{J}\cdot\mathbf{J}-a^2(P_1^2+P_2^2), J_3^2$, $a>0$	Prolate spheroidal $x=a\sinh\eta\sin\alpha\cos\varphi$ $y=a\sinh\eta\sin\alpha\sin\varphi$ $z=a\cosh\eta\cos\alpha$
7	$\mathbf{J}\cdot\mathbf{J}+a^2(P_1^2+P_2^2), J_3^2$, $a>0$	Oblate spheroidal $x=a\cosh\eta\sin\alpha\cos\varphi$ $y=a\cosh\eta\sin\alpha\sin\varphi$ $z=a\sinh\eta\cos\alpha$
8	$\{J_1,P_2\}-\{J_2,P_1\}, J_3^2$	Parabolic $x=\xi\eta\cos\varphi$, $y=\xi\eta\sin\varphi$, $z=(\xi^2-\eta^2)/2$
9	$J_3^2-c^2P_3^2+c(\{J_2,P_1\}+\{J_1,P_2\})$, $c(P_2^2-P_1^2)+\{J_2,P_1\}-\{J_1,P_2\}$	Paraboloidal $x=2c\cosh\alpha\cos\beta\sinh\gamma$ $y=2c\sinh\alpha\sin\beta\cosh\gamma$ $z=c(\cosh 2\alpha+\cos 2\beta-\cosh 2\gamma)/2$
10	$P_1^2+aP_2^2+(a+1)P_3^2+\mathbf{J}\cdot\mathbf{J}$, $J_2^2+a(J_1^2+P_3^2)$, $a>1$	Ellipsoidal $x=\left[\dfrac{(\mu-a)(\nu-a)(\rho-a)}{a(a-1)}\right]^{1/2}$ $y=\left[\dfrac{(\mu-1)(\nu-1)(\rho-1)}{1-a}\right]^{1/2}$ $z=\left[\dfrac{\mu\nu\rho}{a}\right]^{1/2}$
11	$\mathbf{J}\cdot\mathbf{J}, J_1^2+bJ_2^2$, $1>b>0$	Conical $x=r\left[\dfrac{(b\mu-1)(b\nu-1)}{1-b}\right]^{1/2}$ $y=r\left[\dfrac{b(\mu-1)(\nu-1)}{b-1}\right]^{1/2}$, $z=r[b\mu\nu]^{1/2}$

ISBN-0-201-13503-5

We will briefly study each of these systems to determine the form of the separated solutions and the significance of the eigenvalues of the commuting symmetry operators. We begin by considering solutions Ψ of the Helmholtz equation that are eigenfunctions of the operator P_3:

$$P_3\Psi=i\lambda\Psi, \qquad \Psi(x,y,z)=c^{i\lambda z}\Phi(x,y).$$

In this case we can split off the variable z, and equation (1.1) reduces to

$$\left(\Delta_2+\left[\omega^2-\lambda^2\right]\right)\Phi(x,y)=0, \tag{1.16}$$

The Helmholtz equation in two variables. It follows from the results of Section 1.2 (see Table 1) that this reduced equation permits separation of variables in exactly four orthogonal coordinate systems. The corresponding systems for the full equation (1.1) are 1–4 in Table 14.

Next we consider solutions Ψ of (1.1) that are eigenfunctions of J_3:

$$iJ_3\Psi=m\Psi, \qquad \Psi(x,y,z)=e^{im\varphi}\Phi(r,z).$$

Here r,φ,z are cylindrical coordinates 2 and $J_3=-\partial_\varphi$. We now split off the variable φ, and equation (1.1) reduces to

$$(\partial_{rr}+r^{-1}\partial_r-m^2/r^2+\partial_{zz}+\omega^2)\Phi=0. \tag{1.17}$$

This equation is class II, though it arises from a class I equation via partial separation of variables. The reduced equation separates in five coordinate systems, corresponding to systems 2, 5–8.

For spherical coordinates 5 the separated equations in ρ,θ are

$$P''+\frac{2}{\rho}P'+\left(\omega^2-\frac{l(l+1)}{\rho^2}\right)P=0, \tag{1.18a}$$

$$\Theta''+\cot\theta\,\Theta'+\left(l(l+1)-\frac{m^2}{\sin^2\theta}\right)\Theta=0, \tag{1.18b}$$

$$\mathbf{J}\cdot\mathbf{J}\Psi=-l(l+1)\Psi.$$

The separated solutions take the form

$$P(\rho)=\rho^{-1/2}J_{\pm(l+\frac{1}{2})}(\omega\rho), \qquad \Theta(\theta)=P_l^{\pm m}(\cos\theta) \tag{1.19}$$

where $J_\nu(z)$ is a Bessel function and $P_l^m(\cos\theta)$ is a Legendre function (see

ISBN-0-201-13503-5

(B.6iv)). The coordinates ρ,θ,φ vary in the ranges

$$0\leqslant\rho,\qquad 0\leqslant\theta\leqslant\pi,\qquad 0\leqslant\varphi<2\pi$$

to cover the full space R^3.

For prolate spheroidal (or ellipsoidal) coordinates 6 (Table 14) the separated equations in η,α are

$$\begin{gathered}H''+\coth(\eta)H'+(-\lambda+a^2\omega^2\sinh^2\eta-m^2/\sinh^2\eta)H=0,\\ A''+\cot(\alpha)A'+(\lambda+a^2\omega^2\sin^2\alpha-m^2/\sin^2\alpha)A=0,\\ (\mathbf{J}\cdot\mathbf{J}-a^2P_1^2-a^2P_2^2)\Psi=-\lambda\Psi.\end{gathered}\tag{1.20}$$

Equations (1.20) are two forms of the *spheroidal wave equation* [7, 79]. The corresponding solutions Ψ of (1.1) that are bounded and single valued in R^3 are of the form

$$H(\eta)A(\alpha)e^{im\rho}=Ps_n^{|m|}(\cosh\eta,a^2\omega^2)Ps_n^{|m|}(\cos\alpha,a^2\omega^2)e^{im\varphi},\tag{1.21}$$

$$m\text{ integer},\ n=0,1,2,\ldots,\ -n\leqslant m\leqslant n,$$

where $Ps_n^m(z,\gamma)$ is a spheroidal wave function. The discrete eigenvalues $\lambda_n^{|m|}(a^2\omega^2)$ are analytic functions of $a^2\omega^2$. For $a=0$ the spheroidal wave equation reduces to the equation for Legendre functions (1.18b) and $Ps_n^{|m|}(\cos\alpha,0)=P_n^{|m|}(\cos\alpha)$. Furthermore, $\lambda_n^{|m|}(0)=n(n+1)$. The coordinates vary in the range $0\leqslant\alpha<2\pi$, $\eta\geqslant 0$, $0\leqslant\varphi<2\pi$.

For oblate spheroidal (or ellipsoidal) coordinates 7 the separated equations in η,α are

$$\begin{gathered}H''+\tanh(\eta)H'+(-\lambda+a^2\omega^2\cosh^2\eta+m^2/\cosh^2\eta)H=0,\\ A''+\cot(\alpha)A'+(\lambda-a^2\omega^2\sin^2\alpha-m^2/\sin^2\alpha)A=0,\\ (\mathbf{J}\cdot\mathbf{J}+a^2P_1^2+a^2P_2^2)\Psi=-\lambda\Psi.\end{gathered}\tag{1.22}$$

Again these equations are forms of the spheroidal wave equation. The corresponding solutions Ψ of (1.1) that are bounded and single valued in R^3 take the form

$$Ps_n^{|m|}(-i\sinh\eta,\ a^2\omega^2)Ps_n^{|m|}(\cos\alpha,\ -a^2\omega^2)e^{im\varphi},\tag{1.23}$$

$$m\text{ integer},\ n=0,1,2,\ldots,\ -n\leqslant m\leqslant n,$$

with eigenvalues $\lambda_n^{|m|}(-a^2\omega^2)$.

ISBN-0-201-13503-5

For parabolic coordinates 8 the separated equations in ξ,η are

$$\begin{gathered}\Xi''+\xi^{-1}\Xi'+(\omega^2\xi^2-m^2/\xi^2-\lambda)\Xi=0,\\ H''+\eta^{-1}H'+(\omega^2\eta^2-m^2/\eta^2+\lambda)H=0,\\ (\{J_1,P_2\}-\{J_2,P_1\})\Psi=\lambda\Psi,\end{gathered}\tag{1.24}$$

and the separated solutions take the form

$$\begin{aligned}\Xi(\xi)&=\xi^m\exp(\pm i\omega\xi^2/2)\,{}_1F_1\left(\begin{matrix}i\lambda/4\omega+(m+1)/2\\ m+1\end{matrix}\,\middle|\,\mp i\omega\xi^2\right),\\ H(\eta)&=\eta^m\exp(\pm i\omega\eta^2/2)\,{}_1F_1\left(\begin{matrix}-i\lambda/4\omega+(m+1)/2\\ m+1\end{matrix}\,\middle|\,\mp i\omega\eta^2\right).\end{aligned}\tag{1.25}$$

The foregoing eight systems are the only ones whose separated solutions are eigenfunctions of a second-order operator that is the square of a first-order symmetry operator. The remaining three systems are somewhat less tractable.

For paraboloidal coordinates 9 the separated equations in α, β, γ are

$$\begin{gathered}A''+\left(-q-\lambda c\cosh 2\alpha+\frac{\omega^2c^2}{2}\cosh 4\alpha\right)A=0,\\ B''+\left(q+\lambda c\cos 2\beta-\frac{\omega^2c^2}{2}\cos 4\beta\right)B=0,\\ \Gamma''+\left(-q+\lambda c\cosh 2\gamma+\frac{\omega^2c^2}{2}\cosh 4\gamma\right)\Gamma=0,\qquad q=\mu-c^2\omega^2/2,\end{gathered}\tag{1.26}$$

where

$$\begin{gathered}(J_3^2-c^2P_3^2+c\{J_2,P_1\}+c\{J_1,P_2\})\Psi=-\mu\Psi,\\ (cP_2^2-cP_1^2+\{J_2,P_1\}-\{J_1,P_2\})\Psi=\lambda\Psi.\end{gathered}\tag{1.27}$$

Each of the equations (1.26) can be transformed to the Whittaker–Hill equation (6.28), Section 2.6 [127]. Single-valued solutions of (1.1) take the form

$$\begin{aligned}\Psi(\alpha,\beta,\gamma)=&\,gc_n(i\alpha;\,2c\omega,\lambda/2\omega)\,gc_n(\beta;\,2c\omega,\lambda/2\omega)\\ &\times gc_n(i\gamma+\pi/2;\,2c\omega,\lambda/2\omega),\qquad n=0,1,2,\ldots,\mu=\mu_n,\end{aligned}\tag{1.28}$$

or the same form with gc_n replaced by gs_n.

ISBN-0-201-13503-5

For ellipsoidal coordinates μ,ν,ρ where $0<\rho<1<\nu<a<\mu<\infty$ for single-valued coordinates, the separation equations all take the form

$$\left(4(h(\xi))^{1/2}\frac{d}{d\xi}(h(\xi))^{1/2}\frac{d}{d\xi}+\lambda_1\xi+\lambda_2+\omega^2\xi^2\right)E(\xi)=0,$$
$$h(\xi)=(\xi-a)(\xi-1)\xi,\qquad \xi=\mu,\nu,\rho, \tag{1.29}$$

with

$$(\mathbf{J}\cdot\mathbf{J}+P_1^2+aP_2^2+(a+1)P_3^2)\Psi=\lambda_1\Psi,$$
$$(J_2^2+aJ_1^2+aP_3^2)\Psi=\lambda_2\Psi. \tag{1.30}$$

For computational purposes it is more convenient to introduce the equivalent separable coordinates α,β,γ defined by

$$\rho=\mathrm{sn}^2(\alpha,k),\qquad \nu=\mathrm{sn}^2(\beta,k),\qquad \mu=\mathrm{sn}^2(\gamma,k),\qquad k=a^{-1/2}, \tag{1.31}$$

where $\mathrm{sn}(z,k)$ is a Jacobi elliptic function (see Appendix C). The relationship between α,β,γ and x,y,z is

$$x=ik^{-1}k'^{-1}\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma,\qquad y=-kk'^{-1}\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma,$$
$$z=k\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\gamma \tag{1.32}$$

where $\mathrm{cn}\,\alpha$, $\mathrm{dn}\,\alpha$ are elliptic functions and $k'=(1-k^2).^{1/2}$ To obtain real values for x,y,z we choose α real, β complex such that $\mathrm{Re}\,\beta=K$, and γ complex such that $\mathrm{Im}\,\gamma=K'$ where $K(k)$ is defined by (C.3) and $K'=K(k')$. To cover all real values of x,y,z once, it is sufficient to let α vary in the interval $[-K,K]$, β vary in $[K-iK',K+iK']$ (parallel to the imaginary axis), and γ vary in $[-K+iK',K+iK']$ (parallel to the real axis). In these new variables the separation equations take the form of the *ellipsoidal wave equation*

$$\left\{\frac{d^2}{d\xi^2}+k^2\lambda_2+k^2\lambda_1\,\mathrm{sn}^2\xi+k^2\omega^2\,\mathrm{sn}^4\xi\right\}E(\xi)=0,\qquad \xi=\alpha,\beta,\gamma. \tag{1.33}$$

From the periodicity properties of the elliptic functions it follows that if ξ is replaced by $\xi+4Kn+4iK'm$ in (1.32), where n,m are integers and ξ is any one of α,β,γ, then x,y,z remain unchanged. Thus only those solutions $E(\xi)$ of (1.33) that are doubly periodic and single valued in ξ with real period $4K$ and imaginary period $4iK'$ are single-valued functions of x,y,z. The doubly periodic single-valued solutions of (1.33) are called *ellipsoidal wave functions* and are denoted by the symbol $\mathrm{el}(\xi)$ in Arscott's notation [7, Chapter X]. There are eight types of such functions, each expressible in the

ISBN-0-201-13503-5

form

$$\operatorname{sn}^s z \operatorname{cn}^c z \operatorname{dn}^d z\, F(\operatorname{sn}^2 z), \qquad s,c,d=0,1,$$

where F is a convergent power series in its argument. The eigenvalues are countable and discrete.

For conical coordinates r, μ, ν (System 11, Table 14) it is convenient to set $\mu=\operatorname{sn}^2(\alpha,k)$, $\nu=\operatorname{sn}^2(\beta,k)$ where $k=b^{1/2}>0$. Then

$$\begin{aligned} &x=rk'^{-1}\operatorname{dn}(\alpha,k)\operatorname{dn}(\beta,k), \qquad y=irkk'^{-1}\operatorname{cn}(\alpha,k)\operatorname{cn}(\beta,k)\\ &z=rk\operatorname{sn}(\alpha,k)\operatorname{sn}(\beta,k), \end{aligned} \tag{1.34}$$

and the variables have the range $0\leqslant r$, $-2K<\alpha<2K$, $K\leqslant\beta<K+2iK'$ (see [7, p. 24]). The separation equations are

$$\begin{aligned} R''+2r^{-1}R'+\left(\omega-l(l+1)r^{-2}\right)R&=0,\\ A''+\left(\lambda-l(l+1)k^2\operatorname{sn}^2\alpha\right)A&=0,\\ B''+\left(\lambda-l(l+1)k^2\operatorname{sn}^2\beta\right)B&=0,\\ \mathbf{J}\cdot\mathbf{J}\Psi=-l(l+1)\Psi, \qquad (J_1^2+bJ_2^2)\Psi&=\lambda\Psi. \end{aligned} \tag{1.35}$$

The first equation has solutions of the form $R(r)=r^{-1/2}J_{\pm(l+\frac{1}{2})}(\omega r)$, in agreement with (1.18a). The latter two equations are examples of the *Lamé equation*. If α or β is increased by integral multiples of $4K$ or $4iK'$, it follows from (1.34) that x, y, and z are unchanged. Thus only those solutions $A(\alpha), B(\beta)$ of (1.35) that are doubly periodic and single valued in α,β, respectively, lead to single-valued functions of x,y,z. It is known (see [7]) that doubly periodic solutions of Lamé's equation exist only in the cases where $l=0,1,2,\ldots$. Furthermore, for positive integer l there exist exactly $2l+1$ such solutions corresponding to $2l+1$ distinct eigenvalues λ. The solutions, exactly one for each pair of eigenvalues λ,l, can be expressed as finite series called *Lamé polynomials*. There are eight types of Lamé polynomials, each expressible in the form

$$\operatorname{sn}^s\alpha\operatorname{cn}^c\alpha\operatorname{dn}^d\alpha Fp(\operatorname{sn}^2\alpha), \qquad s,c,d=0,1, \qquad s+c+d+2p=l,$$

where $Fp(z)$ is a polynomial of order p in z. In Section 3.3 we shall study these functions in more detail.

3.2 A Hilbert Space Model: The Sphere S_2

In analogy with the methods of Chapter 1 we can introduce a Hilbert space structure on the solution space of (1.1) in such a way that the separated solutions can be interpreted as eigenfunctions of self-adjoint

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operators in the enveloping algebra of $\mathcal{E}(3)$. By an obvious extension of arguments in Section 1.3 we can show that $\Psi(\mathbf{x})$ satisfies $(\Delta_3+\omega^2)\Psi(\mathbf{x})=0$ if it can be represented in the form

$$\Psi(\mathbf{x})=\iint_{S_2} e^{i\omega\mathbf{x}\cdot\hat{\mathbf{k}}}h(\hat{\mathbf{k}})\,d\Omega(\hat{\mathbf{k}})=I(h), \tag{2.1}$$

$$\mathbf{x}=(x_1,x_2,x_3),\ \hat{\mathbf{k}}=(k_1,k_2,k_3).$$

Here $\hat{\mathbf{k}}$ is a unit vector $(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}=1)$ that runs over the unit sphere S_2: $k_1^2+k_2^2+k_3^2=1$, $d\Omega$ is the usual solid-angle measure on the sphere, and h is an arbitrary complex-valued measurable function on S_2 (with respect to $d\Omega$) such that

$$\iint_{S_2}|h(\hat{\mathbf{k}})|^2\,d\Omega(\hat{\mathbf{k}})<\infty.$$

The set $L_2(S_2)$ of such functions h is a Hilbert space with inner product

$$\langle h_1,h_2\rangle=\iint_{S_2} h_1(\hat{\mathbf{k}})\bar{h}_2(\hat{\mathbf{k}})\,d\Omega(\hat{\mathbf{k}}), \tag{2.2}$$

or, in terms of spherical coordinates on S_2

$$\begin{gathered}\hat{\mathbf{k}}=(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta), \qquad 0\leqslant\theta\leqslant\pi,\ -\pi\leqslant\varphi<\pi,\\ d\Omega(\hat{\mathbf{k}})=\sin\theta\,d\theta\,d\varphi\end{gathered} \tag{2.3}$$

and

$$\langle h_1,h_2\rangle=\int_{-\pi}^{\pi}d\varphi\int_0^{\pi}h_1(\theta,\varphi)\bar{h}_2(\theta,\varphi)\sin\theta\,d\theta.$$

The elements $g(A,\mathbf{a})$ of $E(3)$ act on the solutions of the Helmholtz equation via the operators $\mathbf{T}(g)$, (1.9), (1.12). Using (2.1) we find

$$\mathbf{T}(g)\Psi(\mathbf{x})=I(\mathbf{T}(g)h) \tag{2.4}$$

whenever $\Psi=I(h)$, where the operators $\mathbf{T}(g)$ on $L_2(S_2)$ are defined by

$$\mathbf{T}(g)h(\hat{\mathbf{k}})=\exp(i\omega\mathbf{a}\cdot\hat{\mathbf{k}}A)h(\hat{\mathbf{k}}A), \tag{2.5}$$

$$g=(A,\mathbf{a}),\qquad A\in SO(3),\qquad \mathbf{a}\in R^3.$$

Thus the $\mathbf{T}(g)$ acting on Ψ induce operators (which we also call $\mathbf{T}(g)$) acting on h. It is easy to verify directly that the operators (2.5) satisfy the

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group homomorphism property $\mathbf{T}(g_1g_2)=\mathbf{T}(g_1)\mathbf{T}(g_2)$. Moreover, these operators are unitary on $L_2(S_2)$:

$$\langle \mathbf{T}(g)h_1, \mathbf{T}(g)h_2\rangle = \langle h_1, h_2\rangle, \qquad h_j \in L_2(S_2).$$

This result and (2.5) itself depend on the invariance of the measure under rotations: $d\Omega(\hat{\mathbf{k}}A)=d\Omega(\hat{\mathbf{k}})$.

A similar computation shows that the Lie algebra generators on $L_2(S_2)$ induced by the generators (1.2) on the solution space are

$$\begin{aligned}
&P_1 = i\omega k_1 = i\omega\sin\theta\cos\varphi, \qquad P_2 = i\omega k_2 = i\omega \sin\theta\sin\varphi, \qquad P_3 = i\omega k_3 = i\omega\cos\theta,\\
&J_1 = k_3\partial_{k_2} - k_2\partial_{k_3} = \sin\varphi\,\partial_\theta + \cos\varphi\cot\theta\,\partial_\varphi,\\
&J_2 = k_1\partial_{k_3} - k_3\partial_{k_1} = -\cos\varphi\,\partial_\theta + \sin\varphi\cot\theta\,\partial_\varphi,\\
&J_3 = k_2\partial_{k_1} - k_1\partial_{k_2} = -\partial_\varphi.
\end{aligned} \tag{2.6}$$

In analogy with (1.12) these operators are related to the group operators (2.5) by

$$\mathbf{T}(g) = \exp(\varphi' J_3)\exp(\theta' J_1)\exp(\psi' J_3)\exp(a_1P_1 + a_2P_2 + a_3P_3)$$

where φ', θ', ψ' are the Euler angles for A. Furthermore, the operators (2.6) are skew-Hermitian on the dense subspace $\mathfrak{D}$ of $L_2(S_2)$ consisting of infinitely differentiable functions on S_2.

We have shown that the $\mathbf{T}(g)$ define a unitary (irreducible) representation of $E(3)$ on $L_2(S_2)$. The elements of $\mathcal{E}(3)^2$ are easily seen to be symmetric on $\mathfrak{D}$ and we shall show explicitly that their domains can be extended to define self-adjoint operators in dense subspaces of $L_2(S_2)$. Corresponding to each pair of commuting operators listed in Table 14 we shall find a pair of commuting self-adjoint operators S, S' on $L_2(S_2)$ and determine the spectral resolution of this pair. These results will then be used to obtain information about the space $\mathcal{H}$ consisting of solutions Ψ of the Helmholtz equation such that $\Psi = I(h)$ for some $h \in L_2(S_2)$, (2.1). Here $\mathcal{H}$ is a Hilbert space with inner product

$$(\Psi_1, \Psi_2) \equiv \langle h_1, h_2\rangle, \qquad \Psi_j = I(h_j). \tag{2.7}$$

(It is not hard to show that no nonzero $h \in L_2(S_2)$ can be mapped by I to the zero solution of the Helmholtz equation). It follows that I is a unitary transformation from $L_2(S_2)$ to $\mathcal{H}$. Also, the operators $\mathbf{T}(g)$ on $\mathcal{H}$ defined by (1.9), (1.12) are now seen to be unitary.

We can also interpret each function $\Psi(\mathbf{x})$ in $\mathcal{H}$ as an inner product

$$\Psi(\mathbf{x}) = I(h) = \langle h, H(\mathbf{x}, \cdot)\rangle, \qquad H(\mathbf{x}, \hat{\mathbf{k}}) = e^{-i\omega\mathbf{x}\cdot\hat{\mathbf{k}}} \in L_2(S_2). \tag{2.8}$$

Just as we saw in Section 1.3, the existence of the unitary mapping I allows us to transform problems involving $\mathcal{H}$ to problems involving $L_2(S_2)$. In particular, if S, S' are a pair of commuting operators from Table 14, we can interpret them as a pair of commuting self-adjoint operators on $L_2(S_2)$ and compute a basis of eigenfunctions for $L_2(S_2)$:

$$Sf_{\lambda\mu}=\lambda f_{\lambda\mu}, \qquad S'f_{\lambda\mu}=\mu f_{\lambda\mu}, \qquad \langle f_{\lambda\mu},f_{\lambda'\mu'}\rangle=\delta(\lambda-\lambda')\delta(\mu-\mu'). \quad (2.9)$$

Then the functions $\Psi_{\lambda\mu}(\mathbf{x})=I(f_{\lambda\mu})$ will form a corresponding basis in $\mathcal{H}$ for the operators S, S' constructed from the generators (1.2):

$$S\Psi_{\lambda\mu}=\lambda\Psi_{\lambda\mu}, \qquad S'\Psi_{\lambda\mu}=\mu\Psi_{\lambda\mu}. \quad (2.10)$$

These last expressions enable us to evaluate the integral for $\Psi_{\lambda\mu}$, for they guarantee that $\Psi_{\lambda\mu}$ is a solution of the Helmholtz equation that is separable in the coordinates associated with S, S'. Furthermore, if Ψ is any solution of (1.1) such that $\Psi=I(h)$ for some $h\in L_2(S_2)$, we have the expansion

$$\mathbf{T}(g)\Psi(\mathbf{x})=\sum_{\lambda,\mu}\langle\mathbf{T}(g)h,f_{\lambda\mu}\rangle\Psi_{\lambda\mu}(\mathbf{x}), \quad (2.11)$$

which converges both pointwise and in the Hilbert space sense.

We now proceed to analyze our model $L_2(S_2)$. Harmonic analysis involving functions on the sphere is itself a topic of considerable interest. Typically, such studies use only spherical coordinates 5 (Table 14) and lead to theorems concerning expansions in spherical harmonics. However, we shall analyze all eleven coordinate systems on S_2 that follow from Table 14. In some cases we shall employ simpler models of our representations than $L_2(S_2)$ to carry forward the analysis.

Since the spherical coordinate system 5 is treated in detail in so many textbooks (e.g., [40, 45, 85, 128]), we shall here list only the most important facts concerning this system, omitting all proofs. The unitary irreducible representations of $SO(3)$ are all finite dimensional. They are denoted by D_l, $l=0,1,2,\ldots,$where $\dim D_l=2l+1$. If $\{J_1,J_2,J_3\}$ are the operators on the representation space V_l of D_l which correspond to the Lie algebra generators (1.5), then there is an ON basis $\{f_m^{(l)}: m=l,l-1,\ldots,-l\}$ for V_l such that

$$J^0f_m^{(l)}=mf_m^{(l)}, \qquad J^{\pm}f_m^{(l)}=\left[(l\pm m+1)(l\mp m)\right]^{1/2}f_{m\pm1}^{(l)} \quad (2.12)$$

where $J^{\pm}=\mp J_2+iJ_1$, $J^0=iJ_3$. Here, $J^+f_l^{(l)}\equiv J^-f_{-l}^{(l)}\equiv 0$. If the group is parametrized in terms of Euler angles (1.6), the matrix elements of the

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operators $\mathbf{D}(A)=\exp(\varphi J_3)\exp(\theta J_1)\exp(\psi J_3)$ with respect to the ON basis $\{f_m^{(l)}\}$,

$$\mathbf{D}(A)f_m^{(l)}=\sum_{n=-l}^{l} D_{nm}^{l}(A)f_n^{(l)},$$

are given by

$$D_{nm}^{l}(A)=i^{n-m}\left[\frac{(l+m)!(l-n)!}{(l+n)!(l-m)!}\right]^{1/2}\exp[i(n\varphi+m\psi)]P_l^{-n,m}(\cos\theta) \tag{2.13}$$

where

$$P_l^{-n,m}(\cos\theta)=\frac{(\sin\theta)^{m-n}(1+\cos\theta)^{l+n-m}2^{-l}}{\Gamma(m-n+1)} \times {}_2F_1\left(\begin{matrix}-l-n,m-l\\ m-n+1\end{matrix}\,\middle|\,\frac{\cos\theta-1}{\cos\theta+1}\right) \tag{2.14}$$

is a generalized spherical function. (The matrix elements (2.13) are known as the *Wigner D functions* [137].) The D_{nm}^{l} satisfy the usual group homomorphism and unitary properties

$$\begin{aligned} D_{nm}^{l}(AA')&=\sum_{j=-l}^{l} D_{nj}^{l}(A)D_{jm}^{l}(A'), \qquad A,A'\in SO(3),\\ D_{nm}^{l}(A^{-1})&=\bar{D}_{mn}^{l}(A). \end{aligned} \tag{2.15}$$

The special matrix element $D_{0m}^{l}(A)$ is proportional to a spherical harmonic:

$$D_{0m}^{l}(\varphi,\theta,\psi)=i^{m}\left(\frac{4\pi}{2l+1}\right)^{1/2}Y_l^m(\theta,\psi), \tag{2.16}$$

where

$$Y_l^m(\theta,\psi)=\left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{1/2}P_l^m(\cos\theta)e^{im\varphi} \tag{2.17}$$

and $P_l^m(\cos\theta)=P_l^{0,-m}(\cos\theta)$ is an associated Legendre function.

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It follows from (2.12) that on V_l

$$\mathbf{J}\cdot\mathbf{J}=J_1^2+J_2^2+J_3^2=-l(l+1)E \tag{2.18}$$

where E is the identity operator.

Now consider the irreducible representation T of $E(3)$ on $L_2(S_2)$ defined by expression (2.5). The restriction of T to the subgroup $SO(3)$ is no longer irreducible but breaks up into the direct sum

$$T|SO(3)\cong\sum_{l=0}^{\infty}\oplus D_l; \tag{2.19}$$

that is, $L_2(S_2)$ can be decomposed into a direct sum of mutually orthogonal subspaces V_l,

$$L_2(S_2)\cong\sum_{l=0}^{\infty}\oplus V_l,$$

where $\dim V_l=2l+1$ and the action of the operators $\mathbf{T}(A)$ on the invariant subspace V_l is unitary equivalent to D_l. The elements h of V_l are characterized as the solutions of the equation $\mathbf{J}\cdot\mathbf{J}h=-l(l+1)h$, or

$$\left(\partial_{\theta\theta}+\cot\theta\,\partial_\theta+\sin^{-2}\theta\,\partial_{\varphi\varphi}\right)h(\theta,\varphi)=-l(l+1)h(\theta,\varphi), \tag{2.20}$$

in terms of the coordinates (2.3). Here $\mathbf{J}\cdot\mathbf{J}$ is known as the *Laplace operator on the sphere* S_2. It follows from the foregoing results that the self-adjoint extension of this operator (which we also denote $\mathbf{J}\cdot\mathbf{J}$) has discrete spectrum $-l(l+1)$, $l=0,1,2,,\ldots$, each eigenvalue occurring with multiplicity $2l+1$.

There exists a basis for V_l, consisting of eigenfunctions $f_m^{(l)}(\theta,\varphi)$ of the symmetry operator J^0, which satisfy the relations (2.12) where

$$J^{\pm}=e^{\pm i\varphi}(\pm\partial_\theta+i\cot\theta\,\partial_\varphi),\qquad J^0=-i\,\partial_\varphi. \tag{2.21}$$

Indeed, from the recurrence relations (2.12) and the differential equation (2.20) we find

$$f_m^{(l)}(\theta,\varphi)=Y_l^m(\theta,\varphi),\qquad \langle Y_l^m,Y_{l'}^{m'}\rangle=\delta_{ll'}\delta_{mm'}. \tag{2.22}$$

Furthermore, it is straightforward to show that the action of the operators

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P_j on this basis is given by

$$P^0 f_m^{(l)} = -\omega\left[\frac{(l+m+1)(l-m+1)}{(2l+3)(2l+1)}\right]^{1/2} f_m^{(l+1)} - \omega\left[\frac{(l+m)(l-m)}{(2l+1)(2l-1)}\right]^{1/2} f_m^{(l-1)},$$

$$P^+ f_m^{(l)} = \omega\left[\frac{(l+m+1)(l+m+2)}{(2l+3)(2l+1)}\right]^{1/2} f_{m+1}^{(l+1)} - \omega\left[\frac{(l-m)(l-m-1)}{(2l+1)(2l-1)}\right]^{1/2} f_{m+1}^{(l-1)},$$

$$P^- f_m^{(l)} = -\omega\left[\frac{(l-m+2)(l-m+1)}{(2l+3)(2l+1)}\right]^{1/2} f_{m-1}^{(l+1)} + \omega\left[\frac{(l+m)(l+m-1)}{(2l+1)(2l-1)}\right]^{1/2} f_{m-1}^{(l-1)},$$

$$\tag{2.23}$$

where

$$P^0 = iP_3 = -\omega\cos\theta, \qquad P^\pm = \mp P_2 + iP_1 = -\omega e^{\pm i\varphi}\sin\theta \tag{2.24}$$

(see [82]).

The matrix elements of the translation operators $\mathbf{T}(E,\mathbf{a}) = \exp(a_1P_1 + a_2P_2 + a_3P_3)$ are given by

$$\begin{aligned} T_{lm,l'm'}(\mathbf{a}) &= \langle \mathbf{T}(E,\mathbf{a}) f_{m'}^{(l')}, f_m^{(l)} \rangle \\ &= \int_{S_2} e^{i\omega\mathbf{a}\cdot\mathbf{k}} Y_{l'}^{m'}(\hat{\mathbf{k}})\, \overline{Y}_l^m(\hat{\mathbf{k}})\, d\Omega(\hat{\mathbf{k}}), \end{aligned} \tag{2.25}$$

or more explicitly,

$$\begin{aligned} T_{lm,l'm'}(\mathbf{a}) = (4\pi)^{1/2} \sum_{s=0}^{\infty} \left[\frac{(2s+1)(2l+1)}{(2l'+1)}\right]^{1/2} i^s j_s(\omega a) \\ \times Y_s^{m'-m}(\alpha,\beta)\, C(s,0;\, l,0|l',0)\, C(s,m'-m;\, l,m|l',m') \end{aligned} \tag{2.26}$$

where

$$\mathbf{a} = (a\sin\alpha\cos\beta,\ a\sin\alpha\sin\beta,\ a\cos\alpha), \qquad a \geqslant 0,$$

and $C(\cdot)$ is a Clebsch–Gordan coefficient for $SO(3)$ [82, 124, 128]. (In (2.26) the sum is actually finite because the Clebsch–Gordan coefficients vanish except for finitely many values of s. The *spherical Bessel functions* $j_n(z)$ are defined by

$$j_n(z) = (\pi/2z)^{1/2} J_{n+1/2}(z), \qquad n = 0, 1, 2, \ldots. \tag{2.27}$$

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Applying the integral transformation I to our ON basis $\{f_m^{(l)}\}$ for $L_2(S_2)$, we obtain an ON basis $\{\Psi_m^{(l)}=I(f_m^{(l)})\}$ of solutions for the Helmholtz equation that satisfy the eigenvalue equations

$$\mathbf{J}\cdot\mathbf{J}\Psi_m^{(l)}=-l(l+1)\Psi_m^{(l)},\qquad J_3\Psi_m^{(l)}=-im\Psi_m^{(l)}.$$

The eigenfunctions separate in the spherical coordinate system 5 listed in Table 14 and are explicitly given by

$$\Psi_m^{(l)}(r,\theta,\varphi)=4\pi i^l j_l(\omega r)Y_l^m(\theta,\varphi),\qquad l=0,1,2,\ldots,m=l,l-1,\ldots,-l. \tag{2.28}$$

These functions are frequently called (*standing*) *spherical waves*. They necessarily satisfy the recurrence relations (2.12) and (2.23) where now the operators are given by (1.2). Furthermore, the matrix elements (2.13) and (2.26) can be used directly to expand the function $\mathbf{T}(g)\Psi_M^{(L)}$ in terms of the spherical basis. In particular, the special case in which $g=(E,\mathbf{a})$ leads to the addition theorem for spherical waves:

$$\Psi_M^{(L)}(R,\Theta,\Phi)=\sum_{l,m}T_{lm,LM}(\mathbf{a})\Psi_m^{(l)}(r,\theta,\varphi) \tag{2.29}$$

where R,Θ,Φ are spherical coordinates for the three-vector $\mathbf{R}=\mathbf{x}+\mathbf{a}$. Expression (2.29) was first derived in [39].

It is easy to show that the recurrence relations (2.12), (2.23) are also satisfied by the non-Hilbert space solutions

$$\Psi'^{(l)}_m(\rho,\theta,\varphi)=4\pi i^l j_{-l-1}(\omega\rho)Y_l^m(\theta,\varphi), \tag{2.30}$$

hence by any linear combination $\alpha\Psi_m^{(l)}+\beta\Psi'^{(l)}_m$ [124, p. 229]. As a consequence, the matrix elements (2.13), (2.26) are valid for all of these basis sets, and expansion formulas such as (2.29) hold for the set $\{\Psi'^{(l)}_m\}$ as well as for the Hilbert space basis $\{\Psi_m^{(l)}\}$.

Next we compute the spectral decompositions of the operators corresponding to systems 1–4 in Table 14, via our $L_2(S_2)$ model. These systems are characterized by the fact that P_3 is diagonal. From (2.6) it follows immediately that the bounded self-adjoint operator $iP_3=-\omega\cos\theta$ has continuous spectrum covering the interval $[-\omega,\omega]$ with multiplicity one. Fixing an eigenvalue of iP_3 corresponds to fixing the coordinate θ. The remaining coordinate φ can still vary and sweeps out a circle in S_2 as it goes from $-\pi$ to π. For each of the systems 1–4 the remaining second-order symmetry operator commutes with P_3; hence it leaves the functions

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on these circles invariant and reduces to one of the four cases studied in Section 1.3. The work of that section carries over immediately to yield the following results:

1. Cartesian System

The eigenvalue equations are

$$iP_3 f^{(1)}_{\alpha,\gamma} = -\omega\cos(\gamma) f^{(1)}_{\alpha,\gamma}, \qquad iP_2 f^{(1)}_{\alpha,\gamma} = -\omega\sin(\gamma)\sin(\alpha) f^{(1)}_{\alpha,\gamma}, \tag{2.31}$$

with basis eigenfunctions

$$f^{(1)}_{\alpha,\gamma}(\theta,\varphi) = \frac{\delta(\varphi-\alpha)\delta(\theta-\gamma)}{(\sin\gamma)^{1/2}}, \qquad -\pi \leqslant \alpha < \pi, \quad 0 \leqslant \gamma \leqslant \pi,$$
$$\langle f^{(1)}_{\alpha,\gamma}, f^{(1)}_{\alpha',\gamma'}\rangle = \delta(\alpha-\alpha')\delta(\gamma-\gamma'). \tag{2.32}$$

The corresponding solutions of the Helmholtz equation are the plane waves

$$\Psi^{(1)}_{\alpha,\gamma}(\mathbf{x}) = I\left(f^{(1)}_{\alpha,\gamma}\right) = (\sin\gamma)^{1/2}\exp\left[i\omega(x_1\sin\gamma\cos\alpha + x_2\sin\gamma\sin\alpha + x_3\cos\gamma)\right]. \tag{2.33}$$

2. Cylindrical System

The eigenvalue equations are

$$iP_3 f^{(2)}_{n,\gamma} = -\omega\cos(\gamma) f^{(2)}_{n,\gamma}, \qquad iJ_3 f^{(2)}_{n,\gamma} = n f^{(2)}_{n,\gamma}, \tag{2.34}$$

and the basis of eigenfunctions is

$$f^{(2)}_{n,\gamma}(\theta,\varphi) = \frac{e^{in\varphi}\delta(\gamma-\theta)}{(2\pi\sin\gamma)^{1/2}}, \qquad n = 0, \pm 1, \pm 2, \ldots, 0 \leqslant \gamma \leqslant \pi,$$
$$\langle f^{(2)}_{n,\gamma}, f^{(2)}_{n',\gamma'}\rangle = \delta_{nn'}\delta(\gamma-\gamma'). \tag{2.35}$$

Furthermore

$$\Psi^{(2)}_{n,\gamma}(\mathbf{x}) = I(f^{(2)}_{n,\gamma}) = i^n(2\pi\sin\gamma)^{1/2} J_n(\omega\sin(\gamma)r)\exp[i(n\varphi + \omega z\cos\gamma)], \tag{2.36}$$
$$x = r\cos\varphi, \; y = r\sin\varphi, \; z = z.$$

These are *cylindrical wave* solutions of the Helmholtz equation.

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3. Parabolic Cylindrical System

The eigenvalue equations are

$$iP_3 f^{(3)}_{\mu\pm,\gamma} = -\omega\cos(\gamma) f^{(3)}_{\mu\pm,\gamma}, \qquad \{J_3, P_2\} f^{(3)}_{\mu\pm,\gamma} = 2\mu\omega\sin(\gamma) f^{(3)}_{\mu\pm,\gamma}, \tag{2.37}$$

and the basis of eigenfunctions is

$$f^{(3)}_{\mu+,\gamma}(\theta,\varphi) = \begin{cases} (2\pi\sin\gamma)^{-1/2}(1+\cos\varphi)^{-i\mu/2-\frac{1}{4}}(1-\cos\varphi)^{i\mu/2-\frac{1}{4}}\delta(\theta-\gamma), \\ \qquad 0<\varphi<\pi, \\ 0, \qquad -\pi<\varphi<0, \end{cases}$$

$$\begin{aligned} &f^{(3)}_{\mu-,\gamma}(\theta,\varphi) = f^{(3)}_{\mu+,\gamma}(\theta,-\varphi), \qquad -\infty<\mu<\infty, 0\leqslant\gamma\leqslant\pi, \\ &\langle f^{(3)}_{\mu\pm,\gamma}, f^{(3)}_{\mu'\pm,\gamma'}\rangle = \delta(\mu-\mu')\delta(\gamma-\gamma'), \qquad \langle f^{(3)}_{\mu\pm,\gamma}, f^{(3)}_{\mu'\mp,\gamma'}\rangle = 0. \end{aligned} \tag{2.38}$$

The corresponding solutions of the Helmholtz equation are

$$\begin{aligned} \Psi^{(3)}_{\mu+,\gamma}(\mathbf{x}) = I(f^{(3)}_{\mu+,\gamma}) = \left(\frac{\sin\gamma}{2}\right)^{1/2} &\sec(i\mu\pi)[D_{i\mu-\frac{1}{2}}(\sigma\xi)D_{-i\mu-\frac{1}{2}}(\sigma\eta) \\ &+ D_{i\mu-\frac{1}{2}}(-\sigma\xi)D_{-i\mu-\frac{1}{2}}(-\sigma\eta)]e^{i\omega z\cos\gamma}, \end{aligned}$$

$$\Psi^{(3)}_{\mu-,\gamma}(\xi,\eta,z) = \Psi^{(3)}_{\mu+,\gamma}(\xi,-\eta,z), \qquad \sigma = e^{i\pi/4}(2\omega\sin\gamma)^{1/2}, \tag{2.39}$$

$$x = (\xi^2-\eta^2)/2, y = \xi\eta, z = z.$$

4. Elliptic Cylindrical System

The eigenvalue equations are

$$iP_3 f^{(4)}_{nt,\gamma} = -\omega\cos(\gamma) f^{(4)}_{nt,\gamma}, \qquad (J_3^2 + d^2P_1^2) f^{(4)}_{nt,\gamma} = \lambda_{nt} f^{(4)}_{nt,\gamma}, \qquad t = s,c, \tag{2.40}$$

and the basis of eigenfunctions is

$$\begin{aligned} &f^{(4)}_{nc,\gamma}(\theta,\varphi) = (\pi\sin\gamma)^{-1/2}\mathrm{ce}_n(\varphi,q)\delta(\theta-\gamma), \quad n = 0,1,2,\ldots, \\ &f^{(4)}_{ns,\gamma}(\theta,\varphi) = (\pi\sin\gamma)^{-1/2}\mathrm{se}_n(\varphi,q)\delta(\theta-\gamma), \quad n = 1,2,\ldots, \\ &q = \frac{d^2\omega^2}{4}\sin^2\gamma, \qquad 0\leqslant\gamma\leqslant\pi. \end{aligned} \tag{2.41}$$

The eigenvalues $\lambda_{n\pm}$ are discrete, of multiplicity one, and related to the eigenvalues a of the Mathieu equation (B.25) by $a = -\lambda - \frac{1}{2}d^2\omega^2\sin^2\gamma$. The $\{f^{(4)}_{nt,\gamma}\}$ form a basis for $L_2(S_2)$ satisfying

$$\langle f^{(4)}_{nt,\gamma}, f^{(4)}_{n't',\gamma'}\rangle = \delta_{nn'}\delta_{tt'}\delta(\gamma-\gamma'), \qquad t,t' = s,c. \tag{2.42}$$

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The corresponding solutions of the Helmholtz equation are

$$\Psi^{(4)}_{nc,\gamma}(\mathbf{x}) = C_n(\sin\gamma)^{1/2}\,\mathrm{Ce}_n(\alpha,q)\,\mathrm{ce}_n(\beta,q)\exp[i\omega z\cos\gamma], \qquad n=0,1,2,\ldots,$$

$$\Psi^{(4)}_{ns,\gamma}(\mathbf{x}) = S_n(\sin\gamma)^{1/2}\,\mathrm{Se}_n(\alpha,q)\,\mathrm{se}_n(\beta,q)\exp[i\omega z\cos\gamma], \qquad n=1,2,\ldots, \tag{2.43}$$

where Ce_n and Se_n are modified Mathieu functions ((3.40), Section 1.3) and C_n, S_n are constants to be determined from the integral equations $\Psi^{(4)}_{nt,\gamma} = I(f^{(4)}_{nt,\gamma})$. The elliptic cylindrical coordinates α,β,z are defined by

$$x = d\cosh\alpha\cos\beta, \qquad y = d\sinh\alpha\sin\beta, \qquad z = z.$$

The spectral decompositions for systems 6–10 were first computed in [22], though 11 was studied earlier in [106]. The results are as follows.

6. Prolate Spheroidal System

The eigenfunction equations are

$$\left(\mathbf{J}\cdot\mathbf{J} - a^2P_1^2 - a^2P_2^2\right)f^{(6)}_{n,m} = -\lambda_n^m f^{(6)}_{n,m}, \qquad iJ_3 f^{(6)}_{n,m} = m f^{(6)}_{n,m}, \tag{2.44}$$

and the ON basis of eigenfunctions is

$$f^{(6)}_{n,m}(\theta,\varphi) = \left[\frac{(n-|m|)!(2n+1)}{(n+|m|)!(4\pi)}\right]^{1/2} Ps_n^{|m|}(\cos\theta, a^2\omega^2)e^{im\varphi}. \tag{2.45}$$

(The first eigenvalue equation (2.44) takes the form of the second equation (1.20).) Here $n=0,1,2,\ldots$, $m=n,n-1,\ldots,-n$ and the discrete eigenvalues are denoted $\lambda_n^m(a^2\omega^2)$. We have $\langle f^{(6)}_{n,m}, f^{(6)}_{n',m'}\rangle = \delta_{nn'}\delta_{mm'}$ in the normalization adopted by Meixner and Schäfke [79]. The spheroidal wave functions are frequently defined by their expansions in terms of associated Legendre functions:

$$Ps_n^{|m|}(x,a^2\omega^2) = \sum_{2k\geqslant |m|-n} (-1)^k a^{|m|}_{n,2k}(a^2\omega^2)P^{|m|}_{n+2k}(x) \tag{2.46}$$

(see [7, p. 169]). Indeed, substituting (2.46) into the spheroidal wave equation, one can easily derive a recurrence formula for the coefficients $a^m_{n,2k}$.

The corresponding basis of solutions for the Helmholtz equation is

$$\Psi^{(6)}_{n,m}(\mathbf{x}) = I\left(f^{(6)}_{n,m}\right) = C_n^m(a^2\omega^2)Ps_n^{|m|}(\cosh\eta, a^2\omega^2)Ps_n^{|m|}(\cos\alpha, a^2\omega^2)e^{im\varphi} \tag{2.47}$$

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where $C_n^m(a^2\omega^2)$ is a constant to be determined from the integral equation. This result is easily obtained from the fact that $\Psi_{n,m}^{(6)}$ must be separable in the coordinates

$$x = a\sinh\eta\sin\alpha\cos\varphi, \qquad y = a\sinh\eta\sin\alpha\sin\varphi, \qquad z = a\cosh\eta\cos\alpha.$$

(See the corresponding argument for expression (3.38) in Section 1.3.)

7. Oblate Spheroidal System

The eigenvalue equations are

$$\left(\mathbf{J}\cdot\mathbf{J} + a^2P_1^2 + a^2P_2^2\right)f_{n,m}^{(7)} = -\lambda_n^m f_{n,m}^{(7)}, \qquad iJ_3 f_{n,m}^{(7)} = m f_{n,m}^{(7)}, \tag{2.48}$$

and the ON basis of eigenfunctions is

$$f_{n,m}^{(7)}(\theta,\varphi) = \left[\frac{(n-|m|)!(2n+1)}{(n+|m|)!4\pi}\right]^{1/2} Ps_n^{|m|}(\cos\theta, -a^2\omega^2)e^{im\varphi}, \tag{2.49}$$

$$n = 0,1,2,\ldots, m = n, n-1,\ldots,-n.$$

(Here the first eigenvalue equation (2.48) takes the form of the second equation (1.22).) The discrete eigenvalues are $\lambda_n^{|m|}(-a^2\omega^2)$.

The corresponding solutions of the Helmholtz equation are

$$\begin{aligned}\Psi_{n,m}^{(7)}(\mathbf{x}) &= I\left(f_{n,m}^{(7)}\right)\\ &= C_n^m(a^2\omega^2)Ps_n^{|m|}(-i\sinh\eta, a^2\omega^2)Ps_n^{|m|}(\cos\alpha, -a^2\omega^2)e^{im\varphi}\end{aligned} \tag{2.50}$$

where $C_n^m(a^2\omega^2)$ is a constant to be determined from the integral and

$$x = a\cosh\eta\sin\alpha\cos\varphi, \qquad y = a\cosh\eta\sin\alpha\sin\varphi, \qquad z = a\sinh\eta\cos\alpha.$$

8. Parabolic System

The eigenvalue equations are

$$\left(\{J_1,P_2\} - \{J_2,P_1\}\right)f_{\lambda,m}^{(8)} = 2\lambda\omega f_{\lambda,m}^{(8)}, \qquad iJ_3 f_{\lambda,m}^{(8)} = m f_{\lambda,m}^{(8)}. \tag{2.51}$$

Here $\{J_1,P_2\} - \{J_2,P_1\} = 2i\omega(\cos\theta + \sin\theta\,\partial_\theta)$ is first order and has a unique self-adjoint extension. The eigenfunctions are

$$f_{\lambda,m}^{(8)}(\theta,\varphi) = (2\pi)^{-1}\frac{\left[\tan(\theta/2)\right]^{-i\lambda}}{\sin\theta}e^{im\varphi},$$

$$m = 0, \pm 1, \pm 2, \ldots, \quad -\infty < \lambda < \infty, \tag{2.52}$$

$$\langle f_{\lambda,m}^{(8)}, f_{\lambda',m'}^{(8)}\rangle = \delta(\lambda-\lambda')\delta_{mm'}.$$

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The corresponding solutions of the Helmholtz equation are

$$\Psi^{(8)}_{\lambda,m}(\mathbf{x}) = I\left(f^{(8)}_{\lambda,m}\right)$$
$$= \frac{i^m\sqrt{2}}{\xi\eta\omega}\Gamma\left(\frac{1-m+i\lambda}{2}\right)\Gamma\left(\frac{1-m-i\lambda}{2}\right)\mathfrak{M}_{i\lambda/2,-m/2}$$
$$\times\left[\frac{\exp(-i\pi/2)\omega\xi^2}{\sqrt{2}}\right]\mathfrak{M}_{i\lambda/2,-m/2}\left[\frac{\exp(i\pi/2)\omega\eta^2}{\sqrt{2}}\right]\exp(im\varphi). \tag{2.53}$$

Here

$$\mathfrak{M}_{\alpha,\mu/2}(z) = \frac{z^{(1+\mu)/2}e^{-z/2}}{\Gamma(1+\mu)}\,{}_1F_1\left(\left.\begin{matrix}(1+\mu)/2-\alpha\\ 1+\mu\end{matrix}\right| z\right) \tag{2.54}$$

is a Whittaker function [26, p. 12], and

$$x = \xi\eta\cos\varphi, \qquad y = \xi\eta\sin\varphi, \qquad z = (\xi^2-\eta^2)/2.$$

9. Paraboloidal System

The eigenvalue equations are

$$\begin{aligned}(J_3^2 - c^2P_3^2 + c\{J_2,P_1\} + c\{J_1,P_2\})f^{(9)}_{nt\lambda} &= -\mu_{n\pm}f^{(9)}_{nt\lambda},\\ (cP_2^2 - cP_1^2 + \{J_2,P_1\} - \{J_1,P_2\})f^{(9)}_{nt\lambda} &= 2\omega\lambda f^{(9)}_{nt\lambda}\end{aligned} \tag{2.55}$$

and the basis of eigenfunctions is

$$f^{(9)}_{nt\lambda}(\theta,\varphi) = (2\pi)^{-1/2}\frac{[\tan(\theta/2)]^{i\lambda}}{\sin\theta}\exp\left(-\frac{ic\omega}{2}\cos\theta\cos 2\varphi\right)$$
$$\left\{\begin{matrix} gc_n(\varphi;2c\omega,\lambda)\\ gs_n(\varphi;2c\omega,\lambda)\end{matrix}\right., \qquad t=c,s,\ n=0,1,2,\ldots,\ -\infty<\lambda<\infty, \tag{2.56}$$

where gc_n and gs_n are the even and odd nonpolynomial solutions of the Whittaker–Hill equation. The normalization of these functions is that adopted by Urwin and Arscott [127]. We have

$$\langle f^{(9)}_{nt\lambda}, f^{(9)}_{n't'\lambda'}\rangle = \delta_{nn'}\delta_{tt'}\delta(\lambda-\lambda').$$

The corresponding solutions of the Helmholtz equation are

$$\begin{aligned}\Psi^{(9)}_{nt\lambda}(\mathbf{x}) = K^t_n(\omega c,\lambda)\,&gt_n(\beta;2c\omega,\lambda)\,gt_n(i\alpha;2c\omega,\lambda)\\ &\times gt_n(i\gamma+\pi/2;2c\omega,\lambda), \quad t=s,c,\end{aligned} \tag{2.57}$$

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where the constants K_n' are to be determined from the integral equation $\Psi_{n\lambda}^{(9)}=I(f_{n\lambda}^{(9)})$. Here,

$$x=2c\cosh\alpha\cos\beta\sinh\gamma, \qquad y=2c\sinh\alpha\sin\beta\cosh\gamma,$$
$$z=c(\cosh 2\alpha+\cos 2\beta-\cosh 2\gamma)/2.$$

10. Ellipsoidal System

We adopt elliptic coordinates on the unit sphere:

$$k_1=\left[\frac{(s-a)(t-a)}{a(a-1)}\right]^{1/2}, \quad k_2=\left[\frac{(s-1)(t-1)}{1-a}\right]^{1/2}, \quad k_3=\left[\frac{st}{a}\right]^{1/2}, \tag{2.58}$$
$$0<t<1<s<a.$$

Then the eigenvalue equations

$$Sf=\lambda f, \qquad S'f=\mu f, \qquad S=P_1^2+aP_2^2+(a+1)P_3^2+\mathbf{J}\cdot\mathbf{J},$$
$$S'=J_2^2+aJ_1^2+aP_3^2, \tag{2.59}$$

become

$$\left[\frac{4}{s-t}(\partial_{\alpha\alpha}+\partial_{\beta\beta})-\omega^2(s+t)-\omega^2(1+a)\right]f=\lambda f,$$
$$\left[\frac{4}{s-t}(t\,\partial_{\alpha\alpha}+s\,\partial_{\beta\beta})-\omega^2 st\right]f=\mu f \tag{2.60}$$

where

$$\partial_\alpha=\left[(a-s)(s-1)s\right]^{1/2}\partial_s, \qquad \partial_\beta=\left[(t-a)(t-1)t\right]^{1/2}\partial_t.$$

We can find solutions of these equations in the form $f(s,t)=E_1(s)E_2(t)$ where

$$(4\,\partial_{\alpha\alpha}-\omega^2s^2+\lambda's+\mu)E_1(s)=0,$$
$$(4\,\partial_{\beta\beta}+\omega^2t^2-\lambda't-\mu)E_2(t)=0, \qquad \lambda'=-\omega^2(1+a)-\lambda. \tag{2.61}$$

These expressions are algebraic forms of the ellipsoidal wave equation (see (1.29)), so the E_j are ellipsoidal functions. Furthermore, if we set $s=\operatorname{sn}^2(\eta,k), t=\operatorname{sn}^2(\psi,k)$ where $k=a^{-1/2}$, then the separated equations take the Jacobian form

$$\left(\partial_{\xi\xi}-k^2\mu-k^2\lambda'\operatorname{sn}^2\xi+k^2\omega^2\operatorname{sn}^4\xi\right)E_j(\xi)=0, \qquad \xi=\eta,\psi, j=1,2, \tag{2.62}$$

of the ellipsoidal wave equation (1.33). The new coordinates η,ψ also have

ISBN-0-201-13503-5

the property that they allow parametrization of the entire sphere S_2 rather than just the first octant. Indeed

$$\begin{aligned} k_1 &= k'^{-1}\operatorname{dn}(\eta,k)\operatorname{dn}(\psi,k), & k_2 &= ikk'^{-1}\operatorname{cn}(\eta,k)\operatorname{cn}(\psi,k), \\ k_3 &= k\operatorname{sn}(\eta,k)\operatorname{sn}(\psi,k), & k' &= (1-k^2)^{1/2} \end{aligned} \tag{2.63}$$

and these coordinates cover S_2 exactly once if η varies in the range $-2K<\eta<2K$ and ψ varies in the range $K\leqslant\psi<K+2iK'$ where $K=K(k)$ is defined by (C.3) and $K'=K(k')$.

Since k_1,k_2,k_3 remain unchanged when integral multiples of $4K$ and $4iK'$ are added to η or ψ, we are interested only in those single-valued solutions E_j of (2.62) which are also fixed under these substitutions: $E_j(\xi+4Kn+4iK'n)=E_j(\xi), n,m$ integers. As we noted in the preceding section, these doubly periodic functions are called the ellipsoidal wave functions. They have been studied in detail by Arscott [7]. The spectrum of S and S' is discrete, each pair of eigenvalues denoted $\lambda_{nm}\mu_{nm}$. The corresponding ellipsoidal wave functions are $\mathrm{el}_n^m(\xi), \xi=\eta,\psi$, and the eigenfunctions of S and S' are denoted

$$f_{nm}^{(10)}(\eta,\psi)=\mathrm{el}p_n^m(\eta,\psi)=\mathrm{el}_n^m(\eta)\,\mathrm{el}_n^m(\psi) \tag{2.64}$$

where $n=0,1,\ldots$ and the integer m runs over $2n+1$ values. We assume the basis $\{\mathrm{el}p_n^m\}$ is normalized to be ON:

$$\langle \mathrm{el}p_n^m, \mathrm{el}p_{n'}^{m'}\rangle=\delta_{nn'}\delta_{mm'}.$$

(This determines the solutions (2.64) only to within a factor of absolute value one. An essentially unique normalization is given in [7, p. 240]. Note also that $d\Omega(\hat{\mathbf{k}})=ik^2(\operatorname{sn}^2\eta-\operatorname{sn}^2\psi)\,d\eta\,d\psi$.) In general these functions are rather intractable and very little is known about their explicit construction.

The corresponding solutions of the Helmholtz equation $\Psi_{nm}^{(10)}(\mathbf{x})=I(f_{nm}^{(10)})$ are

$$\Psi_{nm}^{(10)}(\mathbf{x})=\mathrm{El}_n^m(\alpha,\beta,\gamma)=K_n^m(\omega,k)\,\mathrm{el}_n^m(\alpha)\,\mathrm{el}_n^m(\beta)\,\mathrm{el}_n^m(\gamma) \tag{2.65}$$

where the constant K_n^m is to be evaluated from the integral. Moreover, this integral reads

$$\begin{aligned} \mathrm{El}_n^m(\alpha,\beta,\gamma)=\iint_{S_2}\exp\Bigg[w\bigg(&-\frac{1}{kk'^2}\operatorname{dn}\alpha\operatorname{dn}\beta\operatorname{dn}\gamma\operatorname{dn}\eta\operatorname{dn}\psi \\ &+\frac{k^2}{k'^2}\operatorname{cn}\alpha\operatorname{cn}\beta\operatorname{cn}\gamma\operatorname{cn}\eta\operatorname{cn}\psi \\ &+ik^2\operatorname{sn}\alpha\operatorname{sn}\beta\operatorname{sn}\gamma\operatorname{sn}\eta\operatorname{sn}\psi\bigg)\Bigg]\,\mathrm{el}p(\eta,\psi)\,d\Omega(\hat{\mathbf{k}}), \end{aligned} \tag{2.66}$$

ISBN-0-201-13503-5

a nontrivial equation expressing the product of three ellipsoidal wave functions as an integral over a product of two such functions. Here the coordinates α,β,γ are related to x,y,z by expressions (1.32). We were able to evaluate the integral (2.66) to within a constant multiple because we knew in advance that it was separable in α,β,γ.

3.3 Lamé Polynomials and Functions on the Sphere

The eigenvalue problem corresponding to the conical coordinate system 11 (Table 14) is of special interest even though it is relatively intractable. Only for conical and spherical coordinates does the eigenvalue problem become finite dimensional; that is, only in these two cases is the problem reduced to finding the eigenvalues of an $n\times n$ matrix.

For functions f on the sphere S_2 the eigenvalue equations associated with system 11 in Table 14 are

$$\mathbf{J}\cdot\mathbf{J}f=-l(l+1)f, \qquad (J_1^2+bJ_2^2)f=\lambda f, \qquad 1>b>0. \tag{3.1}$$

It follows from (2.19) that

$$L_2(S_2)\cong\sum_{l=0}^{\infty}\oplus V_l$$

where $\dim V_l=2l+1$ and V_l transforms irreducibly under the representation D_l of $SO(3)$. Thus $\mathbf{J}\cdot\mathbf{J}$ has the spectrum $-l(l+1), l=0,1,2,\ldots$, each eigenvalue occurring with multiplicity $2l+1$. Since $S=\mathbf{J}\cdot\mathbf{J}$ and $S'=J_1^2+bJ_2^2$ commute, it follows that the subspaces V_l are invariant under the second operator. Thus we can reduce our search for eigenvalues of S' to the $(2l+1)$-dimensional space V_l. This space has an ON basis $\{f_m^{(l)}\}$, (2.12), and the restriction of S' to V_l can be represented by the $(2l+1)\times(2l+1)$ real symmetric matrix $\mathbb{S}'$ with respect to the basis $\{f_m^{(l)}\}$. The $2l+1$ eigenvalues of $\mathbb{S}'$ are the eigenvalues of S' in V_l.

There is another way to look at this problem. The elements h of V_l are characterized as the solutions of the partial differential equation $\mathbf{J}\cdot\mathbf{J}h=-l(l+1)h$, (2.20). It is straightforward to show that the symmetry algebra $so(3)$ of this equation is three dimensional (neglecting the identity symmetry E) with basis $\{J_1,J_2,J_3\}$, (2.6). The corresponding symmetry group is $SO(3)$. The space $\mathbb{S}^{(2)}/\mathfrak{q}$ of symmetric second-order symmetries modulo the multiples of $\mathbf{J}\cdot\mathbf{J}$ is five-dimensional with basis $J_2^2, J_3^2, \{J_1,J_2\}$, $\{J_1,J_3\},\{J_2,J_3\}$. Under the adjoint action of $SO(3)$ this space is decomposed into two oribt types, one orbit with representative J_3^2 and one orbit type with representative $J_1^2+bJ_2^2$, $1>b>0$. Moreover, it is known that the differential equation (2.20) for the Laplace operator on S_2 permits separation in exactly two coordinate systems [106]. One is the spherical coordi-

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nate system $\{\theta,\varphi\}$ in which we have originally expressed (2.20). It corresponds to the diagonalization of J_3^2. The second is the elliptic coordinate system $\{s,t\}$, (2.58), which corresponds to the diagonalization of $J_1^2+bJ_2^2$. The elliptic system was first studied from the group-theoretical point of view in [106] (see also [58]).

Whichever point of view is adopted, we need to compute the matrix $\mathcal{S}'$ of the operator

$$S'=J_1^2+bJ_2^2=\tfrac{1}{4}(b-1)\big((J^+)^2+(J^-)^2\big)+\tfrac{1}{2}(b+1)\big((J^0)^2-l(l+1)\big)$$

with respect to the basis $\{f_m^{(l)}\}$ and compute the $2l+1$ eigenvalues λ of this matrix. As is well known [69, p. 96], this problem is equivalent to computing the roots of the characteristic equation

$$\det(\mathcal{S}'-\lambda\mathcal{E})=0 \tag{3.2}$$

where $\mathcal{E}$ is the $(2l+1)\times(2l+1)$ identity matrix. As shown in [106], for $l\leqslant 7$ one can explicitly find the eigenvalues λ as roots of polynomials of at most fourth order. However, for $l\geqslant 8$ the polynomials are of higher order than four and numerical methods must be used to approximate the roots.

We can use group theory to further aid in the classification of these eigenvalues. Note that both the Helmholtz equation (1.1) and the Laplace equation on the sphere (2.20) are invariant under the full rotation group $O(3)$. (This group is generated by $SO(3)$ and the space inversion operator P: $\mathbf{x}\to-\mathbf{x}$. A matrix realization is the group of all 3×3 real matrices A such that $AA'=E_3$. Here $\det A=\pm 1$ and $\det A=+1$ if and only if $A\in SO(3)$.) The elements of $O(3)$ that do not belong to $SO(3)$ (the rotation-inversions) are bounded away from the identity and are not obtainable by exponentiation of elements from the Lie algebra $so(3)$. The existence of these inversion symmetries must be verified by inspection.

In addition to P we shall be especially interested in the operators Z: $(x,y,z)\to(x,y,-z)$, reflection in the $x-y$ plane; X: $(x,y,z)\to(-x,y,z)$; reflection in the y–z plane; and Y: $(x,y,z)\to(x,-y,z)$, reflection in the x–z plane. Using (2.1) to transfer the action of these operators to the sphere, we find

$$\begin{aligned}&Ph(\hat{\mathbf{k}})=h(-\hat{\mathbf{k}}), && Zh(\hat{\mathbf{k}})=h(k_1,k_2,-k_3),\\ &Xh(\hat{\mathbf{k}})=h(-k_1,k_2,k_3), && Yh(\hat{\mathbf{k}})=h(k_1,-k_2,k_3), \qquad h\in L_2(S_2).\end{aligned} \tag{3.3}$$

Obviously the square of each of the commuting operators P,Z,X,Y is the identity operator E and each operator is self-adjoint. Moreover, these operators each commute with $S'=J_1^2+bJ_2^2$ and $S=\mathbf{J}\cdot\mathbf{J}$. It follows that there exists an ON basis for V_l consisting of simultaneous eigenvectors of P,Z,X,Y and S'.

ISBN-0-201-13503-5

The possible eigenvalues of $P,\ldots,Y$ are ± 1. To determine the multiplicities of these eigenvalues in V_l, we apply the operators (3.3) to the explicit basis $\{f_m^{(l)}(\theta,\varphi)=Y_l^m(\theta,\varphi)\}$, (2.22). The results are

$$\begin{aligned} Pf_m^{(l)}&=(-1)^l f_m^{(l)}, & Zf_m^{(l)}&=(-1)^{l-m} f_m^{(l)},\\ Xf_m^{(l)}&=f_{-m}^{(l)}, & Yf_m^{(l)}&=(-1)^m f_{-m}^{(l)}. \end{aligned}\tag{3.4}$$

Note that $P=(-1)^l E$ on V_l. To compute the multiplicities of the other eigenspaces we define eigenspaces

$$\mathcal{C}_l^{pq}=\{h\in V_l: Xh=ph, XYh=qh\},\qquad p,q=\pm 1,\tag{3.5}$$

and set $n_l^{pq}=\dim \mathcal{C}_l^{pq}$. Since $Y=X(XY)$ and $Z=XYP$, we have

$$Ph=(-1)^l h,\qquad Zh=(-1)^l qh,\qquad Xh=ph,\qquad Yh=pqh,\tag{3.6}$$

for any $h\in\mathcal{C}_l^{pq}$. Furthermore,

$$V_l=\mathcal{C}_l^{++}\oplus\mathcal{C}_l^{+-}\oplus\mathcal{C}_l^{-+}\oplus\mathcal{C}_l^{--}.$$

Using (3.4) we can count the dimensions of these eigenspaces. The results are presented in Table 15.

Since each eigenspace is invariant under S', we can classify the eigenfunctions of S' by their symmetry properties with respect to X and XY. Thus an ON basis for V_l can be denoted $\{f_\lambda^{pq}\}$:

$$\begin{aligned} \mathbf{J}\cdot\mathbf{J}f_\lambda^{pq}&=-l(l+1)f_\lambda^{pq}, & (J_1^2+bJ_2^2)f_\lambda^{pq}&=\lambda f_\lambda^{pq},\\ Xf_\lambda^{pq}&=pf_\lambda^{pq}, & XYf_\lambda^{pq}&=qf_\lambda^{pq}. \end{aligned}\tag{3.7}$$

(It can be shown that there is no degeneracy; that is, there do not exist two linearly independent solutions of (3.7) for fixed l, p, q, λ.)

In terms of elliptic coordinates on the sphere, (2.63), the eigenvalue equations (3.1) separate to give the ordinary differential equations

$$E_j''(\xi)+(\lambda-l(l+1)k^2\operatorname{sn}^2\xi)E_j(\xi)=0,\qquad j=1,2,\ \xi=\eta,\psi,\ k=b^{1/2},\tag{3.8}$$

where $f(\eta,\psi)=E_1(\eta)E_2(\psi)$. As mentioned in the discussion following expressions (1.35), equation (3.8) is the Lamé equation. It has $2l+1$ linearly

Table 15 Dimensions n_l^{pq} of the Eigenspaces $\mathcal{C}_l^{pq}$

	n_l^{++}	n_l^{+-}	n_l^{-+}	n_l^{--}
l even	$1+l/2$	$l/2$	$l/2$	$l/2$
l odd	$(1+l)/2$	$(1+l)/2$	$(-1+l)/2$	$(1+l)/2$

ISBN-0-201-13503-5

independent solutions (the Lamé polynomials) that are single valued on S_2, each expressible in the form

$$\mathrm{sn}^s\xi\,\mathrm{cn}^c\xi\,\mathrm{dn}^d\xi\,F_p(\mathrm{sn}^2\xi),\qquad s,c,d=0,1,\quad s+c+d+2p=l,\tag{3.9}$$

where $F_p(z)$ is a polynomial of order p in z. The eight types of such polynomials correspond to the eight categories listed in Table 15. Since each eigenspace has multiplicity one, the eigenfunctions must take the form $E(\eta)E(\psi)$ where $E(z)$ is a Lamé polynomial.

Rather than continue our analysis of the operator S' on $L_2(S_2)$, we shall study a simpler one-variable model for the spectral resolution of S'. We consider the $(2l+1)$-dimensional space W_l of polynomials $g(z)$ with order $\leqslant 2l$ in the complex variable z. We introduce a scalar product $(\cdot,\cdot)$ on W_l such that

$$(z^{l-m},z^{l-n})=(l-m)!(l+m)!\delta_{mn},\qquad m,n=l,l-1,\ldots,-l,\tag{3.10}$$

or explicitly,

$$\begin{aligned}(2l+1)!(g_1,g_2)&=\pi^{-1}\iint_{-\infty}^{\infty}dx\,dy\,(1+|z|^2)^{-2l-2}g_1(z)\,\bar g_2(z)\\&=\pi^{-1}\int_0^{\infty}r\,dr\int_0^{2\pi}d\varphi\,(1+r^2)^{-2l-2}g_1(re^{i\varphi})\,\bar g_2(re^{i\varphi})\end{aligned}\tag{3.11}$$

for $g_j\in W_l$. Here $z=x+iy=re^{i\varphi}$ and the integration region is the complex plane. The operators J_1,J_2,J_3 defined by

$$J_1=-\frac{i}{2}(1-z^2)\frac{d}{dz}-ilz,\qquad J_2=\frac{1}{2}(1+z^2)\frac{d}{dz}-lz,\qquad J_3=-iz\frac{d}{dz}+il,\tag{3.12}$$

leave W_l invariant and satisfy the commutation relations $[J_j,J_k]=\sum_p\epsilon_{jkp}J_p$ of $so(3)$. Moreover, $\mathbf{J}\cdot\mathbf{J}=-l(l+1)$ in this model. With each function $g\in W_l$ we associate a function $G\in V_l$, defined by

$$G(\hat{\mathbf{k}})=(g,H(\hat{\mathbf{k}},\cdot))=I'(g),\tag{3.13}$$
$$H(\hat{\mathbf{k}},z)=(l!)^{-1}\left[(2l+1)/4\pi\right]^{1/2}\left[k_1(1-z^2)/2+ik_2(1+z^2)/2+k_3z\right]^l.$$

Here, $\hat{\mathbf{k}}\in S_2$. The transformation I' from W_l to V_l is unitary. Indeed it follows from (3.10) that

$$g_m^l(z)=\frac{z^{l+m}}{\left[(l+m)!(l-m)!\right]^{1/2}},\qquad m=l,l-1,\ldots,-l,\tag{3.14}$$

ISBN-0-201-13503-5

is an ON basis for W_l. Since

$$\overline{H}(\hat{\mathbf{k}},z)=\sum_{m=-l}^{l} Y_l^m(\theta,\psi)\,\bar{g}_m^l(z) \tag{3.15}$$

(see [128, p. 147] for a group-theoretic proof of this fact) for $\hat{\mathbf{k}}=(\sin\theta\cos\psi,\sin\theta\sin\psi,\cos\theta)$ we have

$$I'(g_m^l)=Y_l^m(\theta,\varphi)=f_m^l \tag{3.16}$$

where the spherical harmonics Y_l^m form an ON basis for V_l. From (2.12) and (2.22) we see that the operators (3.12) acting on W_l induce the operators (2.6) on V_l:

$$J_jG(\hat{\mathbf{k}})=I'(J_jg(z)),\qquad j=1,2,3. \tag{3.17}$$

We will now study the eigenvalue problem for S' on W_l: $(J_1^2+bJ_2^2)g(z)=\lambda g(z)$. We find

$$\begin{aligned}S'=&\big[(1-k)z^2-(1+k)\big]\big[(1+k)z^2-(1-k)\big]\frac{d^2}{dz^2}\\&+2(2l-1)z\big[1+k^2-z^2(1-k^2)\big]\frac{d}{dz}\\&+2l\big[1+k^2+(1-k^2)(2l-1)z^2\big],\qquad k=b^{1/2}.\end{aligned}$$

If we now write $g(z)=(k')^l[(\alpha-z^2)(1-\alpha z^2)]^{l/2}\mathcal{G}(w)$, where $k'=(1-k^2)^{1/2}$, $\alpha=(1+k)/(1-k)$, and make the change of variable

$$\operatorname{sn}(w,k)=-i(1+\alpha)z\big[(\alpha-z^2)(1-\alpha z^2)\big]^{-1/2}, \tag{3.18}$$

the eigenvalue equation reduces to

$$\left[\frac{d^2}{dw^2}+\lambda-k^2l(l+1)\operatorname{sn}^2(w,k)\right]\mathcal{G}(w)=0, \tag{3.19}$$

the Lamé equation.

It follows from (3.9) that the $2l+1$ Lamé polynomial solutions of this equation are exactly the solutions that correspond to elements $g(z)$ of W_l. Let us see how the classification of Lamé polynomials into eight types exhibits itself in our new model. From (3.4), (3.14), and (3.16) it follows that X and XY on W_l take the forms

$$Xg(z)=z^{2l}g(z^{-1}),\qquad XYg(z)=(-1)^l g(-z), \tag{3.20}$$

for $g\in W_l$.

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Just as in our discussion of the eigenspaces $\mathcal{C}_l^{pq}$ of V_l ((3.5), (3.6)), we can require that the eigenfunctions $g(z)=(k')^l[(\alpha-z^2)(1-\alpha z^2)]^{l/2}\mathcal{G}(w)$ also satisfy the equations $Xg=pg$, $XYg=qg$, $p,q=\pm 1$. Using these expressions, as well as (3.9) and Table 15, we obtain relations between the exponents a, b, c of (3.9) and the eigenvalues p,q as listed in Table 16. As shown in [7, Chapter 9], the Lamé polynomials in each symmetry class can be labeled by the integer $n=0,1,\ldots,n_l^{pq}-1$ where n is the number of zeros of the polynomial in the interval $0<w<K(k)$. Recurrence relations for the coefficients in the polynomial $F_\rho(\mathrm{sn}^2 w)$ can be obtained by substituting expression (3.9) into (3.19) and equating coefficients of independent monomials of elliptic functions $\mathrm{sn}^s w\,\mathrm{cn}^c w\,\mathrm{dn}^d w\,\mathrm{sn}^{2j} w$. One obtains polynomial solutions if and only if λ is one of the $2l+1$ distinct eigenvalues λ_{ln}^{pq}.

Table 16 Symmetry Classes of Lamé Polynomials
$\mathrm{sn}^s w\,\mathrm{cn}^c w\,\mathrm{dn}^d w F\rho(\mathrm{sn}^2 w)$, $s,c,d=0,1$, $s+c+d+2\rho=l$

	(p,q)	s	c	d	Dimension n_l^{pq}
l even	$+,+$	0	0	0	$1+l/2$
	$+,-$	1	1	0	$l/2$
	$-,+$	0	1	1	$l/2$
	$-,-$	1	0	1	$l/2$
l odd	$+,+$	1	0	0	$(1+l)/2$
	$+,-$	0	1	0	$(1+l)/2$
	$-,+$	1	1	1	$(-1+l)/2$
	$-,-$	0	0	1	$(1+l)/2$

We have shown that these eigenvalues may be obtained in two different ways: either in the traditional manner through the search for polynomial solutions of the Lamé equation as described by Arscott, or by solution of the characteristic equation (3.2). In the second method, the matrix $\mathcal{S}'$ is explicitly determined with respect to the ON basis $\{f_m^{(l)}\}$. Thus, once an eigenvalue is computed, the corresponding eigenvector $f_{\lambda n}^{pq}$ can be directly obtained in terms of its expansion coefficients $a_{n,m}^{pq}$ in the $\{f_m^{(l)}\}$ basis:

$$f_n^{pq}=\sum_m a_{n,m}^{pq} f_m^{(l)}. \tag{3.21}$$

(In practice one obtains three-term recurrence formulas for these coefficients (see [106].) On the other hand, the traditional study of the Lamé equation leads to three-term recurrence formulas for the coefficients in the polynomial $F_\rho(\mathrm{sn}^2 w)=\sum_{j=0}^{\rho} b_j\,\mathrm{sn}^{2j} w$. The coefficients $a_{n,m}^{pq}$ are of special interest to us because they define the overlap function between the Lamé basis $\{f_n^{pq}\}$ and the canonical basis $\{f_m^{(l)}\}$. However, it is the coefficients b_k which are tabulated in the literature on Lamé polynomials.

The W_l model can be used to relate these coefficients. Let $\{\Lambda_n^{pql}(z)\}$ be the ON basis for W_l consisting of eigenfunctions of S' classified by

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symmetry type and number of zeros. Then (3.21) implies

$$\Lambda_n^{pql}(z)=\sum_m a^{pq}_{n,m}g^{(l)}_m=\sum_m a^{pq}_{n,m}z^{l+m}\big[(l+m)!(l-m)!\big]^{-1/2}; \qquad (3.22)$$

that is, the overlaps are essentially the coefficients of z^{l+m}, $-l\leqslant m\leqslant l$. On the other hand

$$\Lambda_n^{pql}(z)=(k')^l\big[(\alpha-z^2)(1-\alpha z^2)\big]^{l/2}\operatorname{sn}^s w\operatorname{cn}^c w\operatorname{dn}^d w\sum_{j=0}^{p}b_j\operatorname{sn}^{2j}w \qquad (3.23)$$

where w is related to z by expression (3.18). Expanding (3.23) as a polynomial in z and equating coefficients of z^{l+m} in (3.22), (3.23), we can express each coefficient $a^{pq}_{n,m}$ as a finite sum of coefficients b_j. Some of the details of the straightforward computation can be found in [58].

The transformation (3.13) can now be used to map our results to V_l. If $\{\Lambda_n^{pql}\}$ is the ON basis of eigenfunctions for S' on W_l, then

$$f^{pq}_{ln}=c^{pq}_{ln}E^{p,q}_{ln}(\eta)E^{p,q}_{ln}(\psi)=I'\big(\Lambda_n^{pql}\big)=\big(\Lambda_n^{pql},H(\hat{\mathbf{k}},\cdot)\big) \qquad (3.24)$$

(where η, ψ are elliptic coordinates on S_2, (2.63)) is an ON basis of eigenfunctions for S' on V_l. Here $E^{p,q}_{ln}(\xi)$ is a Lamé polynomial of the same eigenvalue and symmetry type as Λ_n^{pql}. The constant c is to be determined from the double integral once the explicit normalization of $E^{p,q}_{ln}$ and Λ_n^{pql} is fixed. (The integral in (3.24) can be evaluated because we know in advance that it satisfies the Lamé equation in η and ψ, and we can easily check that the integral is periodic in these variables.) Relation (3.21) can now be interpreted as an expansion of products of Lamé polynomials in terms of spherical harmonics.

The totality of all eigenfunctions (3.24) for $l=0,1,2,\ldots$ forms an ON basis for $L_2(S_2)$. Mapping this basis to the Hilbert space of solutions of the Helmholtz equation via the transformation (2.1), we find

$$\Psi^{pq}_{ln}(\mathbf{x})=I\big(f^{pq}_{ln}\big)=d^{pql}_n j_l(\omega r)E^{p,q}_{ln}(\alpha)E^{p,q}_{ln}(\beta), \qquad (3.25)$$

in terms of the conical coordinates (1.34). Here $j_l(z)$ is a spherical Bessel function, (2.27), and d is a constant that can be determined, in principle, from the integral.

Let us note that (3.24) and (3.25) can also be interpreted as nonlinear integral equations satisfied by Lamé polynomials. In this connection we remark that the evaluation of the integral (5.16) in [58] is in error. This integral should be replaced by (3.24).

The W_l model can also be used to study Ince polynomials (see [21]).

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3.4 Expansion Formulas for Separable Solutions of the Helmholtz Equation

From the discussion in Chapters 1 and 2 it is evident that to expand a solution $\mathbf{T}(g)\Psi_\lambda^{(j)}(\mathbf{x})$ of the Helmholtz equation in terms of the eigenfunctions $\{\Psi_\mu^{(i)}\}$ it is sufficient to compute the expansion coefficients $\langle \mathbf{T}(g)f_\lambda^{(j)}, f_\mu^{(i)}\rangle$ in the $L_2(S_2)$ model:

$$\mathbf{T}(g)\Psi_\lambda^{(j)}(\mathbf{x}) = \sum_\mu \langle \mathbf{T}(g)f_\lambda^{(j)}, f_\mu^{(i)}\rangle \Psi_\mu^{(i)}(\mathbf{x}). \tag{4.1}$$

Here we list some of the more tractable expansion coefficients in the case where $\mathbf{T}(g)$ is the identity operator.

The overlap functions $\langle f_\lambda^{(j)}, f_{\alpha,\gamma}^{(1)}\rangle$ relating any system $\{f_\lambda^{(j)}(\hat{\mathbf{k}})\}$ with the Cartesian system (2.32) are trivial:

$$\langle f_\lambda^{(j)}, f_{\alpha,\gamma}^{(1)}\rangle = (\sin\gamma)^{1/2} f_\lambda^{(j)}(\sin\gamma\cos\alpha, \sin\gamma\sin\alpha, \cos\gamma). \tag{4.2}$$

Moreover, the overlaps relating the eigenfunctions for systems 1–4 in Table 14 can easily be obtained from the corresponding overlaps for solutions of the Helmholtz equation $(\Delta_2+\omega^2)\Psi=0$ listed in Section 1.3. Indeed, the overlaps take the form

$$\langle f_{\lambda,\gamma}^{(j)}, f_{\mu,\gamma'}^{(i)}\rangle = \delta(\gamma-\gamma')\langle f_\lambda^{(j)\prime}, f_\mu^{(i)\prime}\rangle, \qquad 1 \leq i,j \leq 4, \tag{4.3}$$

where $\langle f_\lambda^{(j)\prime}, f_\mu^{(i)\prime}\rangle$ is the corresponding overlap computed in Section 1.3 with the $L_2(S_1)$ model.

The overlaps $\langle f_m^{(l)}, f_{\lambda,m'}^{(8)}\rangle$ between the spherical and parabolic bases were computed in [95]:

$$\begin{aligned}\langle f_m^{(l)}, f_{\lambda,m'}^{(8)}\rangle &= \delta_{mm'}\frac{(-1)^{(m+|m|)/2}}{(|m|!)^2}\left[\frac{(2l+1)(l+|m|)!}{4\pi(l-|m|)!}\right]^{1/2}\\ &\quad\times\Gamma\left(\frac{i\lambda+|m|+1}{2}\right)\Gamma\left(\frac{-i\lambda+|m|+1}{2}\right)\\ &\quad\times{}_3F_2\left(\left.\begin{matrix}|m|-l,\ |m|+l+1,\ (i\lambda+|m|+1)/2\\ |m|+1,\ |m|+1\end{matrix}\right|1\right), \end{aligned}\tag{4.4}$$

$$m=0,\pm1,\ldots,\pm l.$$

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The overlaps between the spherical and prolate spheroidal bases are

$$\langle f_m^{(l)}, f_{n,m'}^{(6)}\rangle = \begin{cases} \delta_{mm'}(-1)^{(2m+l-n)/2}\left[\dfrac{(n-m)!(l+m)!(2n+1)}{(n+m)!(l-m)!(2l+1)}\right]^{1/2} a_{n,l-n}^{m}(a^2\omega^2), \\ \hfill m'\geqslant 0, \\ \delta_{mm'}(-1)^{(l-n)/2}\left[\dfrac{(n+m)!(2n+1)}{(n-m)!(2l+1)}\right]^{1/2} a_{n,l-n}^{|m|}(a^2\omega^2), \\ \hfill m'<0, \end{cases} \tag{4.5}$$

where the coefficients $a_{n,2k}^{|n|}$ are defined by (2.46).

The overlaps between the cylindrical and prolate spheroidal bases are

$$\langle f_{n,m}^{(6)}, f_{m',\gamma}^{(2)}\rangle = \left[\frac{(n-|m|)!(2n+1)}{(n+|m|)!2}\sin\gamma\right]^{1/2} Ps_n^{|m|}(\cos\gamma, a\omega^2)\delta_{mm'} \tag{4.6}$$

and the overlaps between the parabolic cylindrical and prolate spheroidal bases are

$$\langle f_{n,m}^{(6)}, f_{\mu\pm,\gamma}^{(3)}\rangle = \left[\frac{(n-|m|)!(2n+1)}{(n+|m|)!2}\sin\gamma\right]^{1/2} Ps_n^{|m|}(\cos\gamma, a^2\omega^2)\langle f_m^{(2)}, f_{\mu\pm}^{(3)}\rangle \tag{4.7}$$

where the overlap $\langle f_m^{(2)}, f_{\mu\pm}^{(3)}\rangle$ is defined by (3.50), Section 1.3. The overlaps between the elliptic cylindrical and prolate spheroidal bases are

$$\langle f_{n,m}^{(6)}, f_{n'p,\gamma}^{(4)}\rangle = \left[\frac{(n-|m|)!(2n+1)}{(n+|m|)!}\sin\gamma\right]^{1/2} Ps_n^{|m|}(\cos\gamma, a^2\omega^2)A_{n'}^{m} \tag{4.8}$$

where the Fourier coefficient A_n^m is defined in terms of the Mathieu functions $\mathrm{pe}_n(\varphi, q)$, $p = s, c$, by

$$\mathrm{pe}_n(\varphi, q) = \sum_{m=-\infty}^{\infty} A_n^m e^{im\varphi}. \tag{4.9}$$

The corresponding overlaps for oblate spheroidal coordinates can be obtained from the prolate overlaps (4.5)–(4.8) by making the replacement $a^2\omega^2 \to -a^2\omega^2$ in the spheroidal wave functions.

Arscott [7, p. 247] shows how to compute the overlaps between the conical basis 11 and the ellipsoidal basis, $\langle f_{nm}^{(10)}, f_{lm'}^{pq}\rangle$, by exhibiting a three-term recurrence relation obeyed by the overlap function.

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The remaining overlaps are more complicated than those we have listed.

It is easy to construct a bilinear generating function for all basis sets of solutions of the Helmholtz equation listed here. Let $\{f_{\lambda\mu}(\hat{\mathbf{k}})\}$ be one of the eleven bases for $L_2(S_2)$ constructed earlier and let $\{\Psi_{\lambda\mu}(\mathbf{x})\}$ be the corresponding basis for the solution space of $(\Delta_3+\omega^2)\Psi(\mathbf{x})=0$. Then

$$\Psi_{\lambda\mu}(\mathbf{x})=I(f_{\lambda\mu})=\langle f_{\lambda\mu}, H(\mathbf{x},\cdot)\rangle$$

where $H(\mathbf{x},\hat{\mathbf{k}})=\exp[-i\omega\mathbf{x}\cdot\hat{\mathbf{k}}]\in L_2(S_2)$ for each $\mathbf{x}\in R^3$. An explicit computation yields

$$\langle H(\mathbf{x},\cdot), H(\mathbf{x}',\cdot)\rangle=4\pi\big[\sin(\omega R)/\omega R\big],\qquad R^2=(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}'). \tag{4.10}$$

On the other hand

$$\begin{aligned}\langle H(\mathbf{x},\cdot), H(\mathbf{x}',\cdot)\rangle&=\sum_{\lambda,\mu}\langle H(\mathbf{x},\cdot),f_{\lambda\mu}\rangle\langle f_{\lambda\mu},H(\mathbf{x}',\cdot)\rangle\\&=\sum_{\lambda,\mu}\bar{\Psi}_{\lambda\mu}(\mathbf{x})\Psi_{\lambda\mu}(\mathbf{x}'),\end{aligned}\tag{4.11}$$

and comparison of (4.10) and (4.11) shows that $4\pi\sin(\omega R)/\omega R$ is a bilinear generating function for each of our bases.

Finally, as shown in [128] and [95], each of our eleven bases $\{\Psi_{\lambda\mu}\}$ considered as functions of ω, $0<\omega<\infty$, can be used to expand arbitrary functions $f(\mathbf{x})$ on R_3, square integrable with respect to Lebesgue measure.

3.5 Non-Hilbert Space Models for Solutions of the Helmholtz Equation

There are obviously many physically and mathematically interesting solutions of the Helmholtz equation that are not representable in the form $I(h)$, (2.1), for $h\in L_2(S_2)$. We shall investigate a few group-theoretic methods for obtaining such solutions and relating different types of separable non-Hilbert space solutions. These methods are considerably less elegant but more flexible than the techniques discussed earlier. Furthermore, they can be applied to the differential equations treated in Chapters 1 and 2.

We begin by considering transforms $I(h)$, (2.1), where the domain of integration is a complex two-dimensional Riemann surface rather than the real sphere S_2. In particular we set

$$\hat{\mathbf{k}}=(k_1,k_2,k_3)=\Big(-\tfrac{1}{2}(t+t^{-1})(1+\beta^2)^{1/2},\ \tfrac{i}{2}(t-t^{-1})(1+\beta^2)^{1/2},\ i\beta\Big) \tag{5.1}$$

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where t and β range over complex values, and write

$$\Psi(\mathbf{x})=\iint_S d\beta\,\frac{dt}{t}\,h(\beta,t)\exp\Big[-\frac{i\omega}{2}(1+\beta^2)^{1/2}\times\{x(t+t^{-1})+iy(t^{-1}-t)\}-\omega\beta z\Big]=I(h). \tag{5.2}$$

We assume that the integration surface S and the analytic function h are such that $I(h)$ converges absolutely and arbitrary differentiation with respect to x, y, and z is permitted under the integral sign. Since $k_1^2+k_2^2+k_3^2=1$ even for arbitrary complex β and t, $t\neq 0$, it follows that $\Psi(\mathbf{x})$ is a solution of the Helmholtz equation

$$(\Delta_3+\omega^2)\Psi(\mathbf{x})=0. \tag{5.3}$$

Integrating by parts, we find that the operators P_j, J_j, (1.2), acting on the solution space of (5.3) correspond to the operators

$$J^{\pm}=it^{\pm 1}\left(\mp(1+\beta^2)^{1/2}\partial_\beta+\frac{\beta t}{(1+\beta^2)^{1/2}}\partial_t\right),\qquad J^0=t\,\partial_t,$$
$$P^{\pm}=\omega(1+\beta^2)^{1/2}t^{\pm 1},\qquad P^0=-i\omega\beta, \tag{5.4}$$

acting on analytic functions $h(\beta,t)$ provided S and h are chosen such that the boundary terms vanish:

$$J^{\pm}\Psi=I(J^{\pm}h),\qquad P^{\pm}\Psi=I(P^{\pm}h),$$

and so on. Here as usual $J^{\pm}=\mp J_2+iJ_1$, $J^0=iJ_3$, $P^{\pm}=\mp P_2+iP_1$, $P^0=iP_3$.

For our first example we set $h=(2\pi^3)^{-1/2}$ and integrate over the contours C_1 and C_2 in the β and t planes, respectively (Figure 1).

In this case h satisfies the equations $\mathbf{J}\cdot\mathbf{J}h=0$, $J^0h=0$ and it is straightforward to verify that $\Psi(\mathbf{x})=I(h)$ satisfies the same equations for $z>0$. Thus, $\Psi(\mathbf{x})$ is independent of the spherical coordinates θ, φ and is a linear combination of the Bessel functions $\rho^{-1/2}J_{1/2}(\omega\rho)$ and $\rho^{-1/2}J_{-1/2}(\omega\rho)$, (1.19). To determine the correct linear combination we evaluate (5.2) in the special case where $x=y=0$. Then the integral becomes elementary and we find

$$\Psi(0,0,z)=(i/\omega z)(2/\pi)^{1/2}e^{i\omega z},\qquad z>0.$$

Thus we obtain

$$\Psi(\mathbf{x})=-(\omega\rho)^{-1/2}H^{(1)}_{1/2}(\omega\rho) \tag{5.5}$$

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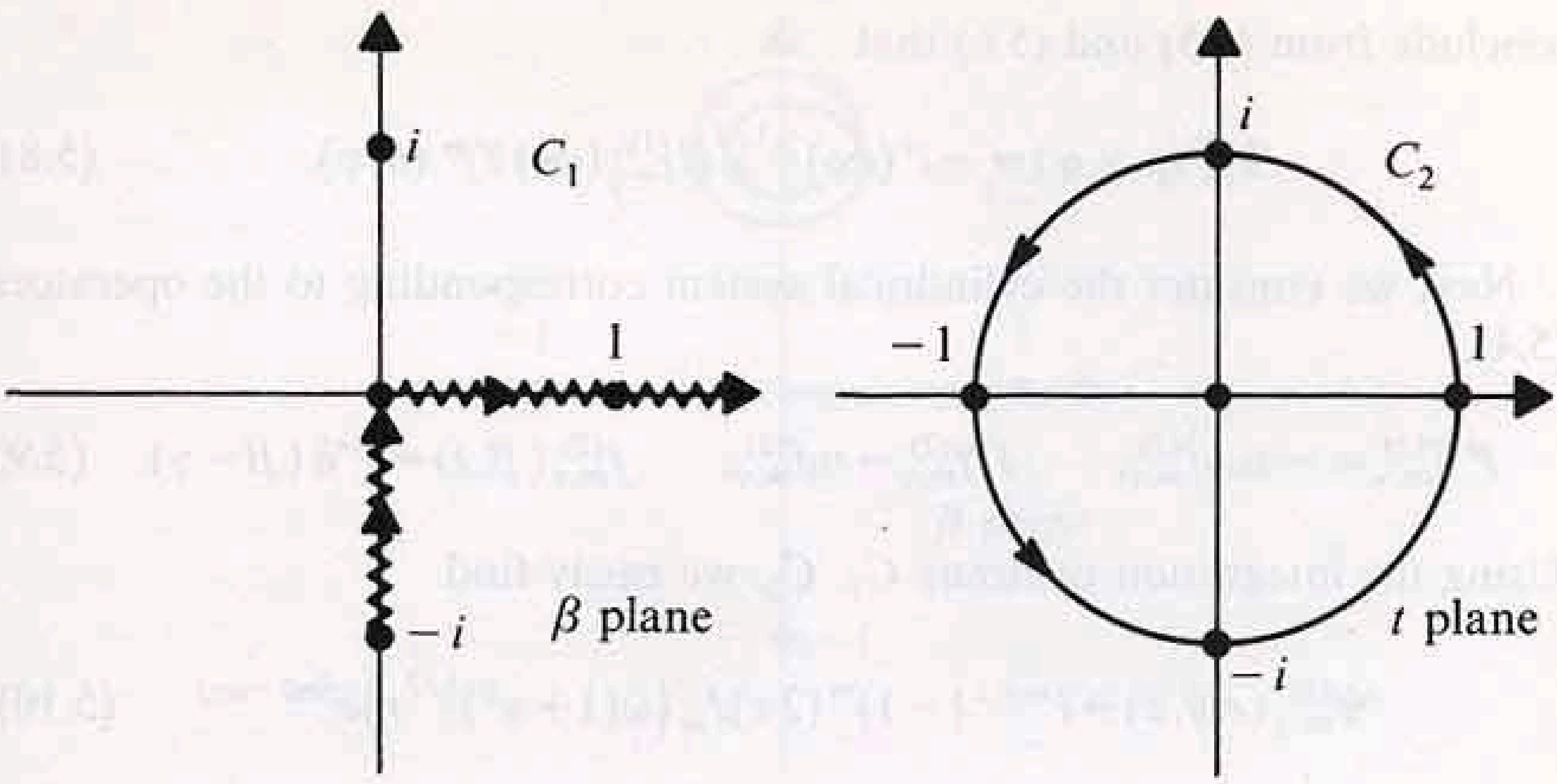

Figure 1

where $H^{(j)}_{n+\frac{1}{2}}(z)$ are Hankel functions of the first $(j=1)$ and the second $(j=2)$ kind:

$$H^{(1)}_\nu(z)=(i\sin\pi\nu)^{-1}\left[J_{-\nu}(z)-J_\nu(z)e^{-i\pi\nu}\right],$$
$$H^{(2)}_\nu(z)=(i\sin\pi\nu)^{-1}\left[J_\nu(z)e^{i\pi\nu}-J_{-\nu}(z)\right], \tag{5.6}$$
$$H^{(1,2)}_{n+\frac{1}{2}}(z)=\mp i(-1)^n\left(\frac{\pi z}{2}\right)^{-1/2}z^{n+1}\left(\frac{d}{z\,dz}\right)^n\frac{e^{\pm iz}}{z},\qquad n=0,1,2,\ldots.$$

The solution (5.5) is a *(traveling) spherical wave.*

More generally, we set

$$h=f^{(l)}_m(\beta,t)=\left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{1/2}P^m_l(i\beta)(-t)^m,$$
$$l=0,1,\ldots,m=l,l-1,\ldots,-l, \tag{5.7}$$

where $P^m_l(z)$ is an associated Legendre function. (This expression makes sense for all $\beta\in C_1$ since, from (B.6iv), for $m\geqslant 0$ $P^{-m}_l(z)$ is a polynomial in $z=i\beta$ times the factor $[(i\beta-1)/(i\beta+1)]^{m/2}$ which remains bounded on C_1 and vanishes at $\beta=-i$. Moreover, $P^{-m}_l(z)=(-1)^m(l-m)!P^m_l(z)/(l+m)!$. It follows from (2.17), (2.22), and (2.24) that the operators (5.4) acting on the functions $\{f^{(l)}_m(\beta,t)\}$ satisfy the recurrence relations (2.12) and (2.23). Thus, the solutions $\Psi^{(l)}_m(\mathbf{x})=I(f^{(l)}_m)$ of the Helmholtz equation also satisfy these relations.

We have already computed the spherical wave $\Psi^{(0)}_0(\mathbf{x})$, (5.5). Using the fact that both the functions (2.28) and (2.30), hence a fixed linear combination of these functions, satisfy recurrence relations (2.12), (2.23), we can

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conclude from (5.5) and (5.6) that

$$\Psi_m^{(l)}(\rho,\theta,\varphi) = -i^l(\omega\rho)^{-1/2}H_{l+\frac{1}{2}}^{(1)}(\omega\rho)Y_l^m(\theta,\varphi). \tag{5.8}$$

Next we consider the cylindrical system corresponding to the operators (5.4),

$$P^0 f_{m,\gamma}^{(2)} = -i\omega\gamma f_{m,\gamma}^{(2)}, \qquad J^0 f_{m,\gamma}^{(2)} = m f_{m,\gamma}^{(2)}, \qquad f_{m,\gamma}^{(2)}(\beta,t) = t^m\delta(\beta-\gamma). \tag{5.9}$$

Using the integration contours C_1, C_2, we easily find

$$\Psi_{m,\gamma}^{(2)}(r,\theta,z) = i^{m+1}(-1)^m(2\pi)J_m\left(\omega(1+\gamma^2)^{1/2}r\right)e^{im\theta-\omega\gamma z} \tag{5.10}$$

for $\gamma\in C_1$. Here $\{r,\theta,z\}$ are cylindrical coordinates (2.36).

From (5.7), (5.9) and the corresponding integral representations $\Psi = I(f)$ there follows easily the expansion

$$\Psi_m^{(l)}(\mathbf{x}) = \left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{1/2}(-1)^m\int_{C_1}P_l^m(i\beta)\Psi_{m,\beta}^{(2)}(\mathbf{x})\,d\beta, \tag{5.11}$$

$$z>0.$$

More generally, if $\Psi_m^{(l)}$ is subjected to a translation $\mathbf{T}(g) = \exp(a_1P_1 + a_2P_2 + a_3P_3)$, we obtain the expansion formula

$$\mathbf{T}(g)\Psi_m^{(l)}(\mathbf{x}) = \left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{1/2}\sum_{n=-\infty}^{\infty}\int_{C_1}(-1)^{n+m}(ie^{-i\alpha})^n$$

$$\times P_l^m(i\beta)J_n[\omega a(1+\beta^2)^{1/2}]\exp(-a_3\omega\beta)\Psi_{m+n,\beta}^{(2)}(\mathbf{x})\,d\beta, \tag{5.12}$$

$$z+a_3>0,\ a_1+ia_2 = ae^{i\alpha},\ a>0.$$

Similar techniques can be used to expand traveling spherical waves in other bases. In each case one derives the expansion for the complex sphere model and then attempts to map the results to the solution space of the Helmholtz equation via the transformation (5.2). The procedure is no longer so straightforward as for our Hilbert space models, and special techniques may have to be developed for each example. Some important cases are worked out (by another method) in [26, Section 16].

We can obtain other expansions by varying the integration contours in (5.2). For example, consider the contour C_1' in the β plane as drawn in Figure 2. We retain the contour C_2 in the t plane as drawn in Figure 1. It is easily verified that the J and P operators on β–t space and on the solution space of the Helmholtz equation correspond, under the mapping (5.2) induced by this choice of contours.

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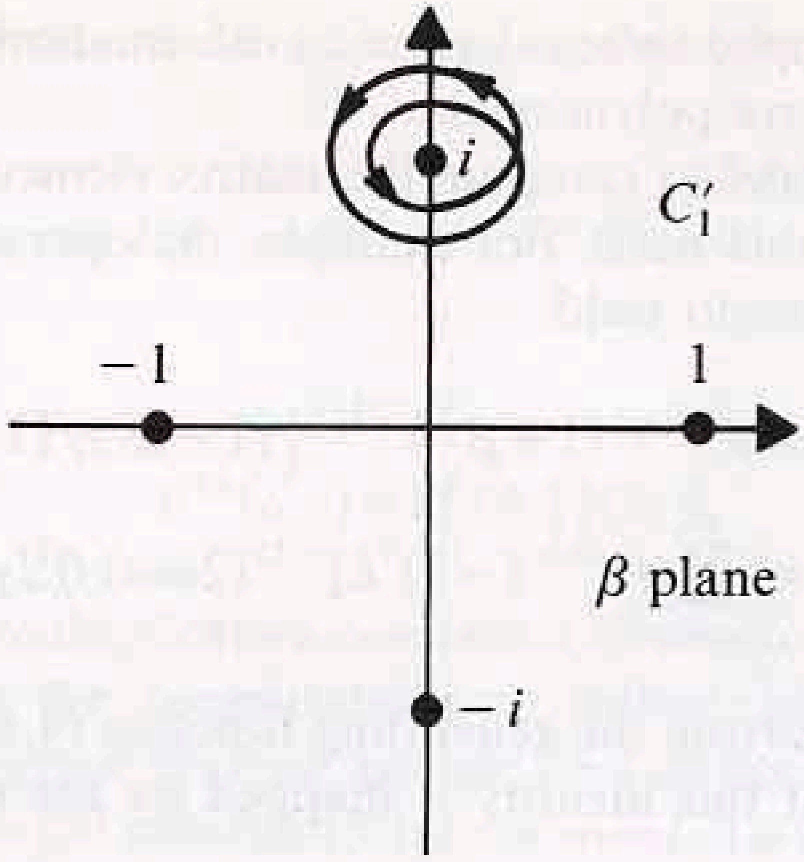

Figure 2

Now consider the eigenvalue equations for the parabolic system 8 (Table 14) in β–t space:

$$(\{J_1,P_2\}-\{J_2,P_1\})f^{(8)}_{\lambda,m}=-2i\lambda\omega f^{(8)}_{\lambda,m},\qquad iJ_3 f^{(8)}_{\lambda,m}=mf^{(8)}_{\lambda,m}.$$

It is straightforward to show that the eigenfunctions are

$$f^{(8)}_{\lambda,m}(\beta,t)=(1+\beta^2)^{-1/2}\left[(1+i\beta)/(1-i\beta)\right]^{\lambda/2}t^m,\qquad \lambda,m\in\mathcal{C}. \tag{5.13}$$

For convenience we restrict ourselves to the case where λ and m are integers. Then, substituting (5.13) into (5.2) for the contours C_1', C_2 and integrating, we find

$$\Psi^{(8)}_{\lambda m}(\mathbf{x})=I\left(f^{(8)}_{\lambda,m}\right)$$
$$=\frac{8\pi^2(i)^{|m|}(-1)^k k!}{(|m|+k)!}(i\omega\xi^2)^{|m|/2}(-i\omega\eta^2)^{|m|/2}$$
$$\times\exp\left[i\omega\frac{(\eta^2-\xi^2)}{2}\right]L_k^{(|m|)}(i\omega\xi^2)L_k^{(|m|)}(-i\omega\eta^2)e^{im\varphi}$$

if

$$\lambda=-|m|-2k-1,\ k=0,1,2,\ldots,\ m=0,\pm1,\pm2,\ldots,$$
$$\Psi^{(8)}_{\lambda m}(\mathbf{x})=0\qquad\text{otherwise.} \tag{5.14}$$

Here ξ,η,φ are parabolic coordinates

$$x=\xi\eta\cos\varphi,\qquad y=\xi\eta\sin\varphi,\qquad z=(\xi^2-\eta^2)/2.$$

(See [95] for the details of this computation.) Note that some nonzero

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functions $f^{(8)}_{\lambda,m}$ are mapped to zero by the transformation I. Here the $L^{(\alpha)}_n(z)$ are generalized Laguerre polynomials.

We can use our model to compute the matrix elements of the operators $\mathbf{T}(g)$ with respect to this basis. For example, the operator $\mathbf{T}(a)=\exp(aP_3)$ acts on the $\{f^{(8)}_{\lambda,m}\}$ basis to yield

$$\begin{aligned}\mathbf{T}(a)f^{(8)}_{\lambda,m}(\beta,t)&=e^{-a\omega\beta}(1+\beta^2)^{-1/2}\left[(1+i\beta)/(1-i\beta)\right]^{\lambda/2}t^m\\&=\sum_{s=0}^{\infty}e^{-i\omega a}(-1)^sL^{(-1)}_s(2ia\omega)f^{(8)}_{\lambda+2s,m}(\beta,t).\end{aligned}$$

This result is obtained from the generating function (4.11), Section 2.4. It is not hard to show that this identity is mapped by the transformation I to the identity

$$\mathbf{T}(a)\Psi^{(8)}_{\lambda m}(\mathbf{x})=\sum_{s=0}^{\infty}e^{-i\omega a}(-1)^sL^{(-1)}_s(2ia\omega)\Psi^{(8)}_{\lambda+2s,m}(\mathbf{x}).\tag{5.15}$$

(Note that the sum is actually finite.) Details of the computation as well as general $E(3)$ matrix elements with respect to the parabolic basis can be found in [95]. The first (nongroup-theoretic) proof of these expansion formulas was given by Hochstadt [50].

Next we consider identities for solutions of the Helmholtz equation which are derivable by Weisner's method. The natural setting for application of this method is the complex Helmholtz equation obtained by allowing all variables in equation (1.1) to assume complex values. To treat this equation systematically we should determine all complex analytic coordinate systems in which variables separate. Here, however, we will consider only a few separable systems that are of particular importance.

Of greatest practical importance is the spherical system

$$\mathbf{J}\cdot\mathbf{J}\Psi=-l(l+1)\Psi,\qquad J^0\Psi=m\Psi.\tag{5.16}$$

We will now study solutions Ψ of the complex Helmholtz equation that satisfy (5.16) in those cases where l and m are complex numbers, not necessarily integers. To treat the group-theoretic properties of these solutions it is convenient first to analyze the corresponding eigenfunctions in our complex sphere model. Thus we begin with the operators (5.4). In terms of the new complex variables τ, ρ where

$$\tau=t(1+\beta^2)^{1/2},\qquad \rho=-i\beta,\tag{5.17}$$

these operators assume the form

$$\begin{aligned}&J^+=-\tau\partial_\rho,\quad J^-=\tau^{-1}((1-\rho^2)\partial_\rho-2\rho\tau\partial_\tau),\quad J^0=\tau\partial_\tau,\\&P^+=\omega\tau,\qquad P^-=\omega(1-\rho^2)\tau^{-1},\qquad P^0=\omega\rho.\end{aligned}\tag{5.18}$$

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It follows easily from these expressions that the solution $f_l^{(l)}$ of the equations

$$J^0 f_l^{(l)} = l f_l^{(l)}, \qquad J^+ f_l^{(l)} = 0 \tag{5.19}$$

is

$$f_l^{(l)}(\rho,\tau) = \Gamma\left(l+\tfrac{1}{2}\right)(2\tau)^l,$$

unique to within a multiplicative constant. (The factor $\Gamma(l+\frac{1}{2})2^l$ is inserted for convenience in the computations to follow. Here l is an arbitrary complex constant except that we assume $l+\frac{1}{2}$ is not an integer. It follows from (5.19) that $\Psi = f_l^{(l)}$ satisfies (5.16) for $m=l$. To obtain more solutions we consider the expansion

$$\exp(\alpha J^-) f_l^{(l)} = \sum_{n=0}^{\infty} (\alpha^n/n!)(J^-)^n f_l^{(l)}. \tag{5.20}$$

Setting $f_{l-n}^{(l)} = [(-1)^n \Gamma(2l-n+1)/\Gamma(2l+1)](J^-)^n f_l^{(l)}$, $n=0,1,2,\ldots$, we see that the commutation relations for the J operators imply

$$\begin{gathered} J^0 f_m^{(l)} = m f_m^{(l)}, \qquad J^+ f_m^{(l)} = (m-l) f_{m+1}^{(l)}, \qquad J^- f_m^{(l)} = -(m+l) f_{m-1}^{(l)}, \\ \mathbf{J}\cdot\mathbf{J} f_m^{(l)} = -l(l+1) f_m^{(l)}, \qquad m = l, l-1, l-2, \ldots . \end{gathered} \tag{5.21}$$

Lie theory arguments applied to the left-hand side of (5.20) yield the generating function

$$\begin{gathered} \Gamma\left(l+\tfrac{1}{2}\right)\left[2\tau - 4\alpha\rho - 2\alpha^2(1-\rho^2)/\tau\right]^l = \sum_{n=0}^{\infty} (-\alpha)^n \binom{2l}{n} j_n^{(l)}(\rho)\tau^{l-n}, \\ f_m^{(l)}(\rho,\tau) = j_n^{(l)}(\rho)\tau^{l-n}, \qquad m = l-n, \end{gathered} \tag{5.22}$$

valid for $\tau \neq 0$ and α in a sufficiently small neighborhood of 0. Comparing coefficients of α^n on both sides of this equation, we find

$$f_m^{(l)}(\rho,\tau) = \Gamma(l-m+1)\Gamma\left(m+\tfrac{1}{2}\right) C_{l-m}^{m+\frac{1}{2}}(\rho)(2\tau)^m \tag{5.23}$$

where $C_n^{\nu}(x)$ is a Gegenbauer (ultraspherical) polynomial (B.6ii). This polynomial is commonly defined by the generating function

$$(1-2\alpha x+\alpha^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)\alpha^n, \tag{5.24}$$

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whose group-theoretic significance will be explained in Section 3.7. For l a positive integer and $m=l,l-1,\ldots,-l$, the functions (5.23) are proportional to complexifications of the spherical harmonics Y_l^m. However, we shall be primarily interested in the case where $2l$ is not an integer.

From the recurrence relation for the Gegenbauer polynomials,

$$xC_n^\nu(x)=\frac{n+1}{2(\nu+n)}C_{n+1}^\nu(x)+\frac{(2\nu+n-1)}{2(\nu+n)}C_{n-1}^\nu(x),$$

which can be verified directly from either (5.24) or (B.6ii), it follows that

$$P^0 f_m^{(l)}=\frac{\omega}{2l+1}f_m^{(l+1)}+\frac{\omega(l+m)(l-m)}{2l+1}f_m^{(l-1)}. \tag{5.25}$$

Furthermore, from the commutation relations $[P^0,J^\pm]=\pm P^\pm$ it follows that

$$\begin{aligned} P^+f_m^{(l)}&=\frac{\omega}{2l+1}f_{m+1}^{(l+1)}-\frac{\omega(l-m)(l-m-1)}{2l+1}f_{m+1}^{(l-1)},\\ P^-f_m^{(l)}&=\frac{-\omega}{2l+1}f_{m-1}^{(l+1)}+\frac{\omega(l+m)(l+m-1)}{2l+1}f_{m-1}^{(l-1)}. \end{aligned} \tag{5.26}$$

Relations (5.21), (5.25), (5.26) determine the action of $\mathcal{E}(3)$ on the basis $\{f_m^{(l)}\}$ where $l=l_0,l_0\pm1,l_0\pm2,\ldots,m=l,l-1,\ldots,$ and $2l_0$ is not an integer. (As is well known, the simple form of (5.25) is related to the fact that the Gegenbauer polynomials are orthogonal with respect to a suitable measure [37, Chapter X]. This property of these polynomials, along with many others, is related to the wave equation and will be studied in the next chapter.)

It is well known that any entire function of x can be expanded uniquely in a series of Gegenbauer polynomials $C_n^\nu(x)$, $n=0,1,2,\ldots$ ($2\nu\neq$ integer), uniformly convergent in compact subsets of the complex plane (e.g., [116, p. 238]). Thus we can exponentiate the P and J operators, and compute the matrix elements of these operators in an $\{f_m^{(l)}\}$ basis. The rather complicated results are presented in [83]. Except for (5.20), (5.22), we present here only one of these results: An induction argument based on the operator relation (5.25) shows that

$$e^{\alpha\rho}=(2/\alpha)^\nu\Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)I_{\nu+n}(\alpha)C_n^\nu(\rho),\qquad \nu,\alpha\in\mathcal{C}; \tag{5.27}$$

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that is,

$$\exp(\alpha P^0) f_l^{(l)} = \left(\frac{2}{\alpha}\right)^{l+\frac{1}{2}} \Gamma\left(l+\frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{\left(l+n+\frac{1}{2}\right)}{n!} I_{l+n+\frac{1}{2}}(\alpha) f_l^{(l+n)}. \quad (5.27')$$

Here, $I_\nu(\alpha) = \exp(-i\nu\pi/2) J_\nu[\alpha \exp(i\pi/2)]$ is a modified Bessel function [37].

Now we consider the relationship between these results and solutions of the complex Helmholtz equation in the spherical basis. Instead of the complex spherical coordinates r, θ, φ (5 in Table 14), it is more convenient to use the equivalent separable coordinates

$$\rho = -\cos\theta, \qquad \tau = -e^{i\varphi}\sin\theta, \qquad s = ir. \quad (5.28)$$

In terms of these coordinates the symmetry operators for the Helmholtz equation are

$$J^+ = -\tau\partial_\rho, \qquad J^- = \tau^{-1}\left((1-\rho^2)\partial_\rho - 2\rho\tau\partial_\tau\right), \qquad J^0 = \tau\partial_\tau,$$

$$P^+ = \tau\partial_s - \frac{\rho\tau}{s}\partial_\rho - \frac{\tau^2}{s}\partial_\tau,$$

$$P^- = \frac{(1-\rho^2)}{\tau}\partial_s - \frac{\rho(1-\rho^2)}{s\tau}\partial_\rho + \frac{(\rho^2+1)}{s}\partial_\tau,$$

$$P^0 = \rho\partial_s + \frac{(1-\rho^2)}{s}\partial_\rho - \frac{\rho\tau}{s}\partial_\tau. \quad (5.29)$$

We search for a set of solutions $\{\Psi_m^{(l)}(\mathbf{x})\}$ of the Helmholtz equation which satisfy the recurrence relations (5.21), (5.25), (5.26) when acted on by the symmetry operators (5.29). Since the J operators in (5.18) and (5.29) are identical, it follows that

$$\Psi_m^{(l)}(x) = S^{(l)}(s) f_m^{(l)}(\rho, \tau).$$

Substituting this expression into (5.25) and (5.26), we find that the $S^{(l)}$ must satisfy the recurrence formulas

$$\left(\frac{d}{ds} - \frac{l}{s}\right) S^{(l)}(s) = \omega S^{(l+1)}(s), \qquad \left(\frac{d}{ds} + \frac{l+1}{s}\right) S^{(l)}(s) = \omega S^{(l-1)}(s). \quad (5.30)$$

It follows that $s^{1/2} S^{(l)}(s)$ is a solution of the modified Bessel equation and

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that the choices

$$S^{(l)}(s)=(\omega s)^{-1/2}I_{l+\frac{1}{2}}(\omega s) \quad \text{or} \quad (\omega s)^{-1/2}I_{-l-\frac{1}{2}}(\omega s) \tag{5.31}$$

separately satisfy the recurrence formulas. Adopting the first of these choices, we conclude that the functions

$$\Psi_m^{(l)}(s,\rho,\tau)=(l-m)!\Gamma\left(m+\tfrac{1}{2}\right)(\omega s)^{-1/2}I_{l+\frac{1}{2}}(\omega s)C_{l-m}^{m+\frac{1}{2}}(\rho)(2\tau)^m \tag{5.32}$$

and the operators (5.29) satisfy the recurrence formulas (5.21), (5.25), (5.26). Thus the matrix elements giving the $E(3)$ group action that were computed for the $\{f_m^{(l)}\}$ basis are also valid for the $\{\Psi_m^{(l)}\}$ basis. For example, (5.27) leads to the addition theorem of Gegenbauer:

$$I_{l+\frac{1}{2}}(sS)(2S)^{-l-\frac{1}{2}}=\Gamma\left(l+\tfrac{1}{2}\right)\sum_{n=0}^{\infty}\left(l+n+\tfrac{1}{2}\right)I_{l+n+\frac{1}{2}}(s)I_{l+n+\frac{1}{2}}(\gamma)C_n^{l+\frac{1}{2}}(\rho),$$
$$S=\left(1+2\gamma\rho/s+\gamma^2/s^2\right)^{1/2}, \qquad |2\gamma\rho/s+\gamma^2/s^2|<1. \tag{5.33}$$

We can also use the complex sphere model to prove operational identities relating solutions of the Helmholtz equation. For example, from (5.18), (5.23) we obtain the virtually trivial identity

$$(l-m)!\,C_{l-m}^{m+\frac{1}{2}}(\omega^{-1}P^0)f_m^{(m)}=f_m^{(l)}, \qquad l-m=0,1,2,\ldots. \tag{5.34}$$

However, for the model (5.29), (5.32) this identity assumes the nontrivial form

$$C_{l-m}^{m+\frac{1}{2}}\left(\rho\partial_s+\frac{(1-\rho^2)}{s}\partial_\rho-\frac{\rho m}{s}\right)I_{m+\frac{1}{2}}(\omega)s^{-1/2}=I_{l+\frac{1}{2}}(s)C_{l-m}^{m+\frac{1}{2}}(\rho)s^{-1/2}. \tag{5.35}$$

Many other operational identities and addition theorems can be found in [83].

Weisner's method in its general form can also be applied to derive identities for spherical waves. For example, consider the (cylindrical wave) solution of the simultaneous equations

$$(\mathbf{P}\cdot\mathbf{P}+\omega^2)\Psi=0, \qquad P^0\Psi=\lambda\Psi, \qquad J^0\Psi=m\Psi, \qquad \lambda,m\in\mathcal{C},$$
$$\Psi(s,\rho,\tau)=\left[\tau(\rho^2-1)^{1/2}(\lambda^2-1)^{1/2}\right]^m e^{\omega\lambda s\rho}I_{\pm m}\left(\omega s(\rho^2-1)^{1/2}(\lambda^2-1)^{1/2}\right). \tag{5.36}$$

ISBN-0-201-13503-5

Choosing the I_m solution, we note the validity of the expansion

$$\Psi(s,\rho,\tau)=(\omega s)^{-1/2}\sum_{n=0}^{\infty} a_n(\lambda) I_{m+n+\frac{1}{2}}(\omega s) C_n^{m+\frac{1}{2}}(\rho)\tau^m$$

expressing Ψ as a sum of spherical wave solutions of the Helmholtz equation. It remains only to compute $a_n(\lambda)$. Since Ψ is symmetric in ρ and λ, we have $a_n(\lambda)=b_n C_n^{m+\frac{1}{2}}(\lambda)$. Furthermore, if $\lambda=1$, then

$$\Psi(s,\rho,\tau)=\frac{(\omega s\tau/2)^m e^{\omega s\rho}}{\Gamma(m+1)}$$

and the identity (5.27) permits computation of the coefficients $a_n(\lambda)$ with the final result ($\omega=1$)

$$\begin{aligned}&\left[(\rho^2-1)(\lambda^2-1)\right]^{-m/2} e^{s\lambda\rho} I_m\left(s\left[(\rho^2-1)(\lambda^2-1)\right]^{1/2}\right)\\ &=\frac{2^{2m+1}}{(2\pi s)^{1/2}}\Gamma\left(m+\tfrac{1}{2}\right)^2\sum_{n=0}^{\infty}\frac{n!\left(m+n+\frac{1}{2}\right)}{\Gamma(2m+n+1)} I_{m+n+\frac{1}{2}}(s) C_n^{m+\frac{1}{2}}(\rho) C_n^{m+\frac{1}{2}}(\lambda),\end{aligned} \tag{5.37}$$

convergent for all $\rho,\lambda\in\mathcal{C}$ (see [37, p. 102]).

Another example is provided by the solutions (5.14) corresponding to the parabolic system 8 in Table 14. Expressing these solutions in terms of coordinates (5.28) and expanding in the spherical basis, we obtain

$$e^{s\rho}L_k^{(m)}(-s(1+\rho))L_k^{(m)}(s(1-\rho))=\sum_{n=0}^{\infty} a_n s^{-m-\frac{1}{2}} I_{m+n+\frac{1}{2}}(s) C_n^{m+\frac{1}{2}}(\rho). \tag{5.38}$$

The coefficients a_n can be determined by setting $\rho=\alpha/s$ and letting $s\to 0$ to obtain

$$2^{m+\frac{1}{2}}\Gamma\left(m+\tfrac{1}{2}\right)e^{\alpha}\left[L_k^{(m)}(-\alpha)\right]^2=\sum_{n=0}^{\infty}\frac{a_n\alpha^n}{n!\left(m+n+\frac{1}{2}\right)}.$$

Use of the transformation formula for a ${}_1F_1$ (Appendix B, Section 3) allows us to explicitly compute the coefficient of α^n on the left-hand side of this equation, with the result

$$\begin{aligned}a_n=&\frac{2^{m+\frac{1}{2}}\left(m+n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)\Gamma(m+k+1)\Gamma(m+k+n+1)}{(k!)^2\Gamma(m+1)\Gamma(m+n+1)}\\ &\times {}_3F_2\left(\left.\begin{matrix}-k,-m-n,-n\\ m+1,-m-k-n\end{matrix}\right|1\right).\end{aligned} \tag{5.39}$$

For $k=0$ this expression reduces to (5.27).

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3.6 The Laplace Equation $\Delta_3\Psi=0$

The known coordinate systems that permit R-separation of variables in the real Laplace equation

$$\Delta_3\Psi(\mathbf{x})=0, \qquad \mathbf{x}=(x_1,x_2,x_3)=(x,y,z), \tag{6.1}$$

are derived and studied in the classic book of Bôcher [17]. However, the explicit relationship between these systems and the symmetry group of (6.1) has been discussed only very recently [22]. Apart from the trivial symmetry E, the symmetry algebra of this equation is ten dimensional with basis

$$\begin{aligned}
&P_j=\partial_j=\partial_{x_j}, \qquad j=1,2,3; \qquad J_3=x_2\partial_1-x_1\partial_2,\\
&J_2=x_1\partial_3-x_3\partial_1, \quad J_1=x_3\partial_2-x_2\partial_3, \quad D=-(\tfrac{1}{2}+x_1\partial_1+x_2\partial_2+x_3\partial_3),\\
&K_1=x_1+(x_1^2-x_2^2-x_3^2)\partial_1+2x_1x_3\partial_3+2x_1x_2\partial_2,\\
&K_2=x_2+(x_2^2-x_1^2-x_3^2)\partial_2+2x_2x_3\partial_3+2x_2x_1\partial_1,\\
&K_3=x_3+(x_3^2-x_1^2-x_2^2)\partial_3+2x_3x_1\partial_1+2x_3x_2\partial_2.
\end{aligned} \tag{6.2}$$

The P_l and J_l operators generate a subalgebra isomorphic to $\mathcal{E}(3)$ and D is the generator of dilatations. The operators K_j are generators of *special conformal transformations* and will be discussed later. Only the elements of the $\mathcal{E}(3)$ subalgebra actually commute with the Laplace operator Δ_3. The remaining elements of the Lie algebra merely leave the solution space of (6.1) invariant.

The symmetry algebra of the Laplace equation is isomorphic to $so(4,1)$, the Lie algebra of all real 5×5 matrices $\mathcal{A}$ such that $\mathcal{A}G^{4,1}+G^{4,1}\mathcal{A}'=0$ where

$$G^{4,1}=\begin{bmatrix}1&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&-1\end{bmatrix}=\sum_{j=1}^{4}\mathcal{E}_{jj}-\mathcal{E}_{55}. \tag{6.3}$$

Here $\mathcal{E}_{ij}$ is the 5×5 matrix with a one in row i, column j, and zeros everywhere else.

$$\mathcal{E}_{ij}=\begin{array}{cc} \begin{array}{ccc} & j & \end{array} & \\ \begin{bmatrix} & & \\ & & \\ & 1 & \end{bmatrix} & \begin{array}{c} \\ \\ i \end{array} \end{array}. \tag{6.4}$$

ISBN-0-201-13503-5

A basis for $so(4,1)$ is provided by the ten elements

$$\begin{aligned} \Gamma_{ab}&=\mathcal{E}_{ab}-\mathcal{E}_{ba}=-\Gamma_{ba}, \quad 1\leqslant a,b\leqslant 4,\\ \Gamma_{a5}&=\mathcal{E}_{a5}+\mathcal{E}_{5a}=\Gamma_{5a} \end{aligned} \tag{6.5}$$

with commutation relations

$$\begin{aligned} &[\Gamma_{ab},\Gamma_{cd}]=\delta_{bc}\Gamma_{ad}+\delta_{ad}\Gamma_{bc}+\delta_{ca}\Gamma_{db}+\delta_{db}\Gamma_{ca},\\ &[\Gamma_{a5},\Gamma_{cd}]=-\delta_{ad}\Gamma_{c5}+\delta_{ac}\Gamma_{d5}, \qquad [\Gamma_{a5},\Gamma_{b5}]=\Gamma_{ab}. \end{aligned} \tag{6.6}$$

One can verify that the correct commutation relations for the operators (6.2) result if the following identifications are made.

$$\begin{aligned} &J_3=\Gamma_{32}, \quad J_2=\Gamma_{24}, \quad J_1=\Gamma_{43}, \quad D=\Gamma_{15},\\ &P_1=\Gamma_{12}+\Gamma_{25}, \quad P_2=\Gamma_{13}+\Gamma_{35}, \quad P_3=\Gamma_{14}+\Gamma_{45},\\ &K_1=\Gamma_{12}-\Gamma_{25}, \quad K_2=\Gamma_{13}-\Gamma_{35}, \quad K_3=\Gamma_{14}-\Gamma_{45}. \end{aligned} \tag{6.7}$$

The symmetry group of (6.1), the *conformal group*, is thus locally isomorphic to $SO(4,1)$, the group of all real 5×5 matrices A such that

$$AG^{4,1}A^{t}=G^{4,1}. \tag{6.8}$$

The identity component of this group consists of those matrices satisfying (6.8), $\det A=1$, and $A_{55}\geqslant 1$. The Lie algebra of $SO(4,1)$ is $so(4,1)$ [46].

Exponentiating the operators (6.2), we can obtain the local action of $SO(4,1)$ as a transformation group of symmetry operators. In particular, the linear momentum and angular momentum operators generate the subgroup of symmetries (1.12) isomorphic to $E(3)$; the dilatation operator generates

$$\exp(\lambda D)\Psi(\mathbf{x})=\exp(-\lambda/2)\Psi[\exp(-\lambda)\mathbf{x}], \qquad \lambda\in R; \tag{6.9}$$

and the K_j generate the special conformal transformations

$$\begin{aligned} &\exp(a_1K_1+a_2K_2+a_3K_3)\Psi(\mathbf{x})\\ &\quad=[1-2\mathbf{x}\cdot\mathbf{a}+(\mathbf{a}\cdot\mathbf{a})(\mathbf{x}\cdot\mathbf{x})]^{-1/2}\Psi\left(\frac{\mathbf{x}-\mathbf{a}(\mathbf{x}\cdot\mathbf{x})}{1-2\mathbf{a}\cdot\mathbf{x}+(\mathbf{a}\cdot\mathbf{a})(\mathbf{x}\cdot\mathbf{x})}\right). \end{aligned} \tag{6.10}$$

In addition, we shall consider the inversion and space reflection symmetries of the Laplace equation:

$$\begin{aligned} I\Psi(\mathbf{x})&=(\mathbf{x}\cdot\mathbf{x})^{-1/2}\Psi(\mathbf{x}/\mathbf{x}\cdot\mathbf{x}), \qquad I=I^{-1},\\ R\Psi(\mathbf{x})&=\Psi(-x_1,x_2,x_3), \qquad R=R^{-1}. \end{aligned} \tag{6.11}$$

ISBN-0-201-13503-5

These are well-known symmetries of (6.1) that are not generated by the infinitesimal operators (6.2) [12, p. 31]. It follows from the definitions of these operators that

$$IP_jI^{-1} = -K_j, \qquad IDI^{-1} = -D, \qquad IJ_jI^{-1} = J_j. \tag{6.12}$$

By a tedious computation we can verify that the Laplace equation is class I. Furthermore, although the space of symmetric second-order operators in the enveloping algebra of $so(4,1)$ is 55 dimensional, on the solution space of (6.1) there are 20 linearly independent relations between these operators. Thus, only 35 operators can be regarded as linearly independent on the solution space. For example, we have the relations

$$\begin{aligned}
&\text{(i)} \quad \mathbf{P}\cdot\mathbf{P} = \mathbf{K}\cdot\mathbf{K} = 0,\\
&\text{(ii)} \quad \mathbf{J}\cdot\mathbf{J} = \tfrac{1}{4} - D^2,\\
&\text{(iii)} \quad \Gamma_{45}^2 + \Gamma_{41}^2 - \Gamma_{51}^2 = \tfrac{1}{4} + \Gamma_{23}^2,\\
&\text{(iv)} \quad \{P_1, K_1\} + \{P_2, K_2\} + \{P_3, K_3\} = 2 + 4D^2.
\end{aligned} \tag{6.13}$$

(Note that these relations are valid only on the solution space of (6.1), not in general. We are considering the $\Gamma_{\alpha\beta}$ as differential operators on this space via the definitions (6.7).)

The reader may be wondering why we have not applied a similar analysis to the Laplace equation $\Delta_2\Psi(\mathbf{x}) = 0$. The reason is that the symmetry algebra of this equation is infinite dimensional. In fact, every transformation $\Psi(x,y) \to \Psi(u(x,y), v(x,y))$, where $u + iv = f(z), z = x + iy$, and $f(z)$ is an analytic function, defines a symmetry of the Laplace equation. The group of all analytic transformations $z \to f(z)$ is the symmetry group of this equation, but it is not a Lie group. (Indeed each group transformation is determined by an infinite number of parameters $\{a_n\}$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n$.) Thus, Lie theory methods are not particularly useful for this Laplace equation. It can be shown that infinite-dimensional symmetry algebras can occur for second-order partial differential equations in n variables only in the case where $n = 2$ [105].

We now return to the separation of variables problem for equation (6.1). We will see that each R-separable coordinate system is characterized by a pair of commuting second-order symmetry operators in the enveloping algebra of $so(4,1)$. As usual, two coordinate systems will be regarded as equivalent if one can be obtained from the other by a transformation from the connected component of the identity of the conformal group, augmented by the discrete symmetries (6.11).

Note first that the eleven separable coordinate systems for the Helmholtz equation, listed in Table 14, are also separable for the Laplace

ISBN-0-201-13503-5

equation. The separation equations can be obtained from the corresponding Helmholtz results by setting $\omega=0$ in Section 3.1. We briefly indicate the form of the separated solutions Ψ of the eigenvalue equations $S_j\Psi=\lambda_j\Psi, \Delta_3\Psi=0$.

For the Cartesian system 1 the solutions take the form

$$\exp(\alpha x+\beta y+\gamma z), \qquad \alpha^2+\beta^2+\gamma^2=0, \tag{6.14}$$

whereas for cylindrical coordinates 2 they are

$$\begin{gathered}\Psi^{(2)}_{\lambda,n}(r,\varphi,z)=J_{\pm n}(\lambda r)\exp(\lambda z+in\varphi),\\ iJ_3\Psi_{\lambda,n}=n\Psi_{\lambda,n}, \qquad P_3\Psi_{\lambda,n}=\lambda\Psi_{\lambda,n}.\end{gathered} \tag{6.15}$$

The results for parabolic cylinder coordinates 3 are

$$\begin{gathered}\Psi^{(3)}_{\lambda,\mu}(\xi,\eta,z)=D_{i\mu-\frac{1}{2}}(\pm\sigma\xi)D_{-i\mu-\frac{1}{2}}(\pm\sigma\eta)e^{\lambda z},\\ \sigma=\exp(i\pi/4)(2\lambda)^{1/2},\\ P_3\Psi_{\lambda,\mu}=\lambda\Psi_{\lambda,\mu}, \qquad \{J_3,P_2\}\Psi_{\lambda,\mu}=2\mu\lambda\Psi_{\lambda,\mu},\end{gathered} \tag{6.16}$$

and for elliptic cylinder coordinates 4 they are

$$\begin{gathered}\Psi^{(4)}_{\lambda,n}(\alpha,\beta,z)=\begin{cases} Ce_n(\alpha,q)ce_n(\beta,q)e^{\lambda z},\\ Se_n(\alpha,q)se_n(\beta,q)e^{\lambda z},\end{cases} \qquad q=d^2\lambda^2/4,\\ (J_3^2+d^2P_1^2)\Psi_{\lambda,n}=\mu_n\Psi_{\lambda,n}, \qquad P_3\Psi_{\lambda,n}=\lambda\Psi_{\lambda,n}.\end{gathered} \tag{6.17}$$

Corresponding to spherical coordinates 5 we have solutions

$$\begin{gathered}\Psi^{(5)}_{l,m}(\rho,\theta,\varphi)=\begin{Bmatrix}\rho^l\\ \rho^{-l-1}\end{Bmatrix}P_l^m(\theta)e^{im\varphi}, \qquad iJ_3\Psi^{(5)}_{l,m}=m\Psi^{(5)}_{l,m},\\ \mathbf{J}\cdot\mathbf{J}\Psi^{(5)}_{l,m}=-l(l+1)\Psi^{(5)}_{l,m}.\end{gathered} \tag{6.18}$$

For prolate spheroidal coordinates 6 the separated equations take the form (1.20) with $\omega=0$ and typical solutions are

$$P_n^m(\cosh\eta)P_n^m(\cos\alpha)e^{im\varphi}; \tag{6.19}$$

that is, (1.21) with $\omega=0$. Similarly, for oblate spheroidal coordinates 7 the separated equations are (1.22) with $\omega=0$ and the eigenfunctions are of the form

$$P_n^m(-i\sinh\eta)P_n^m(\cos\alpha)e^{im\varphi}. \tag{6.20}$$

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For parabolic coordinates 8 the separated equations are (1.24) with $\omega=0$ and the separated solutions are

$$J_{\pm m}(i\sqrt{\lambda}\,\xi)J_{\pm m}(\sqrt{\lambda}\,\eta)e^{im\varphi}. \tag{6.21}$$

For paraboloidal coordinates 9 the separated equations are (1.26) with $\omega=0$ and the separated solutions are Mathieu functions of the form

$$\begin{aligned}&\mathrm{Ce}_n(\alpha,-\lambda c/2)\,\mathrm{ce}_n(\beta,-\lambda c/2)\,\mathrm{Ce}_n(\gamma+i\pi/2,-\lambda c/2),\\&\mathrm{Se}_n(\alpha,-\lambda c/2)\,\mathrm{se}_n(\beta,-\lambda c/2)\,\mathrm{Se}_n(\gamma+i\pi/2,-\lambda c/2).\end{aligned} \tag{6.22}$$

For ellipsoidal coordinates 10 the separation equations take the form (1.29) or (1.33) with $\omega=0$. Thus the three separation equations reduce to the Lamé equation and the single-valued solutions in R^3 are products of three Lamé polynomials (see [7, p. 228]).

Finally, for conical coordinates 11 the separation equations are (1.35) with $\omega=0$. The single-valued solutions in R^3 take the form

$$\left\{\begin{matrix} r^l \\ r^{-l-1}\end{matrix}\right\} E_{ln}^{p,q}(\alpha)E_{ln}^{p,q}(\beta), \qquad l=0,1,2,\ldots, \tag{6.23}$$

where the E functions are Lamé polynomials; see (3.24). Such products of Lamé polynomials are called *ellipsoidal harmonics* in analogy with the spherical harmonics $Y_l^m(\theta,\varphi)$, (6.18) [136a]. The overlap functions relating spherical and ellipsoidal harmonics have already been computed in Section 3.3.

The remaining separable coordinate systems for the Laplace equation are purely R-separable and do not lead to separation for the Helmholtz equation. The coordinate surfaces for these systems are orthogonal families of confocal cyclides. A cyclide is a surface with equation

$$a(x^2+y^2+z^2)^2+P(x,y,z)=0 \tag{6.24}$$

where a is a constant and P is a polynomial of order two. If $a=0$, the cyclide reduces to a quadric surface. Now it is well known that the coordinate surfaces of the eleven separable systems listed in Table 14 are confocal families of quadrics

$$\frac{x^2}{a_1+\lambda}+\frac{y^2}{a_2+\lambda}+\frac{z^2}{a_3+\lambda}=1, \qquad a_j\,\text{constant}, \tag{6.25}$$

and their limiting cases, (see [13, 97, 98, 136a]). In particular, all these coordinates are limiting cases of the ellipsoidal coordinates and the coordinate surfaces are ellipsoids, hyperboloids, and their various limits, such as paraboloids, spheres, and planes.

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We know that under any conformal symmetry of the Laplace equation an R-separable system is mapped to an R-separable system. However, the inversion operator I, (6.11), maps a quadric surface to a cyclide with $a\neq 0$, as the reader can easily verify. Thus one cannot avoid the appearance of cyclides in the study of R-separable coordinate systems for the Laplace equation.

It is straightforward to check that the family of all cyclides is invariant under the action of the conformal group and that this group maps orthogonal surfaces to orthogonal surfaces. Instead of using families of confocal quadrics to construct orthogonal coordinate systems, one can more generally use families of confocal cyclides. By direct computation it can be shown that such families define orthogonal, R-separable coordinate systems for the Laplace equation. Moreover, all separable systems for the Laplace equation can be obtained in this manner.

Since we regard coordinate systems related by a transformation from the conformal group as equivalent, to obtain all distinct cyclidic systems it is obviously necessary to decompose the family of cyclides (6.24) into conformal equivalence classes. Among the equivalence classes of cyclides are some which contain cyclides (6.24) with $a=0$. These correspond to the eleven separable systems listed in Table 14. The remaining classes contain only cyclides with $a\neq 0$ and lead to new R-separable systems. The details of this construction can be found in the classic book of Bôcher [17]. Our primary aim is to provide a group-theoretic characterization of the coordinate systems listed by Bôcher. This characterization was first given in [22] and is contained in Table 17.

For each coordinate system $\{\mu,\nu,\rho\}$ the R-separable solutions of (6.1) take the form $\Psi(\mathbf{x})=\mathfrak{R}^{1/2}(\mu,\nu,\rho)A(\mu)B(\nu)C(\rho)$ and these solutions are characterized by the eigenvalue equations $S_j\Psi=\lambda_j\Psi$ where λ_1,λ_2 are the separation constants.

More specifically, for system 12 the parameters vary over the range

$$0<\rho<1<\nu<b<\mu<a$$

and each factor in the separated solution satisfies the equation

$$\left[(f(\xi))^{1/2}\frac{d}{d\xi}(f(\xi))^{1/2}\frac{d}{d\xi}-\left(\frac{3}{16}\xi^2+\frac{\lambda_1}{4}\xi+\frac{\lambda_2}{4}\right)\right]A(\xi)=0, \qquad (6.26)$$

$$f(\xi)=(\xi-a)(\xi-b)(\xi-1)\xi,\ \xi=\mu,\nu,\rho.$$

Here (6.26) is the standard form of an equation with five elementary singularities [51, p. 500]. Very little is known about the solutions. For

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Table 17 Additional R-Separable Systems for the Laplace Equation

	Commuting operators S_1, S_2	Separable coordinates
12	$S_1=\frac{a+1}{4}(P_2+K_2)^2+\frac{b+1}{4}(P_1+K_1)^2$ $+\frac{a+b}{4}(P_3+K_3)^2+J_3^2+bJ_2^2+aJ_1^2,$ $S_2=\frac{a}{4}(P_2+K_2)^2+\frac{b}{4}(P_1+K_1)^2$ $+\frac{ab}{4}(P_3+K_3)^2$	$x=\Re^{-1}\left[\frac{(\mu-a)(\nu-a)(\rho-a)}{(b-a)(a-1)a}\right]^{1/2},$ $y=\Re^{-1}\left[\frac{(\mu-b)(\nu-b)(\rho-b)}{(a-b)(b-1)b}\right]^{1/2},$ $z=\Re^{-1}\left[\frac{(\mu-1)(\nu-1)(\rho-1)}{(a-1)(b-1)}\right]^{1/2},$ $\Re=1+\left[\frac{\mu\nu\rho}{ab}\right]^{1/2}$
13	$S_1=2\alpha J_3^2+\frac{\alpha+1}{2}\{P_2,K_2\}+\frac{\beta}{2}(P_2^2-K_2^2)$ $+\frac{\alpha}{2}\{P_1,K_1\}+\frac{\beta}{2}(K_1^2-P_1^2),$ $S_2=\frac{\alpha}{2}\{P_2,K_2\}+\frac{\beta}{2}(P_2^2-K_2^2)$ $+(\alpha^2+\beta^2)J_3^2$	$x=\Re^{-1}\left[\frac{(\mu-1)(\nu-1)(\rho-1)}{(a-1)(b-1)}\right]^{1/2},$ $y=\Re^{-1}\left[-\frac{\mu\nu\rho}{ab}\right]^{1/2},$ $z=\Re^{-1}$ $\Re=2\,\mathrm{Re}\left[-\frac{i(\mu-a)(\nu-a)(\rho-a)}{(a-b)(a-1)a}\right]^{1/2},$ $a=\bar{b}=\alpha+i\beta, \alpha,\beta$ real
14	$S_1=J_3^2$ $4S_2=(P_3+K_3)^2-a(P_3-K_3)^2$	$x=\Re^{-1}\cos\varphi,$ $y=\Re^{-1}\sin\varphi,$ $z=\Re^{-1}\left[-\frac{\mu\rho}{a}\right]^{1/2},$ $\Re=\left[\frac{(\mu-a)(a-\rho)}{a(a-1)}\right]^{1/2}-\left[\frac{(\mu-1)(1-\rho)}{a-1}\right]^{1/2}$
15	$S_1=J_3^2$ $4S_2=-4aD^2-(P_3-K_3)^2$	$x=\Re^{-1}\cos\varphi,$ $y=\Re^{-1}\sin\varphi,$ $z=\Re^{-1}\left[\frac{(\mu-a)(a-\rho)}{a(a-1)}\right]^{1/2},$ $\Re=\left[\frac{\mu\rho}{a}\right]^{1/2}+\left[\frac{(\mu-1)(\rho-1)}{(a-1)}\right]^{1/2}$
16	$S_1=J_3^2$ $2S_2=\alpha\{P_3,K_3\}+\beta(K_3^2-P_3^2)$	$x=\Re^{-1}\cos\varphi,$ $y=\Re^{-1}\sin\varphi,$ $z=\Re^{-1}\left[-\frac{\mu\rho}{ab}\right]^{1/2},$ $\Re=2\,\mathrm{Re}\left[\frac{i(\rho-a)(\mu-a)}{a(a-b)}\right]^{1/2},$ $a=\bar{b}=\alpha+i\beta$
17	$S_1=J_3^2$ $4S_2=(P_3+K_3)^2$	$x=\Re^{-1}\sinh\xi\cos\varphi,$ $y=\Re^{-1}\sinh\xi\sin\varphi,$ $z=\Re^{-1}\cos\psi,$ $\Re=\cosh\xi+\sin\psi$

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system 13 the parameters vary in the range

$$-\infty<\rho<0<\mu<1<\nu<\infty.$$

The separated equations are (6.26) with $a=\bar{b}=\alpha+i\beta$. For system 14 the parameters vary in the range $\mu>a>1, \rho<0, 0\leqslant\varphi<2\pi$ and the solutions of Laplace's equation have the form $\Psi=\mathfrak{R}^{1/2}E_1(\mu)E_2(\rho)e^{im\varphi}$ where

$$\left[4(\mathcal{P}(\xi))^{1/2}\frac{d}{d\xi}(\mathcal{P}(\xi))^{1/2}\frac{d}{d\xi}+\left(\frac{1}{4}-m^2\right)\xi-\lambda\right]E_j(\xi)=0,$$

$$j=1,2,\quad \xi=\mu,\rho,\quad \mathcal{P}(\xi)=(\xi-a)(\xi-1)\xi, \tag{6.27}$$

$$iJ_3\Psi=m\Psi,\qquad S_2\Psi=\lambda\Psi.$$

If we set $\mu=\mathrm{sn}^2(\alpha,k), \rho=\mathrm{sn}^2(\beta,k)$ where $a=k^{-2}$, then we find

$$x=\mathfrak{R}^{-1}\cos\varphi,\qquad y=\mathfrak{R}^{-1}\sin\varphi,\qquad z=ik\mathfrak{R}^{-1}\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta,$$

$$\mathfrak{R}=i(k')^{-1}\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta-i(kk')^{-1}\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta \tag{6.28}$$

$$\Psi=\mathfrak{R}^{1/2}\Lambda^p_{m-\frac{1}{2}}(\alpha,k)\Lambda^p_{m-\frac{1}{2}}(\beta,k)e^{im\varphi}$$

where $\Lambda^p_n(z,k)$ is a solution of the Lamé equation

$$\frac{d^2\Lambda}{dz^2}+\left(h^p_n-n(n+1)k^2\mathrm{sn}^2(z,k)\right)\Lambda=0. \tag{6.29}$$

The parameters α,β range over the intervals $\alpha\in[iK',iK'+2K], \beta\in[2K-iK',2K+iK']$ in the complex plane.

For system 15 the parameters vary in the range

$$1<\rho<a<\mu<\infty,\qquad 0\leqslant\varphi<2\pi$$

and the separation equations are (6.27). Making the same elliptic function substitutions as in the previous case, we find

$$x=\mathfrak{R}^{-1}\cos\varphi,\qquad y=\mathfrak{R}^{-1}\sin\varphi,\qquad z=i(k'\mathfrak{R})^{-1}\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta,$$

$$\mathfrak{R}=k(\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta+\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta/k'), \tag{6.30}$$

$$\Psi=\mathfrak{R}^{1/2}\Lambda^p_{m-\frac{1}{2}}(\alpha,k)\Lambda^p_{m-\frac{1}{2}}(\beta,k)e^{im\varphi}$$

where α,β range over the intervals $\alpha\in[iK',iK'+2K], \beta\in[K,K+2iK']$ in the complex plane.

For system 16 the parameters satisfy $\mu>0, \rho<0, 0\leqslant\varphi<2\pi$, and the separation equations are (6.27) with

$$\mathcal{P}(\xi)=(\xi-a)(\xi-b)\xi,\qquad a=\bar{b}=\alpha+i\beta.$$

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Setting $\mu = \mathrm{sn}^2(\gamma, t), \rho = \mathrm{sn}^2(\theta, t)$ where $t = (s + is')(s - is')^{-1}, s^2 = (|a| - \mathrm{Re}\, a)/2|a|$, we obtain solutions

$$\Psi = \mathcal{R}^{1/2} \Lambda^p_{m-\frac{1}{2}}(\gamma, t) \Lambda^p_{m-\frac{1}{2}}(\theta, t) e^{im\varphi} \tag{6.31}$$

where $\gamma \in [-iK', iK'], \theta \in [2K - iK', 2K + iK']$.

Finally, for system 17, toroidal coordinates, the eigenfunctions have the form

$$\Psi = (\cosh\xi + \sin\psi)^{1/2} E(\xi) \exp[i(l\psi + m\varphi)]$$

$$iJ_3 \Psi = m\Psi, \qquad (P_3 + K_3)\Psi = -2il\Psi,$$

$$\left[(\sinh\xi)^{-1} \frac{d}{d\xi} \sinh\xi \frac{d}{d\xi} + \left(1/4 - l^2 - \frac{m^2}{\sinh^2\xi} \right) \right] E(\xi) = 0. \tag{6.32}$$

The associated Legendre functions $P^m_{l-\frac{1}{2}}(\cosh\xi), Q^m_{l-\frac{1}{2}}(\cosh\xi)$ provide a basis of solutions for this last equation.

We can check explicitly that the coordinate surfaces are cyclides in all these cases. For systems 14–17 some of the surfaces are cyclides of revolution. Systems 12–16 are relatively intractable and only the toroidal system 17 has been widely used in studies of the Laplace equation. The toroidal and spherical coordinate systems have much in common. (Indeed, for the complex Laplace equation these two systems become equivalent under the complex conformal group.) Bipolar coordinates [12, p. 108] are frequently used in connection with separation of variables for the Laplace equation but these coordinates are conformally equivalent to spherical coordinates. They are, however, inequivalent to spherical coordinates with respect to the more physical scale Euclidean group, generated by $E(3)$ and dilatations $\exp(\alpha D)$.

Nine of the seventeen R-separable systems for the Laplace equation correspond to diagonalization of the operator J_3: systems 2, 5–8, 14–17. These special systems have the property that their eigenfunctions take the form $\Psi(\mathbf{x}) = \Phi e^{im\varphi}, iJ_3\Psi = m\Psi$, where Φ is a function of the remaining two variables. If we substitute this Ψ into the Laplace equation and factor out $e^{im\varphi}$, we obtain a differential equation for Φ which in cylindrical coordinates is

$$(\partial_{rr} + r^{-1}\partial_r - r^{-2}m^2 + \partial_{zz})\Phi(r, z) = 0. \tag{6.33}$$

Expression (6.33) for fixed $m \geq 0$ is the equation of generalized axial-symmetric potential theory. The real symmetry algebra of this equation is isomorphic to $sl(2, R)$. Indeed, a basis is provided by the operators

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K_3, P_3, D, (6.2), with commutation relations

$$[D,P_3]=P_3, \qquad [D,K_3]=-K_3, \qquad [P_3,K_3]=-2D, \tag{6.34}$$

and, from the identity (6.13iii) it follows that (6.33) can be written in the equivalent operator form

$$\left(\tfrac{1}{2}P_3^2+\tfrac{1}{2}K_3^2-D^2\right)\Phi=\left(\tfrac{1}{4}+m^2\right)\Phi. \tag{6.35}$$

It is shown in [139] (see also [63]) that the space of symmetric second-order symmetry operators in the enveloping algebra of $sl(2,R)$ modulo the subspace generated by the Casimir operator $\frac{1}{2}P_3^2+\frac{1}{2}K_3^2-D^2$ decomposes into nine orbit types under the action of the symmetry group $SL(2,R)$. The nine coordinate systems listed above are exactly those which permit separation of variables in (6.33) and it is straightforward to check that these systems correspond one to one with the nine orbit types. That is, there is perfect correspondence between the list of operators S_2 where J_3^2, S_2 defines each system and a list of representatives of the orbit types.

3.7 Identities Relating Separable Solutions of the Laplace Equation

It is not possible to find a Hilbert space model for the solutions of the Laplace equation such that the action of the conformal group is given by a unitary representation. Indeed, if such a model existed, the momentum operators iP_j, $j=1,2,3$, would be self-adjoint on this Hilbert space. However, the identity $P_1^2+P_2^2+P_3^2=0$ and the spectral theorem for self-adjoint operators imply $P_j=0$, which is a contradiction.

Nevertheless we can use Weisner's method to relate separable solutions of the Laplace equation and we can construct non-Hilbert space models of this equation in a manner analogous to that of Section 3.5. Consider the expression

$$\Psi(x,y,z)=\int_{C_1} d\beta\int_{C_2}\frac{dt}{t}h(\beta,t)\exp\left[\frac{ix\beta}{2}(t+t^{-1})\right.$$
$$\left.+\frac{y\beta}{2}(t-t^{-1})-\beta z\right]=I(h), \tag{7.1}$$

where h is analytic on a domain in $\mathbb{C}\times\mathbb{C}$ that contains the integration contours $C_1\times C_2$ and is chosen such that $I(h)$ converges absolutely and arbitrary differentiation with respect to x,y,z is permitted under the integral sign. It is easy to verify that for each such h, $\Psi=I(h)$ is a solution of the Laplace equation (6.1). Moreover, integrating by parts, we find that the operators P_j, J_j, K_j, D, (6.2), acting on the solution space of (6.1)

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correspond to the operators

$$P^+ = -\beta t, \quad P^- = -\beta t^{-1}, \quad P^0 = -i\beta, \quad D = \beta\partial_\beta + \tfrac{1}{2},$$

$$J^+ = it\beta\partial_\beta - it^2\partial_t, \quad J^- = -i\beta t^{-1}\partial_\beta - i\partial_t, \quad J^0 = t\partial_t,$$

$$K^+ = t\beta^{-1}(\beta\partial_\beta - t\partial_t)(\beta\partial_\beta - t\partial_t - 1),$$

$$K^- = t^{-1}\beta^{-1}(\beta\partial_\beta + t\partial_t)(\beta\partial_\beta + t\partial_t - 1),$$

$$K^0 = i\beta^{-1}((t\partial_t)^2 - (\beta\partial_\beta)^2), \tag{7.2}$$

where

$$J^{\pm} = \mp J_2 + iJ_1, \quad J^0 = iJ_3$$

with similar expressions for $P^{\pm}$, $K^{\pm}$, and so on. Here we are assuming C_1, C_2, and h are chosen such that the boundary terms vanish for each integration by parts:

$$P^{\pm}\Psi = I(P^{\pm}h), \quad J^{\pm}\Psi = I(J^{\pm}h),$$

and so on.

For our first example we choose C_1, C_2 as unit circles in the β and t planes, respectively, with centers at the origin and oriented in the counterclockwise direction. Then for

$$h(\beta,t) = \beta^{-l-1}j(t), \quad j(t) = \sum_{m=-l}^{l} a_m t^m, \quad l = 0,1,2,\ldots, \tag{7.3}$$

we can evaluate the β integral by residues to obtain

$$\Psi(x,y,z) = I(h) = -\frac{2\pi}{l!}\int_0^{2\pi}[ix\cos\alpha + iy\sin\alpha - z]^l j(e^{i\alpha})\,d\alpha. \tag{7.4}$$

From (7.3), h is an eigenfunction of D with eigenvalue $-l-\frac{1}{2}$, so by (6.13ii)

$$\mathbf{J}\cdot\mathbf{J}\Psi = -l(l+1)\Psi.$$

Furthermore, Ψ is a solution of the Laplace equation which is a homogeneous polynomial in x,y,z of order l. In particular, for $j(t) = t^m$, $-l \leqslant m \leqslant l$, we have $J^0\Psi = m\Psi$, so Ψ must be multiple of the solid harmonic $\rho^l Y_l^m(\theta,\varphi)$, expressed in spherical coordinates (5 in Table 14). Evaluating

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the integral in the special case where $\theta=0$, we find

$$I(\beta^{-l-1}t^m)=-\frac{2\pi\rho^l}{l!}\int_0^{2\pi}\left[i\sin\theta\cos(\varphi-\alpha)-\cos\theta\right]^l e^{i\alpha m}\,d\alpha$$
$$=16\pi^3(-1)^{l+1}i^m\rho^l\left[4\pi(2l+1)(l-m)!(l+m)!\right]^{-1/2}Y_l^m(\theta,\varphi). \tag{7.5}$$

Another example is provided by the contour C_2 in the t plane, the contour C_1', which goes from $\beta=0$ to $+\infty$ along the positive real axis in the β plane, and the analytic function $h(\beta,t)=\beta^l t^m, l=0,1,2,\ldots, m=l, l-1,\ldots,-l$. Here $\Psi=I(h)$ satisfies $D\Psi=(l+\frac{1}{2})\Psi, \mathbf{J}\cdot\mathbf{J}\Psi=-l(l+1)\Psi, J^0\Psi=m\Psi$ and it is easy to verify that

$$I(\beta^l t^m)=il!\rho^{-l-1}\int_0^{2\pi}\left[-i\sin\theta\cos(\varphi-\alpha)+\cos\theta\right]^{-l-1}e^{im\alpha}\,d\alpha$$
$$=i^{l-m}\rho^{-l-1}\left[16\pi^3(l-m)!(l+m)!/(2l+1)\right]^{1/2}Y_l^m(\theta,\varphi), \tag{7.6}$$

where ρ,θ,φ are spherical coordinates and $0\leqslant\theta<\pi/2$.

Now consider the equations

$$(\{J_1,P_2\}-\{P_1,J_2\})f=-\lambda f, \qquad J^0f=mf, \tag{7.7}$$

for eigenfunctions corresponding to the parabolic system. In terms of the model (7.2) these eigenfunctions are

$$f^{(8)}_{\lambda,m}(\beta,t)=\exp(-\lambda/2\beta)\,\beta^{-1}t^m. \tag{7.8}$$

Setting $h=f^{(8)}_{\lambda,m}$ in (7.1) and choosing the contours C_1, C_2, we find

$$\Psi^{(8)}_{\lambda,m}=I(f^{(8)}_{\lambda,m})=-2\pi\int_0^{2\pi}J_0\left[i(2\lambda)^{1/2}(z-ix\cos\alpha-iy\sin\alpha)^{1/2}\right]e^{im\alpha}\,d\alpha$$
$$=-4\pi^2J_m(-i\sqrt{\lambda}\,\xi)J_m(\sqrt{\lambda}\,\eta)e^{im\varphi}, \tag{7.9}$$
$$x=\xi\eta\cos\varphi, y=\xi\eta\sin\varphi, z=(\xi^2-\eta^2)/2.$$

As usual, the fact that variables separate enables us to compute the integral. For $h=f^{(8)}_{\lambda,m}$ in (7.1) and the contours C_1', C_2 we obtain

$$\Psi^{(8)\prime}_{\lambda,m}=I(f^{(8)}_{\lambda,m})=2\pi i^{m+1}e^{im\varphi}\int_0^{\infty}J_m(\beta r)\exp(-\beta z-\lambda/2\beta)\,d\beta/\beta$$
$$=2i\int_0^{2\pi}K_0\left[(2\lambda)^{1/2}(z-ix\cos\alpha-iy\sin\alpha)^{1/2}\right]e^{im\alpha}\,d\alpha$$
$$=4\pi iK_m(\sqrt{\lambda}\,\xi)I_m(i\sqrt{\lambda}\,\eta)e^{im\varphi}, \qquad \lambda>0, \xi>|\eta|. \tag{7.10}$$

ISBN-0-201-13503-5

The second and third equalities are obtained by performing only one of the integrations. Note that the second equality yields the expansion of our solution in terms of cylindrical waves.

Similarly, performing the t integration in (7.6) first, we find the expansion

$$I(\beta^l t^m)=2\pi i^{m+1}e^{im\varphi}\int_0^\infty J_m(\beta r)e^{-\beta z}\beta^l\,d\beta, \qquad z>0, \tag{7.6$'$}$$

of a solid spherical harmonic in terms of cylindrical waves.

Applying the transformation I (with contours C_1, C_2) to both sides of the identity

$$f^{(8)}_{\lambda,m}(\beta,t)=t^m\sum_{l=0}^{\infty}(-\lambda/2)^l\beta^{-l-1}/l!,$$

we find the expansion

$$\begin{aligned}\Psi^{(8)}_{\lambda,m}(\mathbf{x})=-\sum_{l=|m|}^{\infty}&16\pi^3 i^m\left[4\pi(2l+1)(l-m)!(l+m)!\right]^{-1/2}\\ &\times(\lambda\rho/2)^l(l!)^{-1}Y_l^m(\theta,\varphi)\end{aligned} \tag{7.9$'$}$$

of products of Bessel functions in terms of spherical harmonics.

Corresponding to the oblate spheroidal system 7, the eigenvalue equations

$$(\mathbf{J}\cdot\mathbf{J}+a^2P_1^2+a^2P_2^2)f=-\lambda f, \qquad J^0 f=mf,$$

in the model (7.2) yield the eigenfunctions

$$f^{(7)}_{\lambda,m}(\beta,t)=\beta^{-1/2}J_\nu(a\beta)t^m, \qquad \nu^2=\lambda+\tfrac{1}{4}. \tag{7.11}$$

Choosing the case where m is a positive integer and $\nu=l+\frac{1}{2}$ (where $l>-1$) and applying the transformation I (contours C_1', C_2), we find

$$\begin{aligned}\Psi^{(7)}_{\lambda,m}(\mathbf{x})=I(f^{(7)}_{\lambda,m})&=2\pi i^{m+1}e^{im\varphi}\int_0^\infty J_m(\beta r)J_\nu(a\beta)e^{-\beta z}\,d\beta/\beta^{1/2}\\ &=2\pi i^{m+1}(a\cosh\eta)^{-1/2}\Gamma(m+l+1)e^{im\varphi}\\ &\quad\times P_l^{-m}(\cos\alpha)P^{-l-\frac{1}{2}}_{m-\frac{1}{2}}(\tanh\eta), \qquad 0<\alpha<\frac{\pi}{2},\ 0<\eta,\end{aligned} \tag{7.12}$$

where α, η, φ are oblate spheroidal coordinates (7 in Table 14). Note that the second equality gives the expansion of our solution in terms of

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cylindrical waves. Again the integrals are rather easy to evaluate because we know in advance that variables separate in the solution. To determine the remaining four constants we need only examine the behavior of the integral near the values $\eta=0$ and $\alpha=0, \pi/2$.

In the case where $\nu=l+\frac{1}{2}$, $l=0,1,2,\ldots$, we can expand (7.11) as a power series in β and apply the transformation I term by term to obtain

$$\Psi^{(7)}_{\lambda,m}(\mathbf{x})=\sum_{n=\max\left(\frac{|m|-l}{2},0\right)}^{\infty}\left(\frac{a}{2}\right)^{l+2n+\frac{1}{2}}(i)^{2n-m+1}\frac{\rho^{-l-2n-1}}{n!\Gamma(l+n+3/2)}$$
$$\times\left[16\pi^3\frac{(l+2n-m)!(l+2n+m)!}{(2l+4n+1)}\right]^{1/2}Y^m_{l+2n}(\theta,\varphi), \quad (7.13)$$

which is an expansion of a spheroidal solution in solid spherical harmonics.

For the toroidal system 17 the eigenvalue equations

$$(P^0+K^0)f=2lf, \qquad J^0f=mf,$$

in the model (7.2) yield the eigenfunctions

$$f^{(17)}_{n,m}(\beta,t)=e^{-i\beta}(\beta t)^m\,{}_1F_1\left(\begin{matrix}-n\\2m+1\end{matrix}\middle|2i\beta\right), \qquad n=-l-m-\tfrac{1}{2}. \quad (7.14)$$

We choose $n, m=0,1,2,\ldots$ and apply I (contours C_1', C_2) to obtain

$$\begin{aligned}\Psi^{(17)}_{n,m}(\mathbf{x})&=I\left(f^{(17)}_{n,m}\right)\\&=2\pi i^{m+1}e^{im\varphi}\int_0^\infty e^{-\beta z-i\beta}J_m(r\beta)\,\beta^m\,{}_1F_1\left(\begin{matrix}-n\\2m+1\end{matrix}\middle|2i\beta\right)d\beta\\&=\sqrt{2}\,\pi\left(-\tfrac{1}{2}\right)^m(-i)^n(2m)!(\cosh\xi+\sin\psi)^{1/2}\\&\quad\times\exp\left[i(m\varphi+l\psi+\pi/4)\right]P^{-m}_{l-\frac{1}{2}}(\cosh\xi). \quad (7.15)\end{aligned}$$

An explicit computation yields

$$\exp(\alpha P_3)f^{(17)}_{n,m}=\sum_{s=m}^{\infty}a^s_{n,m}\beta^s t^m$$
$$a^s_{n,m}=\frac{(-\alpha-i)^{s-m}}{(s-m)!}\,{}_2F_1\left(\begin{matrix}-n,m-s\\2m+1\end{matrix}\middle|\frac{2i}{\alpha+i}\right). \quad (7.16)$$

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so

$$\exp(\alpha P_3)\Psi^{(17)}_{n,m}(\mathbf{x})=\sum_{s=m}^{\infty} a^{s}_{n,m} i^{1-m}\rho^{-s-1}\left[16\pi^3\frac{(s-m)!(s+m)!}{(2s+1)}\right]^{1/2} Y^m_s(\theta,\varphi) \tag{7.17}$$

is the expansion of this toroidal system solution in solid spherical harmonics. (The term-by-term integration used to derive (7.13) and (7.17) can be justified with the Lebesgue-dominated convergence theorem [69].)

As the foregoing examples indicate, the non-Hilbert space model permits us to derive integral representations and expansion formulas for the Laplace separable systems. (In some cases, however, the models yield third- and fourth-order differential operators.) The analysis for systems related to the Lamé and Whittaker–Hill equations proceeds in analogy with Section 3.3. The number of examples can be greatly multiplied by choosing other contours in the β and t planes. In addition, the Hilbert space expansions for solutions of the wave equation (Section 3.9) can be reinterpreted as Laplace equation expansions by replacing t with iz for $z>0$.

The most useful functions for application of Weisner's method are those associated with the spherical system. These functions are characterized as common eigenfunctions of the commuting operators D and J^0. We shall now study the eigenfunctions in greater generality than earlier by first considering the model (7.2). In this model the solutions of the equations

$$J^0 g=mg, \qquad Dg=\left(l+\tfrac{1}{2}\right)g, \qquad m,l\in\mathbb{C},$$

are multiples of $\beta^l t^m$. If the eigenfunctions are normalized so that

$$g^{(l)}_m=i^{l-m}\beta^l t^m, \tag{7.18}$$

it follows easily that the action of the operators (7.2) on this basis is

$$\begin{aligned}
J^{\pm}g^{(l)}_m&=(-l\pm m)\,g^{(l)}_{m\pm1}, & J^0 g^{(l)}_m&=mg^{(l)}_m,\\
P^0 g^{(l)}_m&=-g^{(l+1)}_m, & P^{\pm}g^{(l)}_m&=\mp g^{(l+1)}_{m\pm1},\\
Dg^{(l)}_m&=\left(l+\tfrac{1}{2}\right)g^{(l)}_m, & K^0 g^{(l)}_m&=(l^2-m^2)\,g^{(l-1)}_m,\\
K^{\pm}g^{(l)}_m&=\mp(l\mp m)(l\mp m-1)\,g^{(l-1)}_{m\pm1}.
\end{aligned} \tag{7.19}$$

We shall study our model in the case where $l_0\in\mathbb{C}$ is fixed with $l_0+\frac{1}{2}$ not an integer, $l=l_0,l_0\pm1,l_0\pm2,\ldots$, and $m=l,l-1,l-2,\ldots$. Note that the corresponding set of basis functions $\{g^{(l)}_m\}$ is invariant under the action of $so(4,1)$. In particular, the eigenfunction $g^{(l)}_l$ is mapped to zero by each of the operators J^+,K^0,K^+.

Due to the simplicity of the recurrence relations (7.19), we can easily exponentiate the Lie algebra operators L to obtain the local action $\exp(\alpha L)$ of the conformal group with respect to the basis (7.18). (Indeed one can use local Lie theory to exponentiate all operators (7.2) except the second-order K operators. However, the K operators can be formally exponentiated in the $\{g_m^{(l)}\}$ basis by using the recurrence relations (7.19) and the results will be valid for the Laplace equation model (6.2).) The matrix elements of the group action have been worked out in some detail in [84] and the results applied to derive identities for the Gegenbauer polynomials.

To see how these functions arise, we consider a complex coordinate system $\{w,t,\rho\}$ that is complex equivalent to the complex spherical coordinates $\{\theta,\varphi,\rho\}$ (5 in Table 14). (Since we are interested in analytic expansions, it is now useful to consider solutions of the complex Laplace equation.)

$$\begin{aligned} &w=\cos\theta=z/\rho, \qquad t=e^{i\varphi}(1-w^2)^{1/2}=(x+iy)/\rho,\\ &\rho=(x^2+y^2+z^2)^{1/2}. \end{aligned} \tag{7.20}$$

In terms of these coordinates the operators (6.2) become

$$\begin{aligned} &J^0=t\partial_t, \qquad J^+=-t\partial_w, \qquad J^-=t^{-1}((1-w^2)\partial_w-2wt\partial_t),\\ &D=-(\tfrac{1}{2}+\rho\partial_\rho), \qquad -iP^0=w\partial_\rho+\rho^{-1}(1-w^2)\partial_w-\rho^{-1}wt\partial_t,\\ &-iP^+=t\partial_\rho-\rho^{-1}tw\partial_w-\rho^{-1}t^2\partial_t,\\ &-iP^-=t^{-1}(1-w^2)\partial_\rho-\rho^{-1}t^{-1}w(1-w^2)\partial_w+\rho^{-1}(1+w^2)\partial_t,\\ &-iK^0=\rho w+\rho^2 w\partial_\rho+\rho(w^2-1)\partial_w+\rho tw\partial_t,\\ &-iK^+=\rho t+\rho^2 t\partial_\rho+\rho tw\partial_w+\rho t^2\partial_t,\\ &-iK^-=\rho t^{-1}(1-w^2)+\rho^2 t^{-1}(1-w^2)\partial_\rho-\rho(1+w^2)\partial_t+\rho t^{-1}w(1-w^2)\partial_w. \end{aligned} \tag{7.21}$$

Now we search for functions $\Psi_m^{(l)}(w,t,\rho)$ that satisfy the recurrence relations (7.19) when acted on by operators (7.21). (Since $\mathbf{P}\cdot\mathbf{P}g_m^{(l)}=0$ in model (7.19), the $\Psi_m^{(l)}$ will automatically be solutions of the Laplace equation corresponding to system 5.)

The relations

$$J^0\Psi_l^{(l)}=l\Psi_l^{(l)}, \qquad D\Psi_l^{(l)}=\left(l+\tfrac{1}{2}\right)\Psi_l^{(l)}, \qquad K^0\Psi_l^{(l)}=0$$

imply $\Psi_l^{(l)}=\Gamma(l+\frac{1}{2})(2t)^l(\rho/i)^{-l-1}$ to within a constant multiple. From

(7.19) we have

$$\exp(-i\alpha P^0)\Psi_m^{(l)} = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!}\Psi_m^{(l+n)} \tag{7.22}$$

and from (7.21),

$$\exp(-i\alpha P^0)\Psi_m^{(l)}(w,t,\rho) = \Psi_m^{(l)}\Big[(w+\alpha/\rho)(1+\alpha^2/\rho^2+2\alpha w/\rho)^{-1/2},\\ t\big(1+\alpha^2/\rho^2+2\alpha w/\rho\big)^{-1/2}, \rho\big(1+\alpha^2/\rho^2+2\alpha w/\rho\big)^{1/2}\Big]. \tag{7.23}$$

Substituting (7.23) into (7.22), setting $m=l$, and using our explicit expression for $\Psi_l^{(l)}$, we obtain a simple generating function for the eigenfunctions $\Psi_l^{(l+n)}$. Comparing this expression with (5.24), we find

$$\Psi_m^{(l)}(w,t,\rho) = (l-m)!\,\Gamma\big(m+\tfrac{1}{2}\big)C_{l-m}^{m+\frac{1}{2}}(w)(2t)^m(\rho/i)^{-l-1}. \tag{7.24}$$

Indeed, we can check directly that all of the recurrence relations (7.19) are satisfied by these functions. (The relations coincide exactly with the known differential recurrence relations obeyed by the Gegenbauer polynomials.) The general identity for Gegenbauer polynomials obtained by substituting (7.23) into (7.22) is

$$\begin{aligned}\big[1-2w+\alpha^2\big]^{-\nu-k/2}C_k^{\nu}\Big[(w-\alpha)(1-2\alpha w+\alpha^2)^{-1/2}\Big]\\ = \sum_{n=0}^{\infty}\alpha^n\binom{k+n}{n}C_{n+k}^{\nu}(w), \qquad |\alpha^2-2\alpha w|<1,\end{aligned} \tag{7.25}$$

which reduces to (5.24) when $k=0$.

Similarly, consideration of the expression

$$\exp(-\alpha P^+)\Psi_m^{(l)} = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\Psi_{m+n}^{(l+n)}$$

leads to the identity

$$(1-\alpha)^{-\nu-k/2}C_k^{\nu}\big[w(1-\alpha)^{-1/2}\big] = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\frac{\Gamma(\nu+n)}{\Gamma(\nu)}C_k^{\nu+n}(w), \qquad |\alpha|<1; \tag{7.26}$$

ISBN-0-201-13503-5

consideration of $\exp(\alpha K^0)\Psi_m^{(l)}$ leads to

$$(1+2\alpha w+\alpha^2)^{k/2}C_k^\nu\left[(w+\alpha)(1+2\alpha w+\alpha^2)^{-1/2}\right]$$
$$=\sum_{n=0}^{k}\alpha^l\binom{2\nu+k-1}{n}C_{k-n}^\nu(w), \tag{7.27}$$

and so on. For a more complete list of such expansions see [84].

Another type of identity obtainable from (7.19) is closely related to the Maxwell theory of poles. The identity $(P^0)^n g_l^{(l)}=(-1)^n g_l^{(l+n)}$, obvious from (7.19), leads to

$$n!\rho^{-\nu-n-\frac{1}{2}}C_n^\nu(w)=\left(w\,\partial_\rho+\rho^{-1}(1-w^2)\,\partial_w-\rho^{-1}w(\nu-\tfrac{1}{2})\right)^n\rho^{-\nu-\frac{1}{2}}$$
$$n=0,1,2,\ldots.$$

More generally we can use Weisner's method to derive expansions of the form

$$\mathbf{T}(g)\Psi(w,t,\rho)=\sum_{m,l}a_{m,l}C_{l-m}^{m+\frac{1}{2}}(w)t^m\begin{Bmatrix}\rho^l\\ \rho^{-l-1}\end{Bmatrix} \tag{7.28}$$

even when g is bounded away from the identity element in the conformal group or Ψ is a solution of the Laplace equation not on the spherical orbit. We give one simple example related to the cylindrical orbit. A solution of the equations

$$\mathbf{P}\cdot\mathbf{P}\Psi=0,\qquad -iP^0\Psi=\lambda\Psi,\qquad J^0\Psi=m\Psi,\qquad m,\lambda\in\mathcal{C},$$

is

$$\Psi(w,t,\rho)=\left[t/\left(\lambda(w^2-1)^{1/2}\right)\right]^m e^{\lambda w\rho}I_m\left[\lambda\rho(w^2-1)^{1/2}\right]$$

where $I_m(z)$ is a modified Bessel function. In this case (7.28) yields

$$\Psi(w,t,\rho)=\sum_{n=0}^{\infty}a_n(\lambda)\rho^{m+n}t^m C_n^{m+\frac{1}{2}}(w).$$

The constants $a_n(\lambda)$ can be evaluated by setting $w=1$ on both sides of the equation, yielding the final result

$$\Gamma(m+1)\left[\rho(w^2-1)^{1/2}\right]^{-m}e^{\rho w}I_m\left[\rho(w^2-1)^{1/2}\right]$$
$$=\sum_{n=0}^{\infty}\frac{\Gamma(2m+1)}{\Gamma(2m+n+1)}C_n^{m+\frac{1}{2}}(w)\rho^n. \tag{7.29}$$

ISBN-0-201-13503-5

Every analytic Ψ obtainable from separation of variables in the complex Laplace equation will lead to an expansion (7.28). Such functions can be obtained by analytic continuation of the separable solutions of the real Laplace equation and by continuation of separable solutions of the wave equation

$$(\partial_{tt}-\Delta_2)\Phi(x,y,t)=0$$

to be studied in the following sections. (Set $t=iz$.) Thus there are an enormous number of generating functions for Gegenbauer polynomials that are obtainable in this way. In general (7.28) is a double sum but if $\mathbf{T}(g)\Psi$ is an eigenfunction of J^0, then m is fixed and only l is summed. These functions are just the solutions of (6.33) and can be obtained by choosing Ψ as one of the separable solutions corresponding to this equation and g as an element in the complex group $SL(2,\mathcal{C})$ generated by P^0,K^0,D. In [129], Viswanathan has given a detailed derivation of the generating functions that can be obtained in this manner, with the exception of the difficult Lamé systems. Equation (7.28) also reduces to a single sum when $\mathbf{T}(g)\Psi$ is an eigenfunction of D. Then l is fixed and only m is summed. Coordinate systems in which D is diagonal are discussed in Section 4.3.

Finally, we remark that quadratic transformation formulas for the hypergeometric function ${}_2F_1$ can be obtained from the conformal symmetry of the complex Laplace equation [93].

Exercises

1. Show that $\mathcal{E}(3)$ is decomposed into three orbits under the adjoint action of $E(3)$.

2. Verify that the Helmholtz equation separates in parabolic cylindrical coordinates $x=(\xi^2-\eta^2)/2$, $y=\xi\eta$, $z=z$, and that the corresponding defining operators are $\{J_3,P_2\}$ and P_3^2.

3. Use expressions (4.10), (4.11) to compute the bilinear expansions of the function $\sin(\omega R)/\omega R$ in terms of separable solutions of the Helmholtz equation in spherical and prolate spheroidal coordinates.

4. Compute the symmetry algebra of the Laplace equation $\Delta_3\psi=0$.

5. Show that the change of variables $x=u$, $y-iz=s$, $y+iz=2t$ and the substitution $\Psi=e^{\lambda s}\Phi(t,u)$ reduce the complex Laplace equation $(\partial_{xx}+\partial_{yy}+\partial_{zz})\Psi=0$ to the heat equation for Φ.

ISBN-0-201-13503-5