CHAPTER 4

Lie Theory and Confluent Hypergeometric Functions

In the last chapter fundamental properties of the cylindrical functions were deduced by relating those functions to the representation theory of $\mathscr{G}(0, 0)$. Here, analogous arguments will be used to study confluent hypergeometric functions as they relate to the representation theory of $\mathscr{G}(0, 1)$. Besides treating the general confluent hypergeometric functions ${}_1F_1(a; b; x)$, we shall also consider certain special cases: the Laguerre polynomials $L_n^m(x)$, the parabolic cylinder functions $D_m(x)$, and the Hermite polynomials $H_n(x)$. These special cases arise naturally.

In Section 4-11 we will introduce a real (global) Lie group S_4 whose Lie algebra is a real form of $\mathscr{G}(0, 1)$. We will examine the relationship between the irreducible representations of $\mathscr{G}(0, 1)$ and the unitary irreducible representations of S_4. This study leads to addition theorems and orthogonality relations for the Laguerre polynomials. The problem of decomposing tensor products of unitary representations of S_4 into irreducible parts will also be considered. The solution of this problem leads to identities expressing the product of two Laguerre polynomials as a sum of Laguerre polynomials (Section 4-17), and as an integral over Bessel functions (Section 4-19). Furthermore, the product of a Laguerre polynomial and a Bessel function can be expressed as a sum of Laguerre polynomials (Section 4-20). Each of these expansions will be given an explicit group-theoretical interpretation.

Finally in Section 4-21 we will use the fact that $\mathscr{G}(0, 0)$ is a contraction of $\mathscr{G}(0, 1)$ to derive a formula for Bessel functions as limits of Laguerre polynomials.

Note: There is another real form of $\mathcal{G}(0, 1)$ (the Lie algebra of all **real** matrices (1.30)), distinct from the one we consider here. A study of the representation theory of this new real form leads to integral identities involving confluent hypergeometric functions (see Vilenkin [3]).

4-1 The Representation $R(\omega, m_o, \mu)$

The irreducible representation $R(\omega, m_o, \mu)$ of $\mathcal{G}(0, 1)$ is determined by complex constants ω, m_o, μ such that $\mu \neq 0$, $0 \leqslant \operatorname{Re} m_o < 1$, and $\omega + m_o$ is not an integer (Theorem 2.2). The spectrum of this representation is the set

$$S = \{m_o + n: n \text{ an integer}\},$$

and the representation space V has a basis $\{f_m\}$, $m \in S$, so that

$$\begin{aligned} &\mathrm{J}^3 f_m = m f_m, \qquad \mathrm{E} f_m = \mu f_m, \qquad \mathrm{J}^+ f_m = \mu f_{m+1}, \\ &\mathrm{J}^- f_m = (m + \omega) f_{m-1}, \qquad \mathrm{C}_{0,1} f_m = (\mathrm{J}^+\mathrm{J}^- - \mathrm{E}\mathrm{J}^3) f_m = \mu\omega f_m . \end{aligned} \tag{4.1}$$

The commutation relations satisfied by the infinitesimal operators are

$$[\mathrm{J}^3, \mathrm{J}^\pm] = \pm \mathrm{J}^\pm, \qquad [\mathrm{J}^+, \mathrm{J}^-] = -\mathrm{E}, \qquad [\mathrm{J}^\pm, \mathrm{E}] = [\mathrm{J}^3, \mathrm{E}] = 0. \tag{4.2}$$

We will find a realization of this representation in such a way that the operators $\mathrm{J}^\pm$, J^3, E become the linear differential operators (2.36) acting on a vector space of functions of one complex variable, z. Let the representation space $\mathcal{V}_1$ be the complex vector space consisting of all finite linear combinations of the functions $h_n(z) = z^n$, $n = 0, \pm 1, \pm 2, \ldots$, and set $\lambda = m_o$, $c_3 = m_o + \omega$ in Eqs. (2.36) to obtain the operators

$$\mathrm{J}^3 = m_o + z\frac{d}{dz}, \qquad \mathrm{E} = \mu, \qquad \mathrm{J}^+ = \mu z, \qquad \mathrm{J}^- = \frac{m_o + \omega}{z} + \frac{d}{dz} \tag{4.3}$$

on $\mathcal{V}_1$. Define the basis vectors f_m of $\mathcal{V}_1$ by $f_m(z) = h_n(z)$ where $m = m_o + n$ for all $m \in S$. Then

$$\begin{aligned} \mathrm{J}^3 f_m &= \left(m_o + z\frac{d}{dz}\right) z^n = (m_o + n)\, z^n = m f_m, \\ \mathrm{J}^+ f_m &= (\mu z)\, z^n = \mu z^{n+1} = \mu f_{m+1}, \\ \mathrm{J}^- f_m &= \left(\frac{m_o + \omega}{z} + \frac{d}{dz}\right) z^n = (m + \omega)\, z^{n-1} = (m + \omega) f_{m-1}, \\ \mathrm{E} f_m &= \mu f_m, \end{aligned}$$

and we have a realization of $R(\omega, m_o, \mu)$.

As in Chapter 3 we use this realization by differential operators to obtain a local multiplier representation of the local group $G(0, 1)$ whose Lie algebra is $\mathcal{G}(0, 1)$. The matrix elements of this local representation with respect to the basis $\{f_m\}$ will be confluent hypergeometric functions.

Although the realization of $R(\omega, m_o, \mu)$ given above was constructed from the differential operators (2.36), it could also have been constructed from the operators (2.37). As far as the computation of matrix elements is concerned it makes no difference which operators we use. The matrix elements are uniquely determined by the Lie algebra relations (4.1) and do not depend on the particular realization of these relations.

In Section 1-2 it was shown that $\mathcal{G}(0, 1)$ is the Lie algebra of the local Lie group $G(0, 1)$:

$$G(0,1) \equiv \left\{ \begin{pmatrix} 1 & ce^{\tau} & a & \tau \\ 0 & e^{\tau} & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a, b, c, \tau \in \mathcal{C} \right\}. \tag{4.4}$$

We can extend the realization of $R(\omega, m_o, \mu)$ defined on $\mathcal{V}_1$ to a local multiplier representation of $G(0, 1)$ defined on $\mathcal{O}_1$, where $\mathcal{O}_1$ is the complex vector space of all functions of z analytic in some neighborhood of the point $z = 1$. Clearly $\mathcal{O}_1 \supset \mathcal{V}_1$ and $\mathcal{O}_1$ is invariant under the operators

$$\mathrm{J}^3 = m_o + z\frac{d}{dz}, \qquad \mathrm{E} = \mu, \qquad \mathrm{J}^+ = \mu z, \qquad \mathrm{J}^- = \frac{m_o + \omega}{z} + \frac{d}{dz}.$$

According to Theorem 1.10, these operators generate a Lie algebra, isomorphic to $\mathcal{G}(0, 1)$, which is the algebra of generalized Lie derivatives of a multiplier representation A of $G(0, 1)$ acting on $\mathcal{O}_1$. We will compute the multiplier ν explicitly. From the theorem, the action of the 1-parameter subgroup $\{\exp c\mathcal{J}^-, c \in \mathcal{C}\}$ of $G(0, 1)$ on $\mathcal{O}_1$ is obtained by solving the equations

$$\frac{dz}{dc} = 1, \qquad \frac{d}{dc}\nu(z^o, \exp c\mathcal{J}^-) = \frac{m_o + \omega}{z}\nu(z^o, \exp c\mathcal{J}^-)$$

with initial conditions $z(0) = z^o \neq 0$, $\nu(z^o, \mathbf{e}) = 1$. Here, $\mathbf{e}$ is the identity element of $G(0, 1)$ and $\exp c\mathcal{J}^-$ is given by (1.33). The solution of the differential equations is

$$z(c) = z^o + c, \qquad \nu(z^o, \exp c\mathcal{J}^-) = \left(1 + \frac{c}{z^o}\right)^{m_o+\omega}.$$

Thus, if $f \in \mathcal{O}_1$ is analytic in a neighborhood of z^o then

$$[\mathbf{A}(\exp c\mathcal{J}^-)f](z^o) = \left(1 + \frac{c}{z^o}\right)^{m_o+\omega} f(z^o + c),$$

and $\mathbf{A}(\exp c\mathcal{J}^-)f$ is an element of $\mathcal{O}_1$ for $|c/z^o| < 1$, $z^o + c$ in the domain of f. Similarly we obtain

$$[\mathbf{A}(\exp b\mathcal{J}^+)f](z^o) = \exp(\mu b z^o)f(z^o),$$
$$[\mathbf{A}(\exp \tau\mathcal{J}^3)f](z^o) = e^{m_o\tau} f(e^\tau z^o),$$
$$[\mathbf{A}(\exp a\mathcal{E})f](z^o) = e^{\mu a} f(z^o).$$

If $g \in G(0, 1)$ has coordinates (a, b, c, τ) it is easy to show that

$$g = (\exp b\mathcal{J}^+)(\exp c\mathcal{J}^-)(\exp \tau\mathcal{J}^3)(\exp a\mathcal{E}).$$

Thus, for $|c|$, $|\tau|$ sufficiently small, the operator $\mathbf{A}(g)$ acting on $f \in \mathcal{O}_1$ is given by

$$\begin{aligned}[\mathbf{A}(g)f](z) &= [\mathbf{A}(\exp b\mathcal{J}^+)\,\mathbf{A}(\exp c\mathcal{J}^-)\,\mathbf{A}(\exp \tau\mathcal{J}^3)\,\mathbf{A}(\exp a\mathcal{E})f](z)\\ &= e^{\mu(bz+a)+m_o\tau}\left(1 + \frac{c}{z}\right)^{m_o+\omega} f(e^\tau z + e^\tau c). \qquad (4.5)\end{aligned}$$

To make sense out of this expression we restrict g to the open set $M \subset G(0, 1)$ where $M = \{g \in G(0, 1): |c| < 1\}$. As a local Lie group, M is isomorphic to $G(0, 1)$. For every $f \in \mathcal{O}_1$ let D_f be the domain of f, i.e., the open set in $\mathcal{C}$, containing 1, on which f is defined and analytic. Define $\mathcal{R}_1(g)$, the domain of $g \in M$, by $\mathcal{R}_1(g) = \{f \in \mathcal{O}_1 : e^\tau(1 + c) \in D_f\}$. It follows from these definitions that for any $g \in M$ and $f \in \mathcal{R}_1(g)$ we have $\mathbf{A}(g)f \in \mathcal{O}_1$, where $\mathbf{A}(g)f$ is given by (4.5). In fact, $D_{A(g)f} = \{z \in \mathcal{C}: |c/z| < 1 \text{ and } e^\tau(z + c) \in D_f\}$, so $1 \in D_{A(g)f}$. (The factor $(1 + c/z)^{m_o+\omega}$ in (4.5) is defined by its Laurent expansion about $z = 0$.) The problem of properly defining the domain of the operator $\mathbf{A}(g)$ on $\mathcal{O}_1$ is analogous to the problem of defining the domain of an unbounded operator on a Hilbert space.

We can given a precise interpretation of the representation property of the operators $\mathbf{A}(g)$ as follows: If $g_1, g_2, g_1g_2 \in M$ and $f \in \mathcal{O}_1$ such that $f \in \mathcal{R}_1(g_2)$ and $\mathbf{A}(g_2)f \in \mathcal{R}_1(g_1)$, then $f \in \mathcal{R}_1(g_1g_2)$ and

$$\mathbf{A}(g_1g_2)f = \mathbf{A}(g_1)[\mathbf{A}(g_2)f] \in \mathcal{O}_1. \qquad (4.6)$$

The problem of properly interpreting (4.6) is analogous to the problem of defining the product of two unbounded linear operators on a Hilbert space.

The elements of $\mathscr{V}_1$ are analytic for all $z \neq 0$ so for any $f \in \mathscr{V}_1$, $g \in M$, the expression $\mathbf{A}(g)f$ is well defined as an element of $\mathscr{O}_1$. In fact $D_{A(g)f} = \{z \in \mathscr{C}: |z| > |c|\} \supset \{1\}$. Consequently, if $h \in \bar{\mathscr{V}}_1$ then $D_h = \{|z| > d_h\}$ where d_h is a constant such that $1 > d_h \geqslant 0$. (Recall from Section 2-2 that $\bar{\mathscr{V}}_1$ is the space of all finite linear combinations of functions of the form $\mathbf{A}(g)f$ where $g \in M$ and $f \in \mathscr{V}_1$. By definition, $\bar{\mathscr{V}}_1$ is invariant under A.) Thus, if $h \in \bar{\mathscr{V}}_1$ then h has a unique Laurent expansion

$$h(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad a_n \in \mathscr{C},$$

which converges absolutely for all $|z| > d_h$.

According to the above results the subspace $\bar{\mathscr{V}}_1$ is invariant under A, and the basis functions $f_m(z) = h_n(z) = z^n$, $m = m_o + n$, for $\mathscr{V}_1$ form an analytic basis for $\bar{\mathscr{V}}_1$ (see Section 2-2).

As usual the matrix elements $A_{lk}(g)$ of the operators $\mathbf{A}(g)$ on $\bar{\mathscr{V}}_1$ are defined by

$$[\mathbf{A}(g)h_k](z) = \sum_{l=-\infty}^{\infty} A_{lk}(g)\, h_l(z), \qquad g \in M, \quad k = 0, \pm 1, \ldots, \tag{4.7}$$

or

$$e^{\mu(bz+a)+(m_o+k)\tau}(1 + c/z)^{m_o+\omega+k} z^k = \sum_{l=-\infty}^{\infty} A_{lk}(g) z^l, \qquad |c/z| < 1. \tag{4.8}$$

Furthermore, from the representation property

$$\mathbf{A}(g_1 g_2)h_k = \mathbf{A}(g_1)[\mathbf{A}(g_2)h_k] \tag{4.9}$$

we obtain the addition theorem

$$A_{lk}(g_1 g_2) = \sum_{j=-\infty}^{\infty} A_{lj}(g_1)\, A_{jk}(g_2), \qquad l, k = 0, \pm 1, \pm 2, \ldots. \tag{4.10}$$

Equation (4.9) is defined only for those group elements $g_1, g_2 \in M$ such that $g_1 g_2 \in M$ and $\mathbf{A}(g_2)h_k \in \mathscr{R}_1(g_1)$. However, we shall soon see that the matrix elements $A_{lk}(g)$ are entire functions of the complex group parameters (a, b, c, τ) of g; hence, the matrix elements can be defined by analytic continuation for all $g \in G(0, 1)$. Moreover, the group parameters of $g_1 g_2$ are entire functions of the parameters of g_1 and g_2:

$$\begin{aligned} &g_1(a_1, b_1, c_1, \tau_1)\, g_2(a_2, b_2, c_2, \tau_2) \\ &\qquad = g_1 g_2(a_1 + a_2 + c_1 b_2 e^{\tau_1}, b_1 + e^{\tau_1} b_2, c_1 + e^{-\tau_1} c_2, \tau_1 + \tau_2). \end{aligned} \tag{4.11}$$

Thus, by analytic continuation the addition theorem (4.10) is valid for all $g_1, g_2 \in G(0, 1)$.

To derive an explicit expression for the matrix elements $A_{lk}(g)$ we expand the left-hand side of (4.8) in a Laurent series in z and compute the coefficient of z^l. The result is

$$A_{lk}(g) = e^{\mu a+(m_o+k)\tau}(\mu b)^{l-k} \sum_s \frac{(\mu bc)^s\, \Gamma(\rho + k + 1)}{(l - k + s)!\, s!\, \Gamma(\rho + k + 1 - s)},$$

where the sum extends over all nonnegative integral values of s such that the summand is defined. For convenience we have set $\rho = m_o + \omega$. Comparing with (A. 10) we have

$$\begin{aligned}
A_{lk}(g) &= \frac{\exp[\mu a + (m_o + k)\tau]\, \Gamma(\rho + k + 1)}{(k - l)!\, \Gamma(\rho + l + 1)}\, c^{k-l} \\
&\quad \cdot {}_1F_1(-\rho - l; k - l + 1; -\mu bc) \qquad \text{if} \quad k \geqslant l, \\
A_{lk}(g) &= \frac{\exp[\mu a + (m_o + k)\tau](\mu b)^{l-k}}{(l - k)!} \\
&\quad \cdot {}_1F_1(-\rho - k; l - k + 1; -\mu bc) \qquad \text{if} \quad l \geqslant k.
\end{aligned} \tag{4.12}$$

Here, ρ is not an integer. The functions ${}_1F_1$ are the confluent hypergeometric functions defined by their power series expansions. It is clear from (4.12) that the matrix elements are entire analytic functions of the group parameters.

Perhaps the most convenient form in which to express the matrix elements is in terms of the generalized Laguerre functions $L_\nu^{(\alpha)}$, (A. 15):

$$A_{lk}(g) = e^{\mu a+(m_o+k)\tau} c^{k-l} L_{\rho+l}^{(k-l)}(-\mu bc), \qquad k, l \text{ integers.} \tag{4.13}$$

Substituting (4.13) in (4.8) and simplifying, we obtain the generating function

$$e^{-c/t}(1 + t)^\rho = \sum_{l=-\infty}^{\infty} t^l L_{\rho-l}^{(l)}(c), \qquad 0 < |t| < 1. \tag{4.14}$$

Note that the Laguerre polynomials $L_n^{(\alpha)}$, n an integer, do not occur as matrix elements of $R(\omega, m_o, \mu)$. Laguerre polynomials will arise in the computation of matrix elements of $\uparrow_{\omega,\mu}$ and $\downarrow_{\omega,\mu}$ to be considered later.

We can obtain addition theorems for the Laguerre functions by substituting (4.13) into (4.10). After some simplification there results

$$\begin{aligned}
&e^{-c_1 b_2}(c_1 + c_2)^n L_\rho^{(n)}[(b_1 + b_2)(c_1 + c_2)] \\
&\quad = \sum_{j=-\infty}^{\infty} (c_1)^{j+n} L_\rho^{(j+n)}[b_1 c_1](c_2)^{-j} L_{\rho+j+n}^{(-j)}[b_2 c_2]
\end{aligned} \tag{4.15}$$

where n is an integer, ρ is not an integer, and $b_1, b_2, c_1, c_2 \in \mathcal{C}$. The following limits can be verified from (4.14):

$$c^n L_\rho^{(n)}(bc)\big|_{c=0} = \begin{cases} 0 & \text{if } n > 0, \\ \dfrac{(-b)^{-n}}{(-n)!} & \text{if } n \leqslant 0, \end{cases}$$

$$c^n L_\rho^{(n)}(bc)\big|_{b=0} = \begin{cases} \dbinom{\rho+n}{n} c^n & \text{if } n \geqslant 0, \\ 0 & \text{if } n < 0, \end{cases} \tag{4.16}$$

where n is an integer. Therefore, if $c_1 = b_2 = 0$, $c_2 = 1$, $b_1 = x$ in (4.15), we obtain

$$L_\rho^{(n)}(x) = \begin{cases} \displaystyle\sum_{l=0}^{\infty} \frac{(-x)^l}{l!} \binom{\rho+n+l}{n+l} & \text{if } n \geqslant 0, \\ \displaystyle\sum_{l=0}^{\infty} \frac{(-x)^{l-n}}{(l-n)!} \binom{\rho+l}{l} & \text{if } n \leqslant 0, \end{cases}$$

which is the power series expansion for the generalized Laguerre functions.

If $c_1 = 0$, $c_2 = 1$, $b_1 = x$, $b_2 = y$ in (4.15), one obtains

$$L_\rho^{(n)}(x+y) = \sum_{l=0}^{\infty} \frac{(-x)^l}{l!} L_{\rho-l}^{(l+n)}(y),$$

while for $c_1 = 1$, $c_2 = 0$, $b_1 = x$, $b_2 = y$,

$$L_\rho^{(n)}(x+y) = e^y \sum_{j=0}^{\infty} \frac{(-y)^j}{j!} L_\rho^{(j+n)}(x).$$

If $c_1 = -c_2 = 1$, $b_1 = x$, $b_2 = -y$ we have

$$e^y \frac{(y-x)^n}{n!} = \sum_{j=-\infty}^{\infty} (-1)^j L_\rho^{(j-n)}(x) L_{\rho+j-n}^{(-j)}(y), \qquad n \geqslant 0,$$

and

$$0 = \sum_{j=-\infty}^{\infty} (-1)^j L_\rho^{(j+n)}(x) L_{\rho+j+n}^{(-j)}(y), \qquad n > 0.$$

As a final example, set $c_1 = c_2 = 1$, $b_1 = x$, $b_2 = y$ to obtain

$$e^{-y} 2^n L_\rho^{(n)}[2(x+y)] = \sum_{j=-\infty}^{\infty} L_\rho^{(j+n)}(x) L_{\rho+j+n}^{(-j)}(y).$$

4-2 The Representation $\uparrow_{\omega,\mu}$

The irreducible representation $\uparrow_{\omega,\mu}$ of $\mathcal{G}(0, 1)$ is defined for each ω, $\mu \in \mathcal{C}$ such that $\mu \neq 0$. The spectrum S of $\uparrow_{\omega,\mu}$ is

$$S = \{-\omega + n\colon n \text{ a nonnegative integer}\}$$

and there is a basis $\{f_m, m \in S\}$ for the representation space V with the properties

$$J^3 f_m = m f_m, \qquad E f_m = \mu f_m, \qquad J^+ f_m = \mu f_{m+1},$$
$$J^- f_m = (m + \omega) f_{m-1}, \qquad C_{0,1} f_m = (J^+J^- - EJ^3) f_m = \mu\omega f_m.$$

(Here $f_{-\omega-1} \equiv 0$, so $J^- f_{-\omega} = 0$.)

As in the last section, we can construct a realization of this representation such that $J^\pm$, J^3, E take the form of linear differential operators (2.36) acting on a space of functions of one complex variable. Namely, we designate by $\mathcal{V}_2$ the space of all finite linear combinations of the functions $h_n(z) = z^n$, $n = 0, 1, 2, \ldots$, and define operators on $\mathcal{V}_2$ by setting $\lambda = -\omega$, $c_3 = 0$ in (2.36):

$$J^3 = -\omega + z\frac{d}{dz}, \qquad E = \mu, \qquad J^+ = \mu z, \qquad J^- = \frac{d}{dz}. \tag{4.17}$$

The basis vectors f_m of $\mathcal{V}_2$ are defined by $f_m(z) = h_n(z) = z^n$ where $m = -\omega + n$ and $n \geqslant 0$. These operators and basis vectors satisfy the relations

$$\begin{aligned} J^3 f_m &= \left(-\omega + z\frac{d}{dz}\right) z^n = (-\omega + n)\, z^n = m f_m, \\ J^+ f_m &= (\mu z)\, z^n = \mu z^{n+1} = \mu f_{m+1}, \\ J^- f_m &= \frac{d}{dz} z^n = n z^{n-1} = (\omega + m) f_{m-1}, \\ E f_m &= \mu f_m \end{aligned} \tag{4.18}$$

for all $m \in S$. Relations (4.18) yield a realization of $\uparrow_{\omega,\mu}$.

We can extend this realization to a multiplier representation of $G(0, 1)$ defined on $\mathcal{A}_2$, the complex vector space of all entire analytic functions of z. Obviously, $\mathcal{A}_2$ is invariant under the operators (4.17) and contains $\mathcal{V}_2$ as a subspace. According to Theorem 1.10, the operators (4.17) induce a local multiplier representation B of $G(0, 1)$ acting on the space of all functions analytic in a neighborhood of $z = 0$. However, it will turn out that $\mathbf{B}(g) f \in \mathcal{A}_2$ for all $f \in \mathcal{A}_2$, $g \in G(0, 1)$, so without loss of generality, we can restrict B to $\mathcal{A}_2$.

Since the differential operators

$$J^3 = m_o + z\frac{d}{dz}, \qquad E = \mu, \qquad J^+ = \mu z, \qquad J^- = \frac{d}{dz}$$

are formally identical with the operators (4.2) for the case $m_o + \omega = 0$, the action of $G(0, 1)$ on $\mathcal{O}_2$ induced by these differential operators is obtained from the corresponding result (4.5) in Section 4-1 by setting $m_o = -\omega$. Thus, we have operators $\mathbf{B}(g)$ defined for all $g \in G(0, 1)$, such that

$$[\mathbf{B}(g)f](z) = e^{\mu(bz+a)-\omega\tau} f(e^\tau z + e^\tau c), \qquad f \in \mathcal{O}_2. \tag{4.19}$$

The multiplier ν is given by $\nu(z, g) = \exp[\mu bz + \mu a - \omega\tau]$. Clearly, if $f \in \mathcal{O}_2$, i.e., if f is an entire function of z, then $\mathbf{B}(g)f$ is an entire function of z; therefore the space $\mathcal{O}_2$ is invariant under the operators $\mathbf{B}(g)$. Furthermore, the operators $\mathbf{B}(g)$ are defined on $\mathcal{O}_2$ for all $g \in G(0, 1)$ and satisfy the representation property

$$\mathbf{B}(g_1g_2)f = \mathbf{B}(g_1)[\mathbf{B}(g_2)f] \tag{4.20}$$

for all $f \in \mathcal{O}_2$ and all $g_1, g_2 \in G(0, 1)$. Equation (4.20) is a consequence of Theorem 1.10. However, it is simple to verify its validity directly. Thus, if g_1 and g_2 have coordinates (a_1, b_1, c_1, τ_1) and (a_2, b_2, c_2, τ_2), respectively, then g_1g_2 has coordinates $(a_1 + a_2 + e^{\tau_1}c_1b_2,\ b_1 + e^{\tau_1}b_2,\ c_1 + e^{-\tau_1}c_2,\ \tau_1 + \tau_2)$ and we find

$$\begin{aligned}
\mathbf{B}(g_1)[\mathbf{B}(g_2)f](z) &= \exp[\mu b_1 z + \mu a_1 - \omega\tau_1][\mathbf{B}(g_2)f](e^{\tau_1}z + e^{\tau_1}c_1)\\
&= \exp[\mu(b_1 + e^{\tau_1}b_2)z + \mu(a_1 + a_2 + e^{\tau_1}c_1b_2) - \omega(\tau_1 + \tau_2)]\\
&\quad \cdot f(e^{(\tau_1+\tau_2)}z + e^{(\tau_1+\tau_2)}(c_1 + e^{-\tau_1}c_2))\\
&= [\mathbf{B}(g_1g_2)f](z)
\end{aligned}$$

for all $f \in \mathcal{O}_2$.

Every function f in $\mathcal{O}_2$ has a unique power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n \in \mathcal{C},$$

convergent for all $z \in \mathcal{C}$. Thus, the basis functions $h_n(z) = z^n$, $n \geqslant 0$, of $\mathcal{V}_2$ form an analytic basis for $\mathcal{O}_2$. With respect to this analytic basis the matrix elements $B_{lk}(g)$ are defined by

$$[\mathbf{B}(g)h_k](z) = \sum_{l=0}^{\infty} B_{lk}(g)\, h_l(z), \qquad g \in G(0, 1), \quad k = 0, 1, 2, \ldots, \tag{4.21}$$

or, from (4.19),

$$e^{\mu(bz+a)+(k-\omega)\tau}(z+c)^k = \sum_{l=0}^{\infty} B_{lk}(g) z^l. \tag{4.22}$$

The group property (4.20) leads to the addition theorem

$$B_{lk}(g_1 g_2) = \sum_{j=0}^{\infty} B_{lj}(g_1)\, B_{jk}(g_2), \qquad l, k \geqslant 0, \quad g_1, g_2 \in G(0, 1). \tag{4.23}$$

The matrix elements $B_{lk}(g)$ are easily determined by expanding the left-hand side of (4.22) in a power series in z and computing the coefficient of z^l. The result is

$$B_{lk}(g) = e^{\mu a+(-\omega+k)\tau}(\mu b)^{l-k} \sum_s \frac{(\mu bc)^s\, k!}{(l-k+s)!\, s!(k-s)!}, \qquad k, l \geqslant 0, \tag{4.24}$$

where the (finite) sum is taken over all nonnegative integral values of s such that the summand is defined. We can express these matrix elements in terms of known special functions. Comparison of (4.24) with (A. 10) yields

$$\begin{aligned} B_{lk}(g) &= \frac{\exp[\mu a + (k-\omega)\tau] k!}{(k-l)!\, l!}\, c^{k-l}\, {}_1F_1(-l; k-l+1; -\mu bc) \\ &\qquad \text{if } \ k \geqslant l \geqslant 0, \\ B_{lk}(g) &= \frac{\exp[\mu a + (k-\omega)\tau]}{(l-k)!}\, (\mu b)^{l-k}\, {}_1F_1(-k; l-k+1; -\mu bc) \\ &\qquad \text{if } \ l \geqslant k \geqslant 0. \end{aligned} \tag{4.25}$$

The matrix elements can also be written in the convenient form

$$B_{lk}(g) = e^{\mu a+(k-\omega)\tau} c^{k-l} L_l^{(k-l)}(-\mu bc), \qquad k, l \geqslant 0, \tag{4.26}$$

where the functions $L_l^{(n)}$ are the associated Laguerre polynomials. Substituting (4.26) into (4.22) and simplifying, we obtain the generating function

$$e^{-bz}(z+1)^k = \sum_{l=0}^{\infty} L_l^{(k-l)}(b) z^l. \tag{4.27}$$

Note the great similarity between the expressions for matrix elements of $\uparrow_{\omega,\mu}$ and the matrix elements of $R(\omega, m_o, \mu)$ given by Eqs. (4.12), (4.13). The matrix elements computed here can be obtained from the earlier results by setting $\rho = 0$ and restricting the indices l, k to nonnegative integral values.

In accordance with our usual procedure, we can obtain identities for the associated Laguerre polynomials by substituting (4.26) into the addition theorem (4.23).

The results are (after simplification)

$$e^{-c_1 b_2}(c_1 + c_2)^n L_l^{(n)}[(b_1 + b_2)(c_1 + c_2)]$$
$$= \sum_{j=0}^{\infty} (c_1)^{j-l} L_l^{(j-l)}[b_1 c_1](c_2)^{l+n-j} L_j^{(l+n-j)}[b_2 c_2], \tag{4.28}$$

where n, l are integers, $l \geqslant 0$, $l + n \geqslant 0$, and $b_1, b_2, c_1, c_2 \in \mathbb{C}$.

Comparing (4.26) and (4.24) we can easily derive the relations

$$\begin{aligned} c^n L_l^{(n)}(bc)\big|_{c=0} &= \begin{cases} 0 & \text{if } n > 0, \\ \dfrac{(-b)^{-n}}{(-n)!} & \text{if } n \leqslant 0, \end{cases} \\ c^n L_l^{(n)}(bc)\big|_{b=0} &= \begin{cases} \dbinom{n+l}{n} c^n & \text{if } n \geqslant 0, \\ 0 & \text{if } n < 0. \end{cases} \end{aligned} \tag{4.29}$$

These relations can be used to derive identities for the associated Laguerre polynomials which are simple consequences of (4.28). Thus, if $c_1 = b_2 = 0$, $c_2 = 1$, $b_1 = x$, (4.28) reduces to

$$L_l^{(n)}(x) = \begin{cases} \displaystyle\sum_{j=0}^{l} \frac{(-x)^j}{j!} \binom{l+n}{j+n} & \text{if } n \geqslant 0, \\ \displaystyle(-x)^{-n} \sum_{j=0}^{l+n} \frac{(-x)^j}{(j-n)!} \binom{l+n}{j} & \text{if } n \leqslant 0, \quad l+n \geqslant 0, \end{cases}$$

which is the series expansion for the associated Laguerre polynomials.

For $c_1 = 0$, $c_2 = 1$, $b_1 = x$, $b_2 = y$, Eq. (4.28) reduces to

$$L_l^{(n)}(x + y) = \sum_{j=0}^{\infty} \frac{(-x)^j}{j!} L_{l-j}^{(n+j)}(y),$$

and if $c_1 = 1$, $c_2 = 0$, $b_1 = x$, $b_2 = y$, it becomes

$$L_l^{(n)}(x + y) = e^y \sum_{j=0}^{\infty} \frac{(-y)^j}{j!} L_l^{(n+j)}(x).$$

For $c_1 = -c_2 = 1$, $b_1 = x$, $b_2 = -y$ the resulting expressions are

$$e^y \frac{(x-y)^n}{n!} = \sum_{j=0}^{\infty} (-1)^{l-j} L_l^{(j-l)}(x) L_j^{(l-n-j)}(y),$$

for all $l \geqslant n \geqslant 0$, and

$$0 = \sum_{j=0}^{\infty} (-1)^j L_l^{(j-l)}(x) L_j^{(l+n-j)}(y),$$

for all $l, n \geqslant 0$. If $c_1 = c_2 = 1$, $b_1 = x$, $b_2 = y$, we obtain

$$e^{-y} 2^n L_l^{(n)}[2(x+y)] = \sum_{j=0}^{\infty} L_l^{(j-l)}(x) L_j^{(l+n-j)}(y).$$

4-3 The Representation $\downarrow_{\omega,\mu}$

As was stated in Theorem 2.2, the irreducible representation $\downarrow_{\omega,\mu}$ of $\mathcal{G}(0, 1)$ is defined for each $\omega, \mu \in \mathcal{C}$ such that $\mu \neq 0$. The spectrum of $\downarrow_{\omega,\mu}$ is the set

$$S = \{-\omega - 1 - n \colon n \text{ a nonnegative integer}\},$$

and there is a basis $\{f_m, m \in S\}$ for the representation space V such that

$$\begin{aligned} &\mathrm{J}^3 f_m = m f_m, \qquad \mathrm{E} f_m = -\mu f_m, \qquad \mathrm{J}^+ f_m = -(m + \omega + 1) f_{m+1}, \\ &\mathrm{J}^- f_m = \mu f_{m-1}, \qquad \mathrm{C}_{0,1} f_m = (\mathrm{J}^+\mathrm{J}^- - \mathrm{E}\mathrm{J}^3) f_m = -\mu\omega f_m. \end{aligned} \tag{4.30}$$

(We set $f_{-\omega} \equiv 0$, so $\mathrm{J}^+ f_{-\omega-1} = 0$.)

It is impossible to find a realization of this representation in terms of differential operators of the form (2.36). In fact, if $\mathrm{J}^+ = \mu z$ it is impossible to find a nonzero analytic function $f_{-\omega-1}(z)$ such that $\mathrm{J}^+ f_{-\omega-1} = 0$. We could find a realization in terms of the operators (2.37). However, it is more convenient to use the following modification of (2.36):

$$\mathrm{J}^3 = -\omega - 1 - z\frac{d}{dz}, \qquad \mathrm{E} = -\mu, \qquad \mathrm{J}^+ = \frac{d}{dz}, \qquad \mathrm{J}^- = \mu z. \tag{4.31}$$

As is easily seen, these operators satisfy the commutation relations (4.2), hence they generate a Lie algebra isomorphic to $\mathcal{G}(0, 1)$. We will define a realization of $\downarrow_{\omega,\mu}$ such that $\mathrm{J}^\pm$, J^3, E take the form of the differential operators (4.31) acting on the vector space $\mathcal{V}_2$ of all finite linear combinations of the functions $h_n(z) = z^n$, $n = 0, 1, 2, \ldots$. Namely, we define the basis vectors f_m of $\mathcal{V}_2$ by $f_m(z) = h_n(z) = z^n$ where $m = -n - \omega - 1 \in S$. Then we have

$$\mathrm{J}^3 f_m = \left(-\omega - 1 - z\frac{d}{dz}\right) z^n = (-\omega - 1 - n)\, z^n = m f_m\,,$$

$$\mathrm{J}^+ f_m = \frac{d}{dz} z^n = n z^{n-1} = -(m + \omega + 1) f_{m+1}\,,$$

$$\mathrm{J}^- f_m = (\mu z)\, z^n = \mu z^{n+1} = \mu f_{m-1}\,,$$

$$\mathrm{E} f_m = -\mu z^n = -\mu f_m$$

for all $m \in S$. Comparison with (4.30) shows that our choice of differential operators and basis vectors leads to a realization of $\downarrow_{\omega,\mu}$.

As in Section 4-2 we can extend this realization to a multiplier representation of $G(0, 1)$ defined on the complex vector space $\mathcal{O}\!l_2$ of all entire analytic functions of z. Since the details of the computation are so similar to the computation for $\uparrow_{\omega,\mu}$ it will be sufficient to merely list the results.

The generalized Lie derivatives (4.31) induce a multiplier representation of $G(0, 1)$ defined by operators $\mathbf{C}(g)$, $g \in G(0, 1)$, on $\mathcal{O}\!l_2$:

$$[\mathbf{C}(g)f](z) = \exp[\mu(cz + bc - a) - (\omega + 1)\tau]\, f(e^{-\tau}z + e^{-\tau}b) \tag{4.32}$$

for all $f \in \mathcal{O}\!l_2$, $z \in \mathcal{C}$. The representation property is

$$\mathbf{C}(g_1 g_2) f = \mathbf{C}(g_1)[\mathbf{C}(g_2) f], \qquad g_1, g_2 \in G(0, 1), \quad f \in \mathcal{O}\!l_2\,,$$

and the matrix elements $C_{lk}(g)$ are defined by

$$[\mathbf{C}(g) h_k](z) = \sum_{l=0}^{\infty} C_{lk}(g)\, h_l(z), \qquad k = 0, 1, 2, \ldots, \tag{4.33}$$

or

$$\exp[\mu(cz + bc - a) - (\omega + k + 1)\tau](z + b)^k = \sum_{l=0}^{\infty} C_{lk}(g) z^l,$$

$$g \in G(0, 1), \quad k \geqslant 0. \tag{4.34}$$

The addition theorem for the matrix elements is

$$C_{lk}(g_1 g_2) = \sum_{j=0}^{\infty} C_{lj}(g_1)\, C_{jk}(g_2), \qquad l, k \geqslant 0, \quad g_1, g_2 \in G(0, 1). \tag{4.35}$$

From the generating function (4.34) we can obtain the following explicit formula for the matrix elements:

$$C_{lk}(g) = \exp[\mu(bc - a) - (\omega + k + 1)\tau]b^{k-l}L_l^{(k-l)}(-\mu bc), \quad k, l \geqslant 0. \quad (4.36)$$

We can derive addition theorems for the associated Laguerre polynomials by substituting (4.36) into (4.35). However, the resulting expressions are identical with expressions (4.28) derived in the last section, so we will omit this computation.

4-4 Differential Equations for the Matrix Elements

The individual matrix elements of the representations $R(\omega, m_0, \mu)$, $\uparrow_{\omega,\mu}$, and $\downarrow_{\omega,\mu}$ are all entire functions of the complex group parameters a, b, c, τ and can be considered as analytic functions on the group manifold of $G(0, 1)$.

Let $\mathcal{A}[G(0, 1)]$ or $\mathcal{A}$ for short, be the complex vector space of all entire analytic functions on $G(0, 1)$. There is a natural action of $G(0, 1)$ on $\mathcal{A}$ as a transformation group. For every $g' \in G(0, 1)$ we can define a linear operator $\mathbf{P}(g')$: $\mathcal{A} \to \mathcal{A}$ by

$$[\mathbf{P}(g')f](g) = f(gg'), \quad f \in \mathcal{A}, \quad g \in G(0, 1). \quad (4.37)$$

From this definition it is evident that $\mathbf{P}(g_1g_2)f = \mathbf{P}(g_1)[\mathbf{P}(g_2)f]$ for all $g_1, g_2 \in G(0, 1)$.

Clearly each of the matrix elements $A_{jk}(g)$, $B_{jk}(g)$, $C_{jk}(g)$ computed above is a member of $\mathcal{A}$. Furthermore, the action of $\mathbf{P}$ on these matrix elements is easily determined. We have

$$[\mathbf{P}(g')A_{jk}](g) = A_{jk}(gg') = \sum_{l=-\infty}^{\infty} A_{lk}(g')\,A_{jl}(g),$$

j, k integers,

$$[\mathbf{P}(g')B_{jk}](g) = B_{jk}(gg') = \sum_{l=0}^{\infty} B_{lk}(g')\,B_{jl}(g), \quad (4.38)$$

j, k nonnegative integers,

$$[\mathbf{P}(g')C_{jk}](g) = C_{jk}(gg') = \sum_{l=0}^{\infty} C_{lk}(g')\,C_{jl}(g),$$

j, k nonnegative integers.

Comparing these expressions with (4.7), (4.21), and (4.33) we see that for fixed j, the functions $\{A_{jk}\}$ form a basis for the representation $R(\omega, m_o, \mu)$, the functions $\{B_{jk}\}$ form a basis for the representation $\uparrow_{\omega,\mu}$, and the functions $\{C_{jk}\}$ form a basis for the representation $\downarrow_{\omega,\mu}$. Therefore, the Lie derivatives J^+, J^-, J^3, E defined on $\mathcal{O}\!\mathcal{l}$ by

$$\begin{aligned}
\mathrm{J}^+f(g) &= \frac{d}{dt}\left[\mathbf{P}(\exp t\mathscr{J}^+)f\right](g)\Big|_{t=0}, \\
\mathrm{J}^-f(g) &= \frac{d}{dt}\left[\mathbf{P}(\exp t\mathscr{J}^-)f\right](g)\Big|_{t=0}, \\
\mathrm{J}^3f(g) &= \frac{d}{dt}\left[\mathbf{P}(\exp t\mathscr{J}^3)f\right](g)\Big|_{t=0}, \\
\mathrm{E}f(g) &= \frac{d}{dt}\left[\mathbf{P}(\exp t\mathscr{E})f\right](g)\Big|_{t=0}, \qquad f\in\mathcal{O}\!\mathcal{l},
\end{aligned}\tag{4.39}$$

must satisfy the commutation relations (4.2), and their action on the matrix elements A_{jk}, B_{jk}, C_{jk} must be given by

$$\begin{aligned}
&\mathrm{J}^3A_{jk}(g) = (m_o + k)\,A_{jk}(g), \qquad \mathrm{J}^+A_{jk}(g) = \mu A_{j\,k+1}(g), \\
&\mathrm{J}^-A_{jk}(g) = (m_o + \omega + k)\,A_{j\,k-1}(g), \qquad \mathrm{E}A_{jk}(g) = \mu A_{jk}(g), \\
&\mathrm{C}_{0,1}A_{jk}(g) = (\mathrm{J}^+\mathrm{J}^- - \mathrm{E}\mathrm{J}^3)\,A_{jk}(g) = \mu\omega A_{jk}(g)
\end{aligned}\tag{4.40}$$

for all integers j, k,

$$\begin{aligned}
&\mathrm{J}^3B_{jk}(g) = (-\omega + k)\,B_{jk}(g), \qquad \mathrm{J}^+B_{jk}(g) = \mu B_{j\,k+1}(g), \\
&\mathrm{J}^-B_{jk}(g) = kB_{j\,k-1}(g), \qquad \mathrm{E}B_{jk}(g) = \mu B_{jk}(g), \\
&\mathrm{C}_{0,1}B_{jk}(g) = \mu\omega B_{jk}(g)
\end{aligned}\tag{4.41}$$

for all nonnegative integers j, k,

$$\begin{aligned}
&\mathrm{J}^3C_{jk}(g) = (-\omega - 1 - k)\,C_{jk}(g), \qquad \mathrm{J}^+C_{jk}(g) = kC_{j\,k-1}(g), \\
&\mathrm{J}^-C_{jk}(g) = \mu C_{j\,k+1}(g), \qquad \mathrm{E}C_{jk}(g) = -\mu C_{jk}(g), \\
&\mathrm{C}_{0,1}C_{jk}(g) = -\mu\omega C_{jk}(g)
\end{aligned}\tag{4.42}$$

for all nonnegative integers j, k.

These expressions yield recursion relations and differential equations for the matrix elements. To evaluate them we must compute the Lie derivatives $\mathrm{J}^\pm$, J^3, E defined by (4.39). In terms of the usual coordinates (a, b, c, τ) for elements of $G(0, 1)$ the Lie derivatives will be linear differential operators in these four complex variables. However, the confluent hypergeometric functions and Laguerre functions associated with the matrix elements A_{jk}, B_{jk}, C_{jk} are functionally dependent only on the product bc of the group parameters. For this reason it is convenient to adopt new coordinates $[a, r, c, \tau]$ where a, c, τ are defined as before, but $r = bc$. This new coordinate system is not uniquely defined over the whole group, but only for those group elements such that $c \neq 0$.

If $g \in G(0, 1)$ has coordinates $[a, r, c, \tau]$ and

$$g' = \begin{pmatrix} 1 & c'e^{\tau'} & a' & \tau' \\ 0 & e^{\tau'} & b' & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then the coordinates of gg' are

$$\left[a + a' + e^{\tau}cb', r\left(1 + \frac{e^{-\tau}c'}{c} + \frac{b'c' + e^{\tau}b'c}{r}\right), c + e^{-\tau}c', \tau + \tau'\right]$$

in the new coordinate system. Using this result and the definition (4.39) of the Lie derivatives, we obtain

$$\mathrm{J}^+ = e^{\tau}c\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial r}\right), \qquad \mathrm{J}^- = e^{-\tau}\left(\frac{\partial}{\partial c} + \frac{r}{c}\frac{\partial}{\partial r}\right),$$

$$\mathrm{J}^3 = \frac{\partial}{\partial \tau}, \qquad \mathrm{E} = \frac{\partial}{\partial a}, \tag{4.43}$$

$$\mathrm{C}_{0,1} = r\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} + c\frac{\partial^2}{\partial c\,\partial r} + r\frac{\partial^2}{\partial r\,\partial a} + c\frac{\partial^2}{\partial c\,\partial a} - \frac{\partial^2}{\partial \tau\,\partial a}.$$

In terms of the new coordinates, A_{jk} and B_{jk} are given by

$$A_{jk}(g) = e^{\mu a + (m_0 + k)\tau}c^{k-j}L_{\rho+j}^{(k-j)}(-\mu r), \qquad \rho = m_0 + \omega,$$

$$B_{jk}(g) = e^{\mu a + (-\omega + k)\tau}c^{k-j}L_j^{(k-j)}(-\mu r).$$

Substituting these expressions into (4.40), (4.41) and simplifying we obtain the relations

$$\frac{\partial}{\partial r}L_\rho^{(k)}(r) - L_\rho^{(k)}(r) = -L_\rho^{(k+1)}(r),$$

$$r\frac{\partial}{\partial r}L_\rho^{k}(r) + kL_\rho^{k}(r) = (\rho + k)\,L_\rho^{(k-1)}(r), \tag{4.44}$$

$$r\frac{\partial^2}{\partial r^2}L_\rho^{(k)}(r) + (k + 1 - r)\frac{\partial}{\partial r}L_\rho^{(k)}(r) + \rho L_\rho^{(k)}(r) = 0,$$

valid for $\rho \in \mathcal{C}$, k an integer. In fact, these relations follow from (4.40) when ρ is not an integer and from (4.41) when ρ is an integer.

Similar results can be derived for the matrix elements $C_{jk}(g)$. However, we will omit this computation.

4-5 The Representation $\uparrow_{\omega_1,\mu_1} \otimes \uparrow_{\omega_2,\mu_2}$

As was demonstrated in Section 4-2, the representation $\uparrow_{\omega,\mu}$, $\mu \neq 0$, of $\mathcal{G}(0, 1)$ can be realized on the space $\mathscr{V}_2$ of all finite linear combinations of the basis functions $h_k(z) = z^k$, $k = 0, 1, 2, \ldots$. In terms of this basis the matrix elements $B_{lk}^{(\omega,\mu)}(g)$, $g \in G(0, 1)$, are defined by (4.26). (The superscripts have been added to denote the particular representation $\uparrow_{\omega,\mu}$.)

We will define a representation $\uparrow_{\omega_1,\mu_1} \otimes \uparrow_{\omega_2,\mu_2}$ of $\mathcal{G}(0, 1)$ on the space $\mathscr{V}_2 \otimes \mathscr{V}_2$ of all finite linear combinations of the basis functions $h_{k,l}(z, w) = z^k w^l$, $k, l = 0, 1, 2, \ldots$. Note that $\mathscr{V}_2 \otimes \mathscr{V}_2$ is contained in the vector space $\mathcal{A}_2 \otimes \mathcal{A}_2$ consisting of all entire functions $f(z, w)$ in the complex variables z and w. The functions $h_{k,l}(z, w)$ form an analytic basis for $\mathcal{A}_2 \otimes \mathcal{A}_2$ since every element f of this space has a unique power series expansion

$$f(z, w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l, \qquad a_{kl} \in \mathcal{C},$$

convergent for all $z, w \in \mathcal{C}$.

The operators $\mathbf{T}(g)$,

$$\begin{aligned}[\mathbf{T}(g)f](z, w) &= \exp[b(\mu_1 z + \mu_2 w) - (\omega_1 + \omega_2)\tau + (\mu_1 + \mu_2)a] \\ &\quad \cdot f(e^\tau z + e^\tau c, e^\tau w + e^\tau c), \qquad f \in \mathcal{A}_2 \otimes \mathcal{A}_2, \end{aligned} \tag{4.45}$$

where $g = g(a, b, c, \tau) \in G(0, 1)$, clearly satisfy the group property

$$\mathbf{T}(g_1 g_2)f = \mathbf{T}(g_1)[\mathbf{T}(g_2)f]$$

for all $g_1, g_2 \in G(0, 1)$ and thus define a multiplier representation of $G(0, 1)$ on $\mathcal{A}_2 \otimes \mathcal{A}_2$. This multiplier representation of $G(0, 1)$ induces a representation of $\mathcal{G}(0, 1)$ in terms of the generalized Lie derivatives

$$\begin{aligned} J^3 &= -\omega_1 - \omega_2 + z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}, \qquad J^+ = \mu_1 z + \mu_2 w, \\ J^- &= \frac{\partial}{\partial z} + \frac{\partial}{\partial w}, \qquad E = \mu_1 + \mu_2. \end{aligned} \tag{4.46}$$

The representation of $\mathcal{G}(0, 1)$ on $\mathscr{V}_2 \otimes \mathscr{V}_2$ defined by these operators, which we denote $\uparrow_{\omega_1,\mu_1} \otimes \uparrow_{\omega_2,\mu_2}$, is not irreducible. We will decompose

$\mathscr{V}_2 \otimes \mathscr{V}_2$ into a direct sum of subspaces such that each subspace is irreducible under the action of $\mathscr{G}(0, 1)$ as given by (4.46). Successful completion of this task will enable us to derive formulas expressing the product of two associated Laguerre polynomials as a sum of Laguerre polynomials. In order to carry out this decomposition, however, we must require $\mu_1, \mu_2 \neq 0$, $\mu_1 + \mu_2 \neq 0$.

Let $\mathbf{Q}[.,.]$ be the complex symmetric bilinear form on $\mathscr{V}_2 \otimes \mathscr{V}_2$ defined by

$$\mathbf{Q}[h_{k,l}, h_{k',l'}] = \delta_{k,k'}\,\delta_{l,l'}\,\frac{k!\,l!}{\mu_1^k \mu_2^l}, \qquad k, k', l, l' = 0, 1, 2, \ldots . \tag{4.47}$$

Thus, $\mathbf{Q}[f_1, f_2] \in \mathscr{C}$ for all $f_1, f_2 \in \mathscr{V}_2 \otimes \mathscr{V}_2$ and

$$\begin{aligned} \mathbf{Q}[f_1, f_2] &= \mathbf{Q}[f_2, f_1], \\ \mathbf{Q}[a_1 f_1 + a_2 f_2, f_3] &= a_1\mathbf{Q}[f_1, f_3] + a_2\mathbf{Q}[f_2, f_3] \end{aligned} \tag{4.48}$$

for all $a_1, a_2 \in \mathscr{C}$, $f_1, f_2, f_3 \in \mathscr{V}_2 \otimes \mathscr{V}_2$. Equation (4.47) together with (4.48) completely determines $\mathbf{Q}$. We will use this bilinear form as a bookkeeping device.

Lemma 4.1 If J^+, J^-, J^3, E are defined by (4.47) and f_1, $f_2 \in \mathscr{V}_2 \otimes \mathscr{V}_2$, then

(i) $\mathbf{Q}[J^3 f_1, f_2] = \mathbf{Q}[f_1, J^3 f_2]$,
(ii) $\mathbf{Q}[J^+ f_1, f_2] = \mathbf{Q}[f_1, J^- f_2]$,
(iii) $\mathbf{Q}[E f_1, f_2] = \mathbf{Q}[f_1, E f_2]$.

PROOF Properties (i) and (iii) are trivial. Since $\mathbf{Q}$ is bilinear, it is sufficient for the proof of (ii) to consider the case where $f_1 = h_{k,l}$ and $f_2 = h_{k',l'}$. From (4.47) we have

$$\begin{aligned} \mathbf{Q}[J^+ h_{k,l}, h_{k',l'}] &= \mathbf{Q}[(\mu_1 z + \mu_2 w)\, z^k w^l, z^{k'} w^{l'}] \\ &= \mu_1 \mathbf{Q}[z^{k+1} w^l, z^{k'} w^{l'}] + \mu_2 \mathbf{Q}[z^k w^{l+1}, z^{k'} w^{l'}] \\ &= \delta_{k+1,k'}\,\delta_{l,l'}\,\frac{(k+1)!\,l!}{\mu_1^k \mu_2^l} + \delta_{k,k'}\,\delta_{l+1,l'}\,\frac{k!\,l'!}{\mu_1^k \mu_2^{l-1}} \end{aligned}$$

and

$$\begin{aligned} \mathbf{Q}[h_{k,l}, J^- h_{k',l'}] &= \mathbf{Q}\left[z^k w^l, \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial w}\right) z^{k'} w^{l'}\right] \\ &= k'\mathbf{Q}[z^k w^l, z^{k'-1} w^{l'}] + l'\mathbf{Q}[z^k w^l, z^{k'} w^{l'-1}] \\ &= \delta_{k+1,k'}\,\delta_{l,l'}\,\frac{(k+1)!\,l!}{\mu_1^k \mu_2^l} + \delta_{k,k'}\,\delta_{l+1,l'}\,\frac{k!\,l'!}{\mu_1^k \mu_2^{l-1}}, \end{aligned}$$

which proves (ii).

The bilinear form **Q** was defined by (4.47) just so Lemma 4.1 would be valid.

We will decompose $\mathscr{V}_2 \otimes \mathscr{V}_2$ into subspaces, each of which forms the basis space for a representation $\uparrow_{\omega,\mu}$ of $\mathscr{G}(0, 1)$. One way to determine this decomposition is to compute the eigenvectors f of J^3 which are of lowest weight, i.e., the eigenvectors of J^3 such that $J^- f = 0$ where J^3 and J^- are given by (4.46). Thus, we look for solutions of the equations

$$J^3 f(z, w) = \left(-\omega_1 - \omega_2 + z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}\right) f(z, w) = \lambda f(z, w),$$

$$J^- f(z, w) = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial w}\right) f(z, w) = 0$$

where the eigenvalue λ is to be determined. To within an arbitrary constant the solutions are the functions

$$f_{s,0}(z, w) = (z - w)^s, \qquad s = 0, 1, 2, \ldots . \tag{4.49}$$

In fact,

$$J^3 f_{s,0} = (s - \omega_1 - \omega_2) f_{s,0}, \qquad J^- f_{s,0} = 0. \tag{4.50}$$

From (4.47) and (4.49) it is a straightforward computation to obtain the result

$$\mathbf{Q}[f_{s,0}, f_{s,0}] = s! \left(\frac{\mu_1 + \mu_2}{\mu_1 \mu_2}\right)^s. \tag{4.51}$$

(Recall the assumption $\mu_1 + \mu_2 \neq 0$.)

We now define basis vectors $f_{s,k} \in \mathscr{V}_2 \otimes \mathscr{V}_2$ by

$$f_{s,k} = (\mu_1 + \mu_2)^{-k} (J^+)^k f_{s,0} \tag{4.52}$$

for all integers s, $k \geqslant 0$. It is easy to see that each function $f_{s,k}$ is a polynomial of order $s + k$ in z and w. Moreover, for every positive integer l there are exactly $l + 1$ functions $f_{s,k}$ such that $s + k = l$, i.e., there are $l + 1$ such functions which are of order l in z and w. Hence, if the set $\{f_{s,k}\}$ is linearly independent it forms a basis for $\mathscr{V}_2 \otimes \mathscr{V}_2$ and an analytic basis for $\mathscr{A}_2 \otimes \mathscr{A}_2$.

Lemma 4.2

(i) $J^+ f_{s,k} = (\mu_1 + \mu_2) f_{s,k+1}$,

(ii) $J^- f_{s,k} = k f_{s,k-1}$,

(iii) $J^3 f_{s,k} = (-\omega_1 - \omega_2 + s + k) f_{s,k}$,

(iv) $E f_{s,k} = (\mu_1 + \mu_2) f_{s,k}$.

PROOF (i) $J^+ f_{s,k} = (\mu_1 + \mu_2)^{-k}(J^+)^{k+1} f_{s,o} = (\mu_1 + \mu_2) f_{s,k+1}$.

(iii) The identity $[J^3, (J^+)^k] = k(J^+)^k$ for all $k \geq 0$ is easily proved by induction. Then

$$\begin{aligned} J^3 f_{s,k} &= (\mu_1 + \mu_2)^{-k} J^3 (J^+)^k f_{s,0} \\ &= (\mu_1 + \mu_2)^{-k} (J^+)^k J^3 f_{s,0} + (\mu_1 + \mu_2)^{-k} k (J^+)^k f_{s,0} \\ &= (-\omega_1 - \omega_2 + s + k)(\mu_1 + \mu_2)^{-k} (J^+)^k f_{s,0} \\ &= (-\omega_1 - \omega_2 + s + k) f_{s,k} . \end{aligned}$$

(iv) Obvious. (ii) We use induction on k. From (4.50), $J^- f_{s,o} = 0$, so the equation is valid for $k = 0$. Assume it is valid for $k \leq k_o$. Then $J^- f_{s,k_o+1} = (\mu_1 + \mu_2)^{-1} J^- J^+ f_{s,k_o}$ from property (i). But, $J^- J^+ = J^+ J^- + E$, so from (i), (iv) and the induction hypothesis we have

$$\begin{aligned} J^- J^+ f_{s,k_o} &= J^+ J^- f_{s,k_o} + E f_{s,k_o} \\ &= k_o J^+ f_{s,k_o-1} + (\mu_1 + \mu_2) f_{s,k_o} \\ &= (k_o + 1)(\mu_1 + \mu_2) f_{s,k_o} . \end{aligned}$$

Therefore, $J^- f_{s,k_o+1} = (k_o + 1) f_{s,k_o}$. Q.E.D.

Lemma 4.2 implies that for a fixed value of s, the vectors $f_{s,k}$, $k = 0, 1, 2, \ldots$, form a basis for the representation $\uparrow_{\omega_1+\omega_2-s,\mu_1+\mu_2}$ of $\mathcal{G}(0, 1)$ (compare Lemma 4.2 with Eqs. (4.18)). Thus, the action of $G(0, 1)$ on the vectors $f_{s,k}$ is

$$[\mathbf{T}(g) f_{s,k}](z, w) = \sum_{l=0}^{\infty} B_{lk}^{(\omega_1+\omega_2-s,\mu_1+\mu_2)}(g) f_{s,l}(z, w),$$

$$k, s \geq 0, \quad g \in G(0, 1), \tag{4.53}$$

where the operator $\mathbf{T}(g)$ is defined by (4.45) and the matrix elements $B_{lk}^{(\omega_1+\omega_2-s,\mu_1+\mu_2)}(g)$ are given by (4.26) ($\omega = \omega_1 + \omega_2 - s$, $\mu = \mu_1 + \mu_2$).

Lemma 4.3 $Q[f_{s,k}, f_{s'k'}] = \delta_{s,s'} \delta_{k,k'} M_{sk}$ where

$$M_{sk} = \frac{s!\, k!\, (\mu_1 + \mu_2)^{s-k}}{(\mu_1 \mu_2)^s} .$$

PROOF From Lemmas 4.1 and 4.2 we find

$$\begin{aligned} Q[f_{s,k}, f_{s,k}] &= (\mu_1 + \mu_2)^{-2} Q[J^+ f_{s,k-1}, J^+ f_{s,k-1}] \\ &= (\mu_1 + \mu_2)^{-2} Q[f_{s,k-1}, J^- J^+ f_{s,k-1}] \\ &= (\mu_1 + \mu_2)^{-1} k Q[f_{s,k-1}, f_{s,k-1}]. \end{aligned}$$

Since

$$\mathbf{Q}[f_{s,0}, f_{s,0}] = s! \left(\frac{\mu_1 + \mu_2}{\mu_1 \mu_2}\right)^s$$

by (4.51), it is an easy induction argument to obtain $\mathbf{Q}[f_{s,k}, f_{s,k}] = M_{sk}$ for all $s, k \geqslant 0$. The property $\mathbf{Q}[\mathrm{J}^3 f_{s,k}, f_{s',k'}] = \mathbf{Q}[f_{s,k}, \mathrm{J}^3 f_{s',k'}]$ implies $(s + k - s' - k')\,\mathbf{Q}[f_{s,k}, f_{s',k'}] = 0$. Hence, $\mathbf{Q}[f_{s,k}, f_{s',k'}] = 0$ unless $s + k = s' + k'$. If $k > k'$ it is obvious from (ii), Lemma 4.2, that $(\mathrm{J}^-)^k f_{s',k'} = 0$ for all $s' \geqslant 0$. Thus,

$$\begin{aligned} \mathbf{Q}[f_{s,k}, f_{s',k'}] &= (\mu_1 + \mu_2)^{-k}\,\mathbf{Q}[(\mathrm{J}^+)^k f_{s,0}, f_{s',k'}] \\ &= (\mu_1 + \mu_2)^{-k}\,\mathbf{Q}[f_{s,0}, (\mathrm{J}^-)^k f_{s',k'}] = 0 \qquad \text{for all} \quad s, s' \geqslant 0, \end{aligned}$$

and $\mathbf{Q}[f_{s,k}, f_{s',k'}] = 0$ unless $k = k'$. Collecting all these facts together, we have the lemma.

Among other things Lemma 4.3 proves that the vectors $\{f_{s,k}\}$ are linearly independent. Hence, by our earlier remarks they form a basis for $\mathscr{V}_2 \otimes \mathscr{V}_2$ and an analytic basis for $\mathscr{A}_2 \otimes \mathscr{A}_2$. $\mathscr{V}_2 \otimes \mathscr{V}_2$ can be decomposed into a direct sum of subspaces

$$\mathscr{V}_2 \otimes \mathscr{V}_2 \cong \sum_{s=0}^{\infty} \oplus \mathscr{V}_s'$$

where $\mathscr{V}_s'$ is the subspace spanned by the basis vectors $\{f_{s,k}\}$, $k = 0, 1, 2, \ldots$. The $\mathscr{V}_s'$ transforms under $\mathscr{G}(0, 1)$ according to the irreducible representation $\uparrow_{\omega_1+\omega_2-s,\mu_1+\mu_2}$. This decomposition induces an analogous decomposition of $\mathscr{A}_2 \otimes \mathscr{A}_2$.

Theorem 4.1 $\uparrow_{\omega_1,\mu_1} \otimes \uparrow_{\omega_2,\mu_2} \cong \sum_{s=0}^{\infty} \oplus \uparrow_{\omega_1+\omega_2-s,\mu_1+\mu_2}$ if μ_1, μ_2, $\mu_1 + \mu_2 \neq 0$.

We define Clebsch–Gordan coefficients $\mathrm{H}(\mu_1, l; \mu_2, j \mid s, k)$, nonzero only if $s + k = l + j$, by

$$f_{s,k} = \sum_{l=0}^{s+k} \mathrm{H}(\mu_1, l; \mu_2, s + k - l \mid s, k)\, h_{l,s+k-l}, \qquad k, s \geqslant 0, \quad (4.54)$$

i.e., $f_{s,k}(z, w) = \sum_{l=0}^{s+k} \mathrm{H}(\mu_1, l; \mu_2, s + k - l \mid s, k)\, z^l w^{s+k-l}$. A simple consequence of this definition and (4.47) is the relation

$$\mathbf{Q}[f_{s,k}, h_{l,j}] = \mathrm{H}(\mu_1, l; \mu_2, j \mid s, k)\, \frac{l!\, j!}{\mu_1^l \mu_2^j}\, \delta_{j,s+k-l}, \qquad k, s, l, j \geqslant 0. \quad (4.55)$$

We can use this relation to invert Eq. (4.54). Since $\{f_{s,k}\}$ is a basis for $\mathscr{V}_2 \otimes \mathscr{V}_2$ we can write

$$h_{l,j} = \sum_{q=0}^{l+j} a^q_{l,j} f_{q,l+j-q}, \qquad l, j \geqslant 0,$$

for suitable constants $a^q_{l,j}$. An easy computation using Lemma 4.3 gives

$$\mathrm{Q}[f_{s,k}, h_{l,j}] = a^s_{l,j}\, \delta_{k,l+j-s} M_{sk}. \tag{4.56}$$

Comparing (4.55) and (4.56) we have

$$a^s_{lj} = H(\mu_1, l; \mu_2, j \mid s, l+j-s) \frac{l!\, j!}{\mu_1^l \mu_2^j M_{s,l+j-s}},$$

and finally

$$h_{l,j} = \sum_{s=0}^{l+j} \frac{\mathrm{H}(\mu_1, l; \mu_2, j \mid s, l+j-s)\, l!\, j!}{\mu_1^l \mu_2^j M_{s,l+j-s}} f_{s,l+j-s}. \tag{4.57}$$

It is easy to obtain a generating function for the Clebsch–Gordan coefficients directly from (4.53). In fact, for $k = 0$ and $g = \exp b\mathscr{J}^+$ in (4.53) one obtains the generating function

$$e^{b(\mu_1 z + \mu_2 w)}(z-w)^s = \sum_{l=0}^{\infty} \frac{b^l(\mu_1+\mu_2)^l}{l!} f_{s,l}(z, w)$$

$$= \sum_{l=0}^{\infty} \sum_{j=0}^{s+l} \frac{b^l(\mu_1+\mu_2)^l}{l!} \mathrm{H}(\mu_1, j; \mu_2, s+l-j \mid s, l)\, z^j w^{s+l-j}. \tag{4.58}$$

Equating coefficients of b^l on both sides of this expression we find

$$\left(\frac{\mu_1 z + \mu_2 w}{\mu_1 + \mu_2}\right)^k (z-w)^s = \sum_{j=0}^{s+k} \mathrm{H}(\mu_1, j; \mu_2, s+k-j \mid s, k)\, z^j w^{s+k-j}$$

or

$$\mathrm{H}(\mu_1, j; \mu_2, s+k-j \mid s, k)$$

$$= \frac{k!\, s!}{(\mu_1+\mu_2)^k} \sum_n \frac{\mu_1^n \mu_2^{k-n} (-1)^{s+n-j}}{n!\,(j-n)!\,(k-n)!\,(s-j+n)!}, \quad s+k-j \geqslant 0, \tag{4.59}$$

where the sum on the right-hand side is taken over all integral values of n such that the summand is defined. The Clebsch–Gordan coefficients are independent of ω_1 and ω_2.

These coefficients are closely related to the Jacobi polynomials. In fact,

$$\mathrm{H}(\mu_1, j; \mu_2, s+k-j \mid s, k) = \frac{(-1)^{s+k-j}}{(1+\mu_2/\mu_1)^k} \frac{s!}{(s+k-j)!} \cdot \frac{F(-k, -s-k+j; j-k+1; -\mu_2/\mu_1)}{\Gamma(j-k+1)} \tag{4.60}$$

where we adopt the convention (A. 5) on the right-hand side of this equation.

If we apply the operator $\mathbf{T}(g)$, $g \in G(0, 1)$, to both sides of Eq. (4.57) we obtain

$$\sum_{l',j'=0}^{\infty} B_{l'l}^{(\omega_1,\mu_1)}(g)\, B_{j'j}^{(\omega_2,\mu_2)}(g)\, h_{l',j'}$$

$$= \sum_{k=0}^{\infty} \sum_{s=0}^{l+j} \frac{\mathrm{H}(\mu_1, l; \mu_2, j \mid s, l+j-s)}{\mu_1^l \mu_2^j M_{s,l+j-s}} l!\, j!\, B_{k,l+j-s}^{(\omega_1+\omega_2-s,\mu_1+\mu_2)}(g) f_{s,k}\,. \tag{4.61}$$

The right-hand side of this expression can be expanded in terms of the basis vectors $h_{l,k}$. Equating the coefficients of $h_{l,k}$ on both sides of the resulting expression we get

$$B_{ll'}^{(\omega_1,\mu_1)}(g)\, B_{kk'}^{(\omega_2,\mu_2)}(g) = \sum_{s=0}^{\min[l+k,l'+k']} \cdot \frac{\mathrm{H}(\mu_1, l; \mu_2, k \mid s, l+k-s)}{M_{s,l'+k'-s}} \mathrm{H}(\mu_1, l'; \mu_2, k' \mid s, l'+k'-s)$$

$$\cdot \frac{l'!\, k'!}{(\mu_1)^{l'}(\mu_2)^{k'}} B_{l+k-s,l'+k'-s}^{(\omega_1+\omega_2-s,\mu_1+\mu_2)}(g), \quad g \in G(0,1), \quad l, l', k, k' = 0, 1, 2, \ldots, \tag{4.62}$$

or

$$L_l^{(l'-l)}[\mu_1 r]\, L_k^{(k'-k)}[\mu_2 r] = \sum_{s=0}^{\min[l+k,l'+k']} \frac{(-1)^{k+k'} s!\, l'!}{k!\, (l'+k'-s)!}$$

$$\cdot \frac{(\mu_2/\mu_1)^{s-k'}}{(1+\mu_2/\mu_1)^{l+k}} \frac{F(s-l-k, -k; s-k+1; -\mu_2/\mu_1)}{\Gamma(s-k+1)}$$

$$\cdot \frac{F(s-l'-k', -k'; s-k'+1; -\mu_2/\mu_1)}{\Gamma(s-k'+1)} L_{l+k-s}^{(l'+k'-l-k)}[(\mu_1+\mu_2)r]. \tag{4.63}$$

The reader can work out some of the special cases of this formula for himself.

The above analysis demonstrates in a particular case the close connection between the reduction of tensor products of group representations into irreducible parts and identities expressing products of special functions as sums of special functions. We could carry out the same analysis for the representations $\downarrow_{\omega_1,\mu_1} \otimes \downarrow_{\omega_2,\mu_2}$. However, this procedure would lead to no new results for the associated Laguerre functions, so we omit it.

The method presented here for the reduction of a tensor product of representations works only if the tensor product can be decomposed into a direct sum of irreducible representations. In particular, representations of the form $R(\omega_1, m_1, \mu_1) \otimes R(\omega_2, m_2, \mu_2)$ or $\uparrow_{\omega_1,\mu} \otimes \downarrow_{\omega_2,-\mu}$ cannot be so decomposed and our procedure fails. However, when we consider unitary group representations in Section 4-18, we will be able to use the tools of functional analysis to obtain deeper results relating the properties of Laguerre functions to tensor products of group representations.

4-6 A Realization of $R(\omega, m_o, \mu)$ by *Type D′* Operators

In the first four sections of this chapter the irreducible representations of $\mathscr{G}(0, 1)$ were realized on spaces of analytic functions of one and four complex variables. Now we will construct realizations on a space of two complex variables, x and y. In fact, we will study those realizations of irreducible representations of $\mathscr{G}(0, 1)$ such that the operators $J^\pm$, J^3, E take the form

$$J^3 = \frac{\partial}{\partial y}, \qquad J^\pm = e^{\pm y}\left(\pm\frac{\partial}{\partial x} - \tfrac{1}{2}\mu x\right), \qquad E = \mu \tag{4.64}$$

where μ is a nonzero complex constant. These are the *type D′* operators defined in Section 2-7. (The constant q occurring in the expression for the *type D′* operators can be set equal to zero without any loss of generality.)

To begin with we construct a realization of $R(\omega, m_o, \mu)$ ($0 \leqslant \operatorname{Re} m_o < 1$, $\mu \neq 0$, and $m_o + \omega$ is not an integer). Thus, we look for nonzero functions $f_m(x, y) = Z_m(x)e^{my}$ such that

$$\begin{gathered} J^3 f_m = m f_m, \qquad E f_m = \mu f_m, \\ J^+ f_m = \mu f_{m+1}, \qquad J^- f_m = (m+\omega) f_{m-1}, \\ C_{0,1} f_m = (J^+J^- - EJ^3) f_m = \mu\omega f_m \end{gathered} \tag{4.65}$$

for all $m \in S = \{m_o + n: n \text{ an integer}\}$, where the differential operators $J^{\pm}$, J^3, E are given by (4.64). In terms of the functions $Z_m(x)$, relations (4.65) reduce to

$$\begin{aligned}
&\text{(i)} \quad \left(\frac{d}{dx} - \tfrac{1}{2}\mu x\right) Z_m(x) = \mu Z_{m+1}(x),\\
&\text{(ii)} \quad \left(-\frac{d}{dx} - \tfrac{1}{2}\mu x\right) Z_m(x) = (m+\omega)\, Z_{m-1}(x), \qquad (4.66)\\
&\text{(iii)} \quad \left(-\frac{d^2}{dx^2} + \frac{\mu^2 x^2}{4} - \tfrac{1}{2}\mu - \mu(m+\omega)\right) Z_m(x) = 0.
\end{aligned}$$

The complex constant ω is clearly irrelevant as far as the study of the special functions Z_m is concerned, since we could remove it by relabeling the functions $Z_m \equiv Z'_{m+\omega}$. Hence, without loss of generality we can assume $\omega = 0$. Also, there is no loss of generality for special function theory if we set $\mu = 1$.

The solutions of Eqs. (4.66) are well known. As remarked in Section 2-7, (iii) is the parabolic cylinder equation and its solutions are the parabolic cylinder functions $D_m(x)$, see (A. 16). In fact, for all $m \in S$ the following choices for Z_m satisfy (4.66):

$$\begin{aligned}
&(1) \quad Z_m(x) = (-1)^{m-m_o} D_m(x),\\
&(2) \quad Z_m(x) = D_m(-x).
\end{aligned} \qquad (4.67)$$

Since $\omega = 0$ we must have $0 < \operatorname{Re} m_o < 1$; hence, the elements m of S are not integers.

It follows from the above discussion that if the functions Z_m, $m \in S$, are given by either (1) or (2) in (4.67), then the functions $f_m(x, y) = Z_m(x)e^{my}$ form a basis for a realization of the representation $R(0, m_o, 1)$ of $\mathscr{G}(0, 1)$. By Theorem 1.10 this representation of $\mathscr{G}(0, 1)$ by Lie derivatives can be extended to a local multiplier representation of $G(0, 1)$. Thus, if we denote by $\mathscr{A}$ the space of all entire analytic functions of x and y, the operators (4.64) will uniquely define a multiplier representation T of $G(0, 1)$ on $\mathscr{A}$.

We will compute the induced multiplier representation in the usual manner. Since $J^+ = e^y(\partial/\partial x - \frac{1}{2}x)$ it follows that

$$[\mathbf{T}(\exp b\mathscr{J}^+)f](x, e^y) = \nu(x, e^y, \exp b\mathscr{J}^+) f[x(b), e^{y(b)}], \qquad b \in \mathscr{C}, \quad f \in \mathscr{A},$$

where

$$\frac{d}{db}x(b) = e^{y(b)}, \qquad \frac{d}{db}y(b) = 0,$$

$$\frac{d}{db}\nu(x, e^y, \exp b\mathscr{J}^+) = -\tfrac{1}{2}x(b)\, e^{y(b)} \nu(x, e^y, \exp b\mathscr{J}^+)$$

with initial conditions $x(0) = x, y(0) = y, \nu(x, e^y, \mathbf{e}) = 1$. The solution is

$$[\mathbf{T}(\exp b\mathscr{J}^+)f](x, t) = \exp\left(-\frac{t^2b^2}{4} - \frac{xtb}{2}\right)f(x + bt, t)$$

where $t = e^y$. Similarly we find

$$[\mathbf{T}(\exp c\mathscr{J}^-)f](x, t) = \exp\left(\frac{t^{-2}c^2}{4} - \frac{xt^{-1}c}{2}\right)f(x - ct^{-1}, t),$$

$$[\mathbf{T}(\exp \tau\mathscr{J}^3)f](x, t) = f(x, e^\tau t),$$

$$[\mathbf{T}(\exp a\mathscr{E})f](x, t) = e^a f(x, t).$$

As mentioned in Section 4-1, if $g \in G(0, 1)$ has parameters (a, b, c, τ) then

$$g = (\exp b\mathscr{J}^+)(\exp c\mathscr{J}^-)(\exp \tau\mathscr{J}^3)(\exp a\mathscr{E}).$$

Thus,

$$\mathbf{T}(g)f = \mathbf{T}(\exp b\mathscr{J}^+)\,\mathbf{T}(\exp c\mathscr{J}^-)\,\mathbf{T}(\exp \tau\mathscr{J}^3)\,\mathbf{T}(\exp a\mathscr{E})f$$

for all $f \in \mathscr{A}$. Direct computation gives

$$[\mathbf{T}(g)f](x, t) = \exp\left(-\frac{t^2b^2}{4} - \frac{xtb}{2} - \frac{xt^{-1}c}{2} + \frac{t^{-2}c^2}{4} - \frac{bc}{2} + a\right) \cdot f(x + bt - ct^{-1}, e^\tau t) \quad (4.68)$$

for all $g \in G(0, 1)$, $f \in \mathscr{A}$. (Note that $t = e^y \neq 0$.)

According to Section 2-2 the matrix elements of the operators $\mathbf{T}(g)$ with respect to the basis vectors $f_m(x, t) = Z_m(x)t^m$ are given by the matrices $A_{lk}(g)$, Eq. (4.13) ($\omega = 0, \mu = 1$). This follows from the fact that the functions f_m satisfy condition (B). Thus, we obtain the relations

$$[\mathbf{T}(g)f_{m_0+k}](x, t) = \sum_{l=-\infty}^{\infty} A_{lk}(g)f_{m_0+l}(x, t), \qquad k = 0, \pm 1, \pm 2, \ldots, \quad (4.69)$$

valid for all $g \in G(0, 1)$, or

$$\exp\left(-\frac{t^2b^2}{4} - \frac{xtb}{2} - \frac{xt^{-1}c}{2} + \frac{t^{-2}c^2}{4} - \frac{bc}{2}\right) D_m(x + bt - ct^{-1})$$
$$= \sum_{l=-\infty}^{\infty} (-c)^{-l} L_{m+l}^{(-l)}(-bc)\, D_{m+l}(x)t^l, \quad \text{if} \quad Z_m(x) = (-1)^{m-m_0} D_m(x). \quad (4.70)$$

Various special cases of these relations are worth investigating. If we set $c = 0$, $t = 1$, and make use of the limits (4.16) we obtain

$$\exp\left(-\frac{b^2}{4}-\frac{xb}{2}\right)D_m(x+b)=\sum_{l=0}^{\infty}\frac{(-b)^l}{l!}D_{m+l}(x).$$

If $x = 0$ this relation becomes

$$D_m(b)=\exp\left(\frac{b^2}{4}\right)\sum_{l=0}^{\infty}\frac{(-b)^l}{l!}\frac{\Gamma(\frac{1}{2})(2)^{(m+l)/2}}{\Gamma\left(\dfrac{1-m-l}{2}\right)}.$$

For $b = 0$, $t = 1$, Eq. (4.70) reduces to

$$\exp\left(\frac{c^2}{4}+\frac{xc}{2}\right)D_m(x+c)=\sum_{l=0}^{\infty}\binom{m}{l}c^lD_{m-l}(x)$$

and, for $x = 0$, to

$$D_m(c)=\exp\left(-\frac{c^2}{4}\right)\sum_{l=0}^{\infty}\frac{c^l\Gamma(\frac{1}{2})(2)^{(m-l)/2}}{\Gamma\left(\dfrac{1-m+l}{2}\right)}\binom{m}{l}.$$

4-7 A Realization of $\uparrow_{\omega,\mu}$ by *Type D′* Operators

To obtain a realization of the representation $\uparrow_{\omega,\mu}$ of $\mathcal{G}(0, 1)$ by *type D′* operators acting on $\mathcal{A}$ it is necessary to find nonzero functions $f_m(x, y) = Z_m(x)e^{my}$ ($m = -\omega + k$, $k \geqslant 0$) such that

$$\begin{gathered}\mathrm{J}^3f_m = mf_m, \quad \mathrm{E}f_m = \mu f_m, \quad \mathrm{J}^+f_m = \mu f_{m+1}, \quad \mathrm{J}^-f_m = (m+\omega)f_{m-1}, \\ \mathrm{C}_{0,1}f_m = (\mathrm{J}^+\mathrm{J}^- - \mathrm{E}\mathrm{J}^3)f_m = \mu\omega f_m,\end{gathered} \tag{4.71}$$

where

$$\mathrm{J}^3=\frac{\partial}{\partial y},\qquad \mathrm{J}^{\pm}=e^{\pm y}\left(\pm\frac{\partial}{\partial x}-\tfrac{1}{2}\mu x\right),\qquad \mathrm{E}=\mu.$$

In terms of the functions $Z_m(x)$ these relations take the form (4.66) where, as remarked earlier, without loss of generality we can assume $\omega = 0$, $\mu = 1$. Thus,

$$\begin{gathered}\left(\frac{d}{dx}-\tfrac{1}{2}x\right)Z_m(x)=Z_{m+1}(x),\\ \left(-\frac{d}{dx}-\tfrac{1}{2}x\right)Z_m(x)=mZ_{m-1}(x),\\ \left(-\frac{d^2}{dx^2}+\frac{x^2}{4}-\frac{1}{2}-m\right)Z_m(x)=0,\end{gathered} \tag{4.72}$$

$m = 0, 1, 2,\ldots$. We can use these relations to explicitly compute the functions $Z_m(x)$. In fact, the relation $J^- f_0 = 0$ implies

$$\left(\frac{d}{dx} + \tfrac{1}{2}x\right) Z_0(x) = 0,$$

which has the solution $Z_0(x) = c\exp(-x^2/4)$, $c \in \mathcal{C}$. Choose $c = 1$. For $m > 0$ we define the function $Z_m(x)$ by

$$Z_m(x)\, e^{my} = f_m(x, y) = (J^+)^m f_0(x, y) = (J^+)^m(Z_0(x)).$$

A straightforward induction argument yields

$$Z_m(x) = \exp\left(\frac{x^2}{4}\right)\frac{\partial^m}{\partial x^m}\exp\left(-\frac{x^2}{2}\right), \qquad m = 0, 1, 2,\ldots . \tag{4.73}$$

At this point we have found functions $f_m(x, y) = Z_m(x)e^{my}$, $m \geqslant 0$, such that

$$J^3 f_m = mf_m, \qquad J^+ f_m = f_{m+1}, \qquad J^- f_0 = 0.$$

We can then proceed exactly as in the proof of Theorem 2.2 to show that these functions must actually satisfy **all** the relations (4.71) for $\omega = 0$, $\mu = 1$. Hence, the functions $Z_m(x)$ defined by (4.73) are the solutions of (4.72). The $Z_m(x)$ are easily expressed in terms of parabolic cylinder functions or Hermite polynomials $H_m(x)$. In fact,

$$Z_m(x) = (-1)^m D_m(x) = (-1)^m \exp(-x^2/4)\, 2^{-m/2} H_m(2^{-1/2}x), \qquad m = 0, 1, 2,\ldots . \tag{4.74}$$

According to the above discussion the functions $f_m(x, y) = Z_m(x)e^{my}$, $m \geqslant 0$, form a basis for a realization of the representation $\uparrow_{0,1}$. As usual this representation of $\mathcal{G}(0, 1)$ can be extended to a multiplier representation T of $G(0, 1)$ on $\mathcal{O}$. The action of the operators $\mathbf{T}(g)$ has already been determined and is given by (4.68). The matrix elements of this representation with respect to the analytic basis $\{f_m\}$ are the functions $B_{lk}(g)$ computed in Section 4-2 ($\omega = 0$, $\mu = 1$). Thus, we have

$$[\mathbf{T}(g)f_k](x, t) = \sum_{l=0}^{\infty} B_{lk}(g) f_l(x, t), \qquad k = 0, 1, 2,\ldots,$$

or

$$\exp\left(-\frac{t^2b^2}{4} - \frac{xtb}{2} - \frac{xt^{-1}c}{2} + \frac{t^{-2}c^2}{4} - \frac{bc}{2}\right) D_k(x + bt - ct^{-1})\, t^k$$
$$= \sum_{l=0}^{\infty} (-c)^{k-l} L_l^{(k-l)}(-bc)\, D_l(x)\, t^l, \tag{4.75}$$

where $t = e^y$. This last relation is valid for all $b, c, x, t \in \mathcal{C}$ such that $t \neq 0$. In terms of the Hermite polynomials

$$H_m(x) = \exp(x^2/2)\, 2^{m/2} D_m(2^{1/2}x), \qquad m = 0, 1, 2, \ldots,$$

Eq. (4.75) takes the form

$$\exp(-t^2b^2 - 2xtb) H_k(x + bt - ct^{-1}) t^k = \sum_{l=0}^{\infty} (-c)^{k-l} L_l^{(k-l)}(-bc) H_l(x) t^l. \quad (4.76)$$

If $c = 0$, $t = 1$, this equation reduces to

$$\exp(-b^2 - 2xb)\, H_k(x + b) = \sum_{l=0}^{\infty} \frac{(-b)^l}{l!} H_{k+l}(x),$$

and in the special case $k = 0$, $b = -y$, it becomes

$$\exp(2xy - y^2) = \sum_{l=0}^{\infty} \frac{y^l}{l!} H_l(x) \quad (4.77)$$

which is a well-known generating function for the Hermite polynomials.

Finally, when $b = 0$, $t = -1$, Eq. (4.76) reads

$$H_k(x - c) = \sum_{l=0}^{k} \binom{k}{l} c^{k-l} H_l(x).$$

We could find a realization of the representation $\downarrow_{\omega,\mu}$ using *type D'* operators in exact analogy with the procedure for $\uparrow_{\omega,\mu}$. However, this construction will be omitted since the analysis leads to no new information about special functions.

4-8 Transformations of *Type C'* Operators

The *type C'* operators classified in Section 2-7 take the form

$$\mathrm{J}^3 = \frac{\partial}{\partial y}, \qquad \mathrm{J}^{\pm} = e^{\pm y}\left(\pm \frac{\partial}{\partial x} - \frac{1}{x}\frac{\partial}{\partial y} - \frac{\mu x}{4} + \frac{q}{x}\right), \qquad \mathrm{E} = \mu \quad (4.78)$$

where $q, \mu \in \mathcal{C}$, $\mu \neq 0$. (Without loss of generality, the constant p in the expression for the *type C'* operators can be set equal to 0.) These operators generate a complex Lie algebra isomorphic to $\mathcal{G}(0, 1)$. In analogy with Section 4-6 we try to find a realization of the representation $R(\omega, m_o, \mu)$, where now the operators $\mathrm{J}^{\pm}$, J^3, E have the form (4.78).

Thus, we look for nonzero functions $f_m(x, y) = Z_m(x)e^{my}$, $m \in S = \{m_o + k\colon k \text{ an integer}\}$, such that

$$\begin{aligned} &\mathrm{J}^3 f_m = m f_m\,, \qquad \mathrm{E} f_m = \mu f_m\,, \qquad \mathrm{J}^+ f_m = \mu f_{m+1}\,, \\ &\mathrm{J}^- f_m = (m+\omega) f_{m-1}\,, \qquad \mathrm{C}_{0,1} f_m = (\mathrm{J}^+\mathrm{J}^- - \mathrm{E}\mathrm{J}^3) f_m = \mu\omega f_m \end{aligned} \tag{4.79}$$

where the differential operators are given by (4.78). As was shown in Section 2-7, the functions $Z_m(x)$ must satisfy the equation

$$-\frac{1}{x}\frac{d}{dx}\left[x\frac{d}{dx}Z_m(x)\right] + \left[\frac{(m-q)^2}{x^2} - \frac{1}{2}\mu(m+q+1) + \frac{\mu^2x^2}{16}\right] Z_m(x) = \mu\omega Z_m(x), \qquad m \in S.$$

The solutions of this equation are called the functions of the paraboloid of revolution, Erdélyi *et al.* [1], Vol. II, p. 126. In particular, one such solution is

$$(x^2/4)^{\xi} \exp(-x^2/8)\,{}_1F_1(\xi - \eta + \tfrac{1}{2}\,; 2\xi + 1; x^2/4)$$

where $\xi = \frac{1}{2}(m - q)$ and $\eta = \frac{1}{2}\mu(m + q + 2\omega + 1)$. If $m - q$ is not an integer there is a linearly independent solution

$$(x^2/4)^{-\xi} \exp(-x^2/8)\,{}_1F_1(-\xi - \eta + \tfrac{1}{2}\,; -2\xi + 1; x^2/4).$$

Expressed in terms of generalized Laguerre functions, there are linearly independent solutions

$$\begin{aligned} &(1)\quad (x^2/4)^{\xi} \exp(-x^2/8)\, L^{(2\xi)}_{\eta-\xi-\frac{1}{2}}(x^2/4) \qquad \text{and} \\ &(2)\quad (x^2/4)^{-\xi} \exp(-x^2/8)\, L^{(-2\xi)}_{\eta+\xi-\frac{1}{2}}(x^2/4). \end{aligned} \tag{4.80}$$

If we wish to study directly the properties of the confluent hypergeometric functions (or the generalized Laguerre functions), however, it is more convenient if we can find differential operators whose eigenfunctions are of the form

$$f_m(x, y) = L^{(2\xi)}_{\eta-\xi-\frac{1}{2}}(x)(e^y)^m. \tag{4.81}$$

We shall transform the *type C′* operators into a new set of differential operators to bring about this desirable situation.

To begin with we assume $\omega = 0$, $\mu = 1$. (As far as special function theory is concerned this is no loss of generality.) Then, the first solution (4.80) is $f_m(u, t) = u^{-q/2}e^{-u/2}L_q^{\ (m-q)}(u)t^m$ where the variables u, t are defined by $u = x^2/4$, $t = e^y x/2$. Denote by $\mathscr{F}$ the space of all functions analytic in a neighborhood of the point $(u^o, t^o) = (1, 1)$ and define the transformation φ of $\mathscr{F}$ onto $\mathscr{F}$ by $[\varphi f](u, t) = u^{-q/2}e^{-u/2}f(u, t)$, $f \in \mathscr{F}$. We can consider the mapping φ to be multiplication by the function $\varphi(u, t) = u^{-q/2}e^{-u/2}$. This mapping is invertible with inverse φ^{-1} defined by $[\varphi^{-1}f](u, t) = u^{+q/2}e^{+u/2}f(u, t)$. Under φ^{-1} the basis function $f_m(u, t)$ defined above is mapped into $[\varphi^{-1}f_m](u, t) = L_q^{(m-q)}(u)t^m$, which is in the desired form (4.81).

If J is a linear differential operator on $\mathscr{F}$ define the linear differential operator J^φ on $\mathscr{F}$ by $J^\varphi(\varphi^{-1}f) = (\varphi^{-1}J\varphi)(\varphi^{-1}f) = \varphi^{-1}(Jf)$ for all $f \in \mathscr{F}$, i.e., $J^\varphi = \varphi^{-1}J\varphi$. As a consequence of this definition we have

Lemma 4.4

(1) $[J, K]^\varphi = [J^\varphi, K^\varphi]$,

(2) $(aJ + bK)^\varphi = aJ^\varphi + bK^\varphi, \qquad a, b \in \mathscr{C}$,

(3) $Jf = h \Leftrightarrow J^\varphi(\varphi^{-1}f) = \varphi^{-1}h, \qquad f, h \in \mathscr{F}$,

for all linear differential operators J, K on $\mathscr{F}$.

PROOF (1) For all $f \in \mathscr{F}$,

$$\begin{aligned}[J, K]^\varphi(\varphi^{-1}f) &= \varphi^{-1}([J, K]f) = \varphi^{-1}\{J(Kf) - K(Jf)\} \\ &= \varphi^{-1}\{J(Kf)\} - \varphi^{-1}\{K(Jf)\} = (J)^\varphi\{\varphi^{-1}(Kf)\} - (K)^\varphi\{\varphi^{-1}(Jf)\} \\ &= [J^\varphi, K^\varphi](\varphi^{-1}f).\end{aligned}$$

Properties (2) and (3) are trivial. Q.E.D.

Thus the map $J \to J^\varphi$ is a Lie algebra isomorphism.

If

$$J = f_1(u, t)\frac{\partial}{\partial u} + f_2(u, t)\frac{\partial}{\partial t} + f_3(u, t), \qquad f_1, f_2, f_3 \in \mathscr{F},$$

an explicit computation gives

$$\begin{aligned}J^\varphi = f_1(u, t)\frac{\partial}{\partial u} + f_2(u, t)\frac{\partial}{\partial t} + f_3(u, t) + \varphi^{-1}(u, t) \\ \cdot \left(f_1(u, t)\frac{\partial\varphi}{\partial u} + f_2(u, t)\frac{\partial\varphi}{\partial t}\right),\end{aligned} \tag{4.82}$$

where $\varphi(u, t) = u^{-q/2}e^{-u/2}$, $\varphi^{-1}(u, t) = u^{q/2}e^{u/2}$. (Although we have made a special choice for φ, Lemma 4.4 and Eq. (4.82) are clearly valid for any nonzero function φ in $\mathscr{F}$.)

Armed with these facts, we can compute the operators $(J^3)^\varphi$, $(J^\pm)^\varphi$, $(E)^\varphi$ where J^3, $J^\pm$, E are given by (4.78) ($\mu = 1$). In terms of the coordinates u, t ($u = x^2/4$, $t = e^y x/2$), the results are

$$(J^3)^\varphi = t\frac{\partial}{\partial t}, \qquad (E)^\varphi = 1,$$
$$(J^+)^\varphi = t\left(\frac{\partial}{\partial u} - 1\right), \qquad (J^-)^\varphi = t^{-1}\left(-u\frac{\partial}{\partial u} - t\frac{\partial}{\partial t} + q\right). \tag{4.83}$$

As an example we verify the computation of $(J^+)^\varphi$. Under the change of variables from (x, y) to (u, t), J^+ becomes

$$J^+ = e^y\left(\frac{\partial}{\partial x} - \frac{1}{x}\frac{\partial}{\partial y} + \frac{q}{x} - \frac{x}{4}\right) = t\left(\frac{\partial}{\partial u} + \frac{q}{2u} - \frac{1}{2}\right).$$

Moreover,

$$\varphi^{-1}(u, t)t\frac{\partial}{\partial u}[\varphi(u, t)] = u^{q/2}e^{u/2}\frac{\partial}{\partial u}[u^{-q/2}e^{-u/2}] = -\frac{tq}{2u} - \frac{t}{2}.$$

Therefore,

$$(J^+)^\varphi = J^+ - \left(\frac{tq}{2u} + \frac{t}{2}\right) = t\left(\frac{\partial}{\partial u} - 1\right).$$

The other relations are proved similarly.

4-9 *Type C′* Realizations of $R(\omega, m_o, \mu)$

By construction the differential operators (4.83) generate a Lie algebra isomorphic to $\mathscr{G}(0, 1)$. These operators can be used to find a realization of the representation $R(\omega, m_o, \mu)$ on $\mathscr{F}$. Thus, we look for nonzero functions $f_m(u, t) = Z_m(u)t^m$ defined for all $m \in S = \{m_o + k,\ k$ an integer$\}$, such that relations (4.79) are valid.

Here the differential operators are given by (4.83) and the superscript φ has been omitted. (There is no loss of generality for special function theory if we set $\omega = 0$, $\mu = 1$.) In terms of the functions $Z_m(u)$ these relations are

$$\left(\frac{d}{du} - 1\right)Z_m(u) = Z_{m+1}(u), \qquad \left(-u\frac{d}{du} + q - m\right)Z_m(u) = mZ_{m-1}(u),$$
$$\left(-u\frac{d^2}{du^2} + (q - m - 1 + u)\frac{d}{du} - q\right)Z_m(u) = 0 \tag{4.84}$$

for all $m \in S$. The solutions of these equations are generalized Laguerre functions or, what is the same thing, confluent hypergeometric functions. Indeed, for all $m \in S$ the choice

$$Z_m(u) = (-1)^{m-m_o} L_q^{(m-q)}(u) \tag{4.85}$$

satisfies (4.84), see (A. 13). (It is left to the reader to find additional solutions.) Since $\omega = 0$ we must have $0 < \operatorname{Re} m_o < 1$, so the elements m of S are not integers. In the special case where $m_o - q$ is an integer the functions (4.85) have been obtained as matrix elements of representations $R(\omega, m_o, \mu)$. However, when $m_o - q$ is not an integer these functions do not occur as matrix elements.

The above remarks have established the fact that the functions $f_m(u, t) = Z_m(u)t^m$, $m \in S$, form a basis for a realization of the representation $R(0, m_o, 1)$ of $\mathscr{G}(0, 1)$ if the $Z_m(u)$ satisfy (4.84). In the usual manner this realization can be extended to a local multiplier representation T of $G(0, 1)$ on the space $\mathscr{F}$.

According to Theorem 1.10 and relations (4.83), the local multiplier representation takes the form

$$\begin{aligned}
&[\mathbf{T}(\exp b\mathscr{J}^+)f](u, t) = e^{-bt}f(u + bt, t),\\
&[\mathbf{T}(\exp c\mathscr{J}^-)f](u, t) = (1 - c/t)^{-q}f(u(1 - c/t), t - c), \qquad |c/t| < 1,\\
&[\mathbf{T}(\exp \tau\mathscr{J}^3)f](u, t) = f(u, e^\tau t),\\
&[\mathbf{T}(\exp a\mathscr{E})f](u, t) = e^a f(u, t)
\end{aligned} \tag{4.86}$$

for $f \in \mathscr{F}$. If $g \in G(0, 1)$ has parameters (a, b, c, τ), then

$$\mathbf{T}(g) = \mathbf{T}(\exp b\mathscr{J}^+)\,\mathbf{T}(\exp c\mathscr{J}^-)\,\mathbf{T}(\exp \tau\mathscr{J}^3)\,\mathbf{T}(\exp a\mathscr{E})$$

and

$$[\mathbf{T}(g)f](u, t) = e^{a-bt}(1 - c/t)^{-q} f[(u + bt)(1 - c/t), e^\tau(t - c)] \tag{4.87}$$

for $|c/t| < 1$, $f \in \mathscr{F}$. The matrix elements of $\mathbf{T}(g)$ with respect to the analytic basis $\{f_m(u, t) = Z_m(u)t^m\}$, (4.85), are the functions $A_{lk}(g)$ defined by Eq. (4.13), ($\omega = 0, \mu = 1$). (The fact that this basis is analytic follows from (2.2) and the t dependence of the basis functions.) Thus,

$$[\mathbf{T}(g)f_{m_o+k}](u, t) = \sum_{l=-\infty}^{\infty} A_{lk}(g) f_{m_o+l}(u, t), \qquad k = 0, \pm 1, \pm 2, \ldots, \tag{4.88}$$

which simplifies to the identity

$$e^{-bt}(1+c/t)^p L_q^{(p)}[(u+bt)(1+c/t)] = \sum_{l=-\infty}^{\infty} c^{-l} L_{p+q+l}^{(-l)}(bc) L_q^{(p+l)}(u) t^l,$$

$$|c/t| < 1, \quad p, q \in \mathscr{C}, \quad p+q \text{ not an integer.} \tag{4.89}$$

Using the limits (4.16) we can investigate two special cases of this identity: If $c = 0$, $t = 1$, there follows

$$L_q^{(p)}(u+b) = e^b \sum_{l=0}^{\infty} \frac{(-b)^l}{l!} L_q^{(p+l)}(u),$$

while if $b = 0$, $t = 1$, there follows

$$(1+c)^p L_q^{(p)}[u(1+c)] = \sum_{l=0}^{\infty} \binom{p+q}{l} c^l L_q^{(p-l)}(u), \qquad |c| < 1.$$

4-10 *Type C′* Realizations of $\uparrow_{0,1}$

To obtain a realization of the representation $\uparrow_{0,1}$ of $\mathscr{G}(0, 1)$ by operators

$$\mathrm{J}^3 = t\frac{\partial}{\partial t}, \qquad \mathrm{E} = 1, \qquad \mathrm{J}^+ = t\left(\frac{\partial}{\partial u} - 1\right),$$

$$\mathrm{J}^- = t^{-1}\left(-u\frac{\partial}{\partial u} - t\frac{\partial}{\partial t} + q\right), \qquad q \in \mathscr{C},$$

acting on $\mathscr{F}$ we must find nonzero functions $f_m(u, t) = Z_m(u)t^m$, $m = 0, 1, 2, \ldots$, such that

$$\begin{aligned} \mathrm{J}^3 f_m &= m f_m, \qquad \mathrm{E} f_m = f_m, \qquad \mathrm{J}^+ f_m = f_{m+1}, \\ \mathrm{J}^- f_m &= m f_{m-1}, \qquad \mathrm{C}_{0,1} f_m = (\mathrm{J}^+\mathrm{J}^- - \mathrm{E}\mathrm{J}^3) f_m = 0 \end{aligned} \tag{4.90}$$

for all $m \geqslant 0$. (We assume $f_{-1} = 0$.) Conditions (4.90) will be satisfied if and only if the special functions $Z_m(u)$, $m \geqslant 0$, satisfy the equations

$$\begin{aligned} \left(\frac{d}{du} - 1\right) Z_m(u) = Z_{m+1}(u), \qquad \left(-u\frac{d}{du} - m + q\right) Z_m(u) = m Z_{m-1}(u), \\ \left(-u\frac{d^2}{du^2} + (q - m - 1 + u)\frac{d}{du} - q\right) Z_m(u) = 0. \end{aligned} \tag{4.91}$$

(Here, $Z_{-1}(u) \equiv 0$.) Equations (4.91) determine the functions Z_m to within an arbitrary constant. Thus, the relation $\mathrm{J}^- f_0 = 0$ implies

$(-u\, d/du + q)\, Z_0(u) = 0$, which has the solution $Z_0(u) = cu^q$, c an arbitrary constant. To normalize our solution we set $c = 1$. For $m > 0$ the functions $Z_m(u)$ can be defined recursively by $Z_m(u)t^m = f_m(u, t) = (\mathrm{J}^+)^m f_0(u, t) = (\mathrm{J}^+)^m Z_0(u)$. A straightforward induction argument yields

$$Z_m(u) = e^u \left(\frac{d}{du}\right)^m (u^q e^{-u}), \qquad m = 0, 1, 2, \ldots . \tag{4.92}$$

By definition the functions $f_m(u, t)$, $m \geqslant 0$, satisfy the relations

$$\mathrm{J}^3 f_m = m f_m\,, \qquad \mathrm{J}^+ f_m = f_{m+1}\,, \qquad \mathrm{J}^- f_0 = 0.$$

However, it follows immediately from the proof of Theorem 2.2 that the f_m must actually satisfy all of the relations (4.90), and the functions $Z_m(u)$ must satisfy all the equations (4.91). The special functions Z_m are conveniently expressed in terms of associated Laguerre polynomials $L_m^{(\alpha)}$, m a positive integer. In fact,

$$Z_m(u) = m!\, u^{q-m} L_m^{(q-m)}(u), \qquad m = 0, 1, 2, \ldots . \tag{4.93}$$

This realization of $\mathscr{G}(0, 1)$ can be extended to a multiplier representation T of $G(0, 1)$ on $\mathscr{F}$ where the action of the operators $\mathbf{T}(g)$, $g \in G(0, 1)$ is given by (4.87). The matrix elements of $\mathbf{T}(g)$ with respect to the analytic basis $\{f_m\}$ are the functions $B_{lk}(g)$ studied in Section 4-2 ($\omega = 0, \mu = 1$). Thus,

$$[\mathbf{T}(g) f_k](u, t) = \sum_{l=0}^{\infty} B_{lk}(g) f_l(u, t), \qquad k = 0, 1, 2, \ldots,$$

which leads to the identities

$$\begin{aligned} &k!\, e^{-bt}(1 + bt/u)^{q-k} L_k^{(q-k)}[u(1 + bt/u)(1 - c/t)]t^k \\ &\quad = \sum_{l=0}^{\infty} c^{k-l} L_l^{(k-l)}(-bc)\, l!\, u^{k-l} L_l^{(q-l)}(u) t^l, \end{aligned} \tag{4.94}$$

valid for all $b, c, t, u, q \in \mathcal{C}$ such that $|\, bt/u \,| < 1$. Making use of the limits (4.29) we can derive some simple consequences of (4.94). For $c = 0$, $u = t$,

$$e^{-bu}(1 + b)^{q+k} L_k^{(q)}[u(1 + b)] = \sum_{l=0}^{\infty} b^l \binom{l + k}{l} L_{l+k}^{(q-l)}(u), \qquad |\, b \,| < 1,$$

and, setting $k = 0$ in this expression we obtain a well-known generating function for the associated Laguerre polynomials:

$$e^{-bu}(1 + b)^q = \sum_{l=0}^{\infty} b^l L_l^{(q-l)}(u), \qquad q \in \mathcal{C}, \quad |\, b \,| < 1.$$

When $b = 0$, $u = t$, Eq. (4.94) becomes

$$L_k^{(q)}(u - c) = \sum_{l=0}^{k} \frac{c^l}{l!} L_{k-l}^{(q+l)}(u).$$

We could also use the *type* C' operators to find a realization of the representation $\downarrow_{\omega,\mu}$. However, the identities for Laguerre polynomials obtained from such a realization are just (4.94) again, so this analysis will be omitted.

In Chapter 5 we will derive additional identities for the Laguerre functions by relating these functions to the representation theory of $sl(2)$.

4-11 The Group S_4

S_4 is the real 4-parameter Lie group of matrices

$$g\{w, \alpha, \delta\} = \begin{pmatrix} 1 & \frac{1}{2}e^{-i\alpha}\bar{w} & +i\delta - w\bar{w}/8 \\ 0 & e^{-i\alpha} & -\frac{1}{2}w \\ 0 & 0 & 0 \end{pmatrix} \tag{4.95}$$

where $w = x + iy \in \mathcal{C}$, $0 \leqslant \alpha < 2\pi$, (mod 2π), and δ is real. The group multiplication law for S_4 is

$$\begin{aligned} &g\{w, \alpha, \delta\} \cdot g\{w', \alpha', \delta'\} \\ &\quad = g\left\{w + e^{-i\alpha}w', \alpha + \alpha', \delta + \delta' + \frac{i}{8}(\bar{w}w'e^{-i\alpha} - w\bar{w}'e^{+i\alpha})\right\}. \end{aligned} \tag{4.96}$$

Thus, $g\{0, 0, 0\}$ is the identity and the inverse of a group element is given by

$$g^{-1}\{w, \alpha, \delta\} = g\{-e^{i\alpha}w, -\alpha, -\delta\}. \tag{4.97}$$

As a basis for the real Lie algebra $\mathscr{S}_4$ of S_4 we can choose the elements

$$\begin{aligned} \mathscr{J}_1 &= \begin{pmatrix} 0 & -i/2 & 0 \\ 0 & 0 & i/2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathscr{J}_2 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}, \\ \mathscr{J}_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathscr{Q} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{4.98}$$

with commutation relations

$$[\mathscr{J}_1, \mathscr{J}_2] = \tfrac{1}{2}\mathscr{Q}, \quad [\mathscr{J}_3, \mathscr{J}_1] = \mathscr{J}_2, \quad [\mathscr{J}_3, \mathscr{J}_2] = -\mathscr{J}_1, \quad [\mathscr{J}_k, \mathscr{Q}] = \mathscr{O}, \quad k = 1, 2, 3. \tag{4.99}$$

The 3×3 matrices $\mathscr{J}^+$, $\mathscr{J}^-$, $\mathscr{J}^3$, $\mathscr{E}$,

$$\mathscr{J}^\pm = \mp \mathscr{J}_2 + i\mathscr{J}_1, \qquad \mathscr{J}^3 = i\mathscr{J}_3, \qquad \mathscr{E} = -i\mathscr{Q},$$

must then satisfy the commutation relations

$$[\mathscr{J}^3, \mathscr{J}^\pm] = \pm\mathscr{J}^\pm, \qquad [\mathscr{J}^+, \mathscr{J}^-] = -\mathscr{E}, \qquad [\mathscr{E}, \mathscr{J}^\pm] = [\mathscr{E}, \mathscr{J}^3] = \mathscr{O}.$$

Clearly, the complex Lie algebra generated by the basis elements (4.98) is $\mathscr{G}(0, 1)$. Thus $\mathscr{G}(0, 1)$ is the complexification of $\mathscr{S}_4$ and $\mathscr{S}_4$ is a real form of $\mathscr{G}(0, 1)$ (see Section 3-5). Due to this relationship between the two Lie algebras, the abstract irreducible representations $R(\omega, m_0, \mu)$, $\uparrow_{\omega,\mu}$, and $\downarrow_{\omega,\mu}$ of $\mathscr{G}(0, 1)$ induce irreducible representations of $\mathscr{S}_4$. We shall investigate which of these irreducible representations of $\mathscr{S}_4$ can be extended to an irreducible unitary representation of S_4 on a Hilbert space. The technique for carrying out this investigation was discussed in Section 3-6 and applied to the real Lie group E_3 in Section 3-7.

4-12 Induced Representations of $\mathscr{S}_4$

As in Section 3-6 we consider a unitary irreducible representation U of S_4 on a Hilbert space $\mathscr{H}$ and define the infinitesimal operators J_1, J_2, J_3, Q by

$$\begin{aligned} \mathrm{J}_k f &= \frac{d}{dt} \mathrm{U}(\exp t\mathscr{J}_k) f \Big|_{t=0}, \qquad k = 1, 2, 3, \\ \mathrm{Q} f &= \frac{d}{dt} \mathrm{U}(\exp t\mathscr{Q}) f \Big|_{t=0} \end{aligned} \tag{4.100}$$

for all $f \in \mathscr{D}$. Here, $\mathscr{D}$ is a dense subspace of $\mathscr{H}$ satisfying properties (3.45) and (3.46). On $\mathscr{D}$ these operators obey the commutation relations

$$[\mathrm{J}_1, \mathrm{J}_2] = \tfrac{1}{2}\mathrm{Q}, \qquad [\mathrm{J}_3, \mathrm{J}_1] = \mathrm{J}_2, \qquad [\mathrm{J}_3, \mathrm{J}_2] = -\mathrm{J}_1, \quad [\mathrm{J}_k, \mathrm{Q}] = 0, \quad k = 1, 2, 3,$$

as follows immediately from (4.99). Moreover, the operators $\mathrm{J}^\pm$, J^3, E defined by

$$\mathrm{J}^\pm = \mp \mathrm{J}_2 + i\mathrm{J}_1, \qquad \mathrm{J}^3 = i\mathrm{J}_3, \qquad \mathrm{E} = -i\mathrm{Q} \tag{4.101}$$

satisfy the commutation relations

$$[\mathrm{J}^+, \mathrm{J}^-] = -\mathrm{E}, \qquad [\mathrm{J}^3, \mathrm{J}^\pm] = \pm\mathrm{J}^\pm, \qquad [\mathrm{J}^\pm, \mathrm{E}] = [\mathrm{J}^3, \mathrm{E}] = 0;$$

hence, these operators determine a representation ρ of the complex Lie algebra $\mathscr{G}(0, 1)$ on $\mathscr{D}$.

We shall first investigate which of the representations $R(\omega, m_o, \mu)$ of $\mathcal{G}(0, 1)$ can be induced by ρ on some dense subspace $\mathcal{D}'$ of $\mathcal{D}$. According to Lemma 3.1 we must have

$$\langle \mathrm{J}_k f, h\rangle = -\langle f, \mathrm{J}_k h\rangle, \qquad k = 1, 2, 3,$$
$$\langle \mathrm{Q} f, h\rangle = -\langle f, \mathrm{Q} h\rangle$$

for all $f, h \in \mathcal{D}$, where $\langle .,. \rangle$ is the inner product on $\mathcal{H}$. Using (4.101) we can write these conditions in the form

$$\langle \mathrm{J}^3 f, h\rangle = \langle f, \mathrm{J}^3 h\rangle, \qquad \langle \mathrm{J}^+ f, h\rangle = \langle f, \mathrm{J}^- h\rangle, \qquad \langle \mathrm{E} f, h\rangle = \langle f, \mathrm{E} h\rangle \quad (4.102)$$

for all $f, h \in \mathcal{D}$. The representation $R(\omega, m_o, \mu)$ is determined by the relations

$$\begin{aligned} \mathrm{J}^3 f_m &= m f_m, & \mathrm{E} f_m &= \mu f_m, \\ \mathrm{J}^+ f_m &= \mu f_{m+1}, & \mathrm{J}^- f_m &= (m + \omega) f_{m-1} \end{aligned} \quad (4.103)$$

where $\mu \neq 0$, $0 \leqslant \operatorname{Re} m_o < 1$, $m = m_o + k$ and k runs over the integers. We will assume that the basis vectors f_m are in $\mathcal{D}$ and use conditions (4.102) to find restrictions on ω, μ, and m_o. Thus,

$$\bar{m}\langle f_m, f_n\rangle = \langle \mathrm{J}^3 f_m, f_n\rangle = \langle f_m, \mathrm{J}^3 f_n\rangle = n\langle f_m, f_n\rangle, \quad (4.104)$$

or $(\bar{m} - n)\langle f_m, f_n\rangle = 0$ for all m, n in the spectrum of J^3. Setting $n = m$, we can conclude that m is real, hence $\operatorname{Im} m_o = 0$. If $m \neq n$ we have $\langle f_m, f_n\rangle = 0$. Similarly, the relation $\langle \mathrm{E} f_m, f_n\rangle = \langle f_m, \mathrm{E} f_n\rangle$ proves that μ must be real. Finally, the relation

$$\mu\langle f_m, f_m\rangle = \langle \mathrm{J}^+ f_{m-1}, f_m\rangle = \langle f_{m-1}, \mathrm{J}^- f_m\rangle = (m + \omega)\langle f_{m-1}, f_{m-1}\rangle \quad (4.105)$$

implies $0 < |f_m|^2/|f_{m-1}|^2 = (m + \omega)/\mu$ for all $m \in S$. Since this last condition can never be satisfied for all $m \in S$ it follows that none of the irreducible representations $R(\omega, m_o, \mu)$ of $\mathcal{G}(0, 1)$ are induced by unitary representations of S_4.

However, we shall have better luck with the representations $\uparrow_{\omega,\mu}$ of $\mathcal{G}(0, 1)$, determined by conditions (4.103) where now $\mu \neq 0$, ω is arbitrary, and $m = -\omega + k$ where k runs over the nonnegative integers. As in the preceding case, in order that the operators (4.103) be obtained from a unitary representation of S_4 we must require (i) $\langle f_m, f_n\rangle = 0$ if $m \neq n$; (ii) the spectrum of J^3 is real, i.e., ω is real; (iii) μ is real; and (iv) $0 < |f_{-\omega+k+1}|^2/|f_{-\omega+k}|^2 = (k + 1)/\mu$ for all nonnegative integers k. Condition (iv) implies $\mu > 0$. From (3.46) we can show

$$\mathrm{U}(\exp \alpha \mathcal{J}_3) f_m = \exp(-i\alpha \mathrm{J}^3) f_m = e^{-im\alpha} f_m$$

for all m in the spectrum of J^3. Since $\exp 2\pi \mathscr{J}_3 = \exp \mathscr{O} = \mathbf{e}$ (the identity element of S_4), we must have $e^{-2\pi i m} = 1$ for all m in the spectrum of J^3. Thus, m and ω must be integers. If we define new basis vectors j_k by

$$j_k = \frac{\mu^{k/2}}{(k!)^{1/2}} f_{-\omega+k}\,, \qquad k = 0, 1, 2, \ldots,$$

it is a simple consequence of conditions (i) and (iv) that $\langle j_k\,, j_{k'}\rangle = b\,\delta_{k,k'}$ for all $k, k' \geqslant 0$, where b is a positive constant which without loss of generality can be set equal to 1.

We can draw the following conclusions from this analysis: Any irreducible representation $\uparrow_{\omega,\mu}$ of $\mathscr{G}(0, 1)$ induced by a unitary representation U of S_4 must be such that ω is an integer and $\mu > 0$. Furthermore, the vectors $\{j_k\}$ form an orthonormal basis for $\mathscr{H}$ and satisfy the relations

$$\begin{aligned} \mathrm{J}^3 j_k &= (-\omega + k)\, j_k\,, & \mathrm{E} j_k &= \mu j_k\,, \\ \mathrm{J}^+ j_k &= [\mu(k+1)]^{1/2}\, j_{k+1}\,, & \mathrm{J}^- j_k &= (\mu k)^{1/2}\, j_{k-1} \end{aligned} \tag{4.106}$$

for all $k \geqslant 0$ $(j_{-1} \equiv 0)$.

Conversely, we will soon show that each such representation (4.106) induces a unitary irreducible representation U of S_4 on a Hilbert space.

A similar analysis proves: If the irreducible representation $\downarrow_{\omega,\mu}$ of $\mathscr{G}(0, 1)$ is induced by a unitary representation of S_4 then ω is an integer, $\mu > 0$ and there is an orthonormal basis $\{j_k\}$, $k = 0, 1, 2, \ldots$, for $\mathscr{H}$ satisfying the relations

$$\begin{aligned} \mathrm{J}^3 j_k &= (-\omega - 1 - k) j_k\,, & \mathrm{E} j_k &= -\mu j_k\,, \\ \mathrm{J}^+ j_k &= (\mu k)^{1/2}\, j_{k-1}\,, & \mathrm{J}^- j_k &= [\mu(k+1)]^{1/2}\, j_{k+1} \end{aligned} \tag{4.107}$$

for all $k \geqslant 0$ $(j_{-1} \equiv 0)$. Conversely we will show that each such representation (4.107) induces a unitary irreducible representation of S_4 on a Hilbert space.

From (4.95) and (4.98) it is easy to verify the relations $\exp y\mathscr{J}_1 = g\{iy, 0, 0\}$, $\exp x\mathscr{J}_2 = g\{x, 0, 0\}$, $\exp \alpha\mathscr{J}_3 = g\{0, \alpha, 0\}$, and $\exp \delta\mathscr{Q} = g\{0, 0, \delta\}$ for x, y, α, δ real. If $w = x + iy$ it follows from (4.96) that $g\{w, \alpha, \delta\} = (\exp x\mathscr{J}_2)(\exp y\mathscr{J}_1)(\exp \alpha\mathscr{J}_3)(\exp(\delta + xy/4)\mathscr{Q})$. Thus, S_4 is uniquely determined by the elements $\mathscr{J}_1, \mathscr{J}_2, \mathscr{J}_3, \mathscr{Q}$ of $\mathscr{S}_4$ and according to Eqs. (3.46), the operators $\mathbf{U}(g)$, $g \in S_4$ are uniquely determined by the infinitesimal operators $\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3, \mathrm{Q}$ defined by (4.100). The irreducible representations (4.106) and (4.107) uniquely determine the unitary representations from which they are derived.

Before proceding with the determination of the irreducible unitary representations of S_4 it is useful to examine a realization of the representation $\uparrow_{0,1}$ of $\mathscr{S}_4$ which is of importance in quantum mechanics: the occupation number space formalism for bosons. Let $\mathscr{H}$ be a complex Hilbert space with orthonormal basis vectors $|\,n\rangle$, $n = 0, 1, 2, \ldots$. On $\mathscr{H}$ we consider the annihilation operator $\mathbf{a}$ and its adjoint, the creation operator $\mathbf{a}^+$, defined by

$$\mathbf{a}\,|\,n\rangle = n^{1/2}\,|\,n-1\rangle, \quad \mathbf{a}^+\,|\,n\rangle = (n+1)^{1/2}\,|\,n+1\rangle, \quad n = 0, 1, 2, \ldots, \tag{4.108}$$

and linearity. The number-of-particles operator $\mathbf{N}$ is defined by $\mathbf{N} = \mathbf{a}^+\mathbf{a}$ and has the property

$$\mathbf{N}\,|\,n\rangle = n\,|\,n\rangle, \qquad n = 0, 1, 2, \ldots . \tag{4.109}$$

As the names of the operators suggest, the eigenstates $|\,n\rangle$ of $\mathbf{N}$ are interpreted as eigenstates of n bosons. The vacuum or no-particle state is $|\,0\rangle$. From their definitions it is easy to verify that the four linear operators $\mathbf{a}$, $\mathbf{a}^+$, $\mathbf{N}$, $\mathbf{I}$, ($\mathbf{I}$ is the identity operator on $\mathscr{H}$) satisfy the commutation relations

$$\begin{gathered}[\mathbf{a}^+, \mathbf{a}] = -\mathbf{I}, \qquad [\mathbf{N}, \mathbf{a}^+] = \mathbf{a}^+, \qquad [\mathbf{N}, \mathbf{a}] = -\mathbf{a}, \\ [\mathbf{a}^+, \mathbf{I}] = [\mathbf{a}, \mathbf{I}] = [\mathbf{N}, \mathbf{I}] = 0.\end{gathered} \tag{4.110}$$

Comparison of Eqs. (4.106) and (4.108)–(4.110) shows that the occupation number space formalism gives a realization of the representation $\uparrow_{0,1}$ of $\mathscr{G}(0, 1)$, hence, of $\mathscr{S}_4$. In Section 4-14 this representation will be extended to a unitary representation of S_4.

4-13 The Hilbert Space $\mathscr{F}$

Our aim is now to extend the representations (4.106), (4.107) of $\mathscr{S}_4$ to unitary irreducible representations of S_4. For this purpose we first define the Hilbert space $\mathscr{F}$ on which these representations will operate. $\mathscr{F}$ is the space of entire analytic functions f of the complex variable t such that

$$\int \overline{f(t)}\, f(t)\, d\xi(t) < \infty,$$

where $d\xi(t) = \pi^{-1} \exp(-\bar{t}t)\, dx\, dy, t = x + iy$, and the domain of integration is the complex plane. Every function $f \in \mathscr{F}$ can be expanded uniquely in a power series $f(t) = \sum_{n=0}^{\infty} a_n t^n$ which converges everywhere. The inner product $\langle .,. \rangle$ is

$$\langle f, h \rangle = \int \overline{f(t)}\, h(t)\, d\xi(t), \qquad f, h \in \mathscr{F}. \tag{4.111}$$

It is easy to see that the functions $j_n(t) = t^n/(n!)^{1/2}$, $n = 0, 1, 2,...$, form an orthonormal basis for $\mathscr{F}$. In fact, introducing polar coordinates $t = re^{i\theta}$, we have

$$\langle j_n, j_{n'} \rangle = \frac{1}{\pi(n!\, n'!)^{1/2}} \int_0^{2\pi} e^{i(n-n')\theta}\, d\theta \int_0^\infty r^{n+n'+1} \exp(-r^2)\, dr.$$

Hence,

$$\langle j_n, j_{n'} \rangle = \delta_{n,n'}. \tag{4.112}$$

For any two functions $f(t) = \sum a_n t^n$ and $h(t) = \sum b_n t^n$ in $\mathscr{F}$ we find

$$\langle f, h \rangle = \sum_{n=0}^{\infty} n!\, \bar{a}_n b_n \tag{4.113}$$

and

$$\langle f, f \rangle = \sum_{n=0}^{\infty} n!\, | a_n |^2. \tag{4.114}$$

From (4.114), an entire function f is an element of $\mathscr{F}$ if and only if $\sum_{n=0}^{\infty} n!\, | a_n |^2 < \infty$. Equation (4.112) shows that the vectors $\{j_n\}$ are orthonormal while (4.114) shows that they are complete; hence, they form an orthonormal basis for $\mathscr{F}$.

The space $\mathscr{F}$ was introduced by Segal [1] and studied in detail by Bargmann [3, 4]. We mention here some of the special properties of this space.

I. Define the function $\mathbf{e}_b \in \mathscr{F}$ for some complex constant b by $\mathbf{e}_b(t) = \exp(\bar{b}t) = \sum_{n=0}^{\infty} (\bar{b}t)^n/n!$. It follows from (4.113) that for any $f \in \mathscr{F}$,

$$\langle \mathbf{e}_b, f \rangle = \sum_{n=0}^{\infty} b^n a_n = f(b), \qquad \langle \mathbf{e}_b, \mathbf{e}_b \rangle = \mathbf{e}_b(b) = \exp(\bar{b}b).$$

Thus, $\mathbf{e}_b$ acts like a delta function. Use of the vectors $\mathbf{e}_b$ will greatly simplify the computations to follow.

II. Define the norm of a vector $f \in \mathscr{F}$ by $\| f \| = \langle f, f \rangle^{1/2}$. From the Schwarz inequality, $| f(b) | = | \langle \mathbf{e}_b, f \rangle | \leqslant \| \mathbf{e}_b \| \cdot \| f \| = \exp(\bar{b}b/2) \| f \|$. Thus, if $f,\ h \in \mathscr{F}$, then $| f(b) - h(b) | \leqslant \exp(\bar{b}b/2) \| f - h \|$ which shows that convergence in the norm of $\mathscr{F}$ implies pointwise convergence, uniform on any compact set in $\mathbb{C}$.

III. Finally, we quote without proof a theorem from Bargmann [3] which will be needed to justify several of our computations. Let R^k be

k-dimensional real Euclidean space and $\mathcal{C}^m$ be m-dimensional complex space. We are concerned with integrals of the form

$$F(\mathbf{z}) = \int_D f(\mathbf{z}, \tau)\, d^k\tau,$$

where D is a measurable set in R^k and $\mathbf{z} = (z_1, \ldots, z_m)$ is a point in an open set $\mathcal{O}$ of $\mathcal{C}^m$. Assume that, for every $\mathbf{z}$ in a neighborhood $N(|z_j - b_j| < \rho_j)$ of the point b in $\mathcal{O}$, f is analytic in z for every τ, measurable in τ, and

$$|f(\mathbf{z}, \tau)| < \eta(\tau), \qquad |z_j - b_j| < \rho_j,$$

where η is summable over D. Then $F(\mathbf{z})$ is analytic in N, and its partial derivatives are obtained by differentiating under the sign of integration, the resulting integrals being summable. In the following computations we freely interchange summation and integration without explicitly constructing the appropriate $\eta(\tau)$.

4-14 The Unitary Representation (λ, l)

We will now verify that all of the representations $\uparrow_{\omega,\mu}$ of $\mathcal{G}(0, 1)$ with ω an integer and $\mu > 0$ induce unitary irreducible representations of S_4. To find these representations of S_4 note that for ω an integer the multiplier representation (4.19), induced by $\uparrow_{\omega,\mu}$, depends on the parameter τ only in the form e^τ:

$$[\mathbf{B}(g)f](z) = e^{\mu(bz+a)}(e^\tau)^{-\omega} f(e^\tau z + e^\tau c), \qquad f \in \mathcal{O}_2, \tag{4.115}$$

where $g \in G(0, 1)$ has parameters (a, b, c, τ). (If ω is not an integer $(e^\tau)^{-\omega}$ is not a single-valued function of e^τ.) Thus, the operators $\mathbf{B}(g)$ define a multiplier representation of the 4-parameter matrix group $G(0, 1)'$ with elements

$$g'(a, b, c, \tau) = \begin{pmatrix} 1 & ce^\tau & a \\ 0 & e^\tau & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad a, b, c, \tau \in \mathcal{C}. \tag{4.116}$$

The matrices (4.116) are obtained from expression (4.4) for the matrices of $G(0, 1)$ by eliminating the last row and column. $G(0, 1)'$ is not simply connected. In fact, $g'(a, b, c, \tau) = g'(a, b, c, \tau + 2\pi i)$. However, $G(0, 1)$ and $G(0, 1)'$ are isomorphic as local Lie groups, i.e., they have isomorphic Lie algebras.

Comparison of (4.95) and (4.116) shows that S_4 can be considered as the real subgroup of $G(0, 1)'$ consisting of all matrices $g'(a, b, c, \tau)$ such that $\operatorname{Re} \tau = 0$, $b = -\bar{c}$, and $\operatorname{Re}(-bc/2 + a) = 0$. Thus,

$$g\{w, \alpha, \delta\} = g(i\delta - w\bar{w}/8, -w/2, \bar{w}/2, -i\alpha)$$

$$= \begin{pmatrix} 1 & e^{-i\alpha}\bar{w}/2 & i\delta - w\bar{w}/8 \\ 0 & e^{-i\alpha} & -w/2 \\ 0 & 0 & 1 \end{pmatrix} \in S_4,$$

where $w \in \mathcal{C}$, $0 \leqslant \alpha < 2\pi$, (mod 2π), and δ is real. Since S_4 can be embedded as a subgroup of $G(0, 1)'$ we can obtain a global multiplier representation for S_4 by restricting the operators $\mathbf{B}(g)$, (4.115), to the group elements g in S_4:

$$[\mathbf{B}(g)f](z) = \exp[\mu(-wz/2 + i\delta - w\bar{w}/8) + i\omega\alpha]f(e^{-i\alpha}z + e^{-i\alpha}\bar{w}/2), \quad (4.117)$$

defined for every entire analytic function f and every element g in S_4 with parameters $\{w, \alpha, \delta\}$. It will be convenient to write this expression in a slightly different form. To emphasize the fact that μ and ω can assume only restricted values we set $\mu = l > 0$ and $\omega = \lambda$ where λ is an integer. Further, we introduce the new complex variable t defined by $t = l^{1/2}z$ and the polar coordinates $re^{i\theta} = w/2$. In terms of these new variables (4.117) becomes

$$[\mathbf{B}(g)f](t) = \exp[-l^{1/2}tre^{i\theta} + il\delta - lr^2/2 + i\lambda\alpha]f(te^{-i\alpha} + l^{1/2}re^{-i\alpha-i\theta}) \quad (4.118)$$

where f is an entire function of t.

The generalized Lie derivatives of this multiplier representation of S_4 are readily computed:

$$\mathrm{J}_1 = \frac{il^{1/2}}{2}\left(-t - \frac{d}{dt}\right), \qquad \mathrm{J}_2 = \frac{l^{1/2}}{2}\left(-t + \frac{d}{dt}\right),$$

$$\mathrm{J}_3 = i\left(\lambda - t\frac{d}{dt}\right), \qquad \mathrm{Q} = il.$$

Thus,

$$\mathrm{J}^+ = -\mathrm{J}_2 + i\mathrm{J}_1 = l^{1/2}t, \qquad \mathrm{J}^- = +\mathrm{J}_2 + i\mathrm{J}_1 = l^{1/2}\frac{d}{dt},$$

$$\mathrm{J}^3 = i\mathrm{J}_3 = -\lambda + t\frac{d}{dt}, \qquad \mathrm{E} = -i\mathrm{Q} = l.$$

Applying these operators to the basis functions $j_n(t) = t^n/(n!)^{1/2}$, $n = 0, 1, 2, \ldots$, we have

$$\mathrm{J}^3 j_n = (n - \lambda) j_n , \qquad \mathrm{E} j_n = l j_n ,$$
$$\mathrm{J}^+ j_n = [l(n+1)]^{1/2} j_{n+1} , \qquad \mathrm{J}^- j_n = (ln)^{1/2} j_{n-1} .$$

These relations are identical with the relations (4.106) derived in Section 4-12. Thus to define unitary representations of S_4 induced by $\uparrow_{l,\lambda}$ we need only introduce a Hilbert space such that the functions j_n form an orthonormal basis. The action of the group on this Hilbert space will be given formally by (4.118). As we saw in the last section the Hilbert space $\mathscr{F}$ has the required property.

Using the preceding facts as motivation we now construct the unitary representation (λ, l) of S_4 on $\mathscr{F}$. Here, λ is an integer and $l > 0$. Let g be an element of S_4 with parameters $\{w, \alpha, \delta\} = \{2re^{i\theta}, \alpha, \delta\}$. Define the linear operator $\mathbf{U}(g)$ on $\mathscr{F}$ by

$$[\mathbf{U}(g)f](t) = \exp[-l^{1/2}tre^{i\theta} - lr^2/2 + il\delta + i\lambda\alpha] f(te^{-i\alpha} + l^{1/2}re^{-i\alpha-i\theta}), \quad f \in \mathscr{F}. \tag{4.119}$$

If g_1 and g_2 are elements of S_4 with coordinates $r_1, \theta_1, \alpha_1, \delta_1$ and $r_2, \theta_2, \alpha_2, \delta_2$, respectively, one readily finds the coordinates $r, \theta, \alpha, \delta$, of $g_1 g_2 \in S_4$ to be given by

$$\begin{aligned} re^{i\theta} &= r_1 e^{i\theta_1} + r_2 e^{i(\theta_2 - \alpha_1)}, \qquad \alpha = \alpha_1 + \alpha_2 , \\ \delta &= \delta_1 + \delta_2 + r_1 r_2 \sin(\alpha_1 + \theta_1 - \theta_2). \end{aligned} \tag{4.120}$$

Theorem 4.2 (λ, l) is a unitary representation of S_4.

PROOF (i) Unitarity. Using the inner product (4.111) on $\mathscr{F}$ we have

$$\begin{aligned} \langle \mathbf{U}(g)f, \mathbf{U}(g)h \rangle = \pi^{-1} \int & \exp[-l^{1/2}\bar{t}re^{-i\theta} - l^{1/2}tre^{i\theta} - lr^2] \\ & \cdot \overline{f(te^{-i\alpha} + l^{1/2}re^{-i\alpha-i\theta})} \\ & \cdot h(te^{-i\alpha} + l^{1/2}re^{-i\alpha-i\theta}) \exp(-t\bar{t})\, dx\, dy, \qquad t = x + iy, \end{aligned}$$

for all $g \in S_4$, $f, h \in \mathscr{F}$. With the introduction of the new complex variable $t' = x' + iy' = te^{-i\alpha} + l^{1/2}re^{-i\alpha-i\theta}$ this integral simplifies to

$$\pi^{-1} \int \overline{f(t')}\, h(t') \exp(-t'\bar{t}')\, dx'\, dy' = \langle f, h \rangle.$$

(ii) Representation property. If $g_1, g_2 \in S_4$ with coordinates as given above, we have

$$\begin{aligned}[\mathbf{U}(g_1)\,\mathbf{U}(g_2)f](t) &= \exp[-l^{1/2}tr_1e^{i\theta_1} - lr_1^2/2 + il\delta_1 + i\lambda\alpha_1][\mathbf{U}(g_2)f] \\ &\qquad \cdot (te^{-i\alpha_1} + l^{1/2}r_1e^{-i\alpha_1-i\theta_1}) \\ &= \exp[-l^{1/2}tre^{i\theta} - lr^2/2 + il\delta + i\lambda\alpha]\,f(te^{-i\alpha} + l^{1/2}re^{-i\alpha-i\theta}) \\ &= [\mathbf{U}(g_1g_2)f](t)\end{aligned}$$

for all $f \in \mathscr{F}$, where the variables $re^{i\theta}$, α, δ are defined by (4.120). Q.E.D.

4-15 The Matrix Elements of (λ, l)

Given $g \in S_4$ with coordinates $re^{i\theta}$, α, δ, we shall compute the matrix elements

$$U_{n,m}(g) = \langle j_n\,, \mathbf{U}(g)j_m\rangle, \qquad n, m = 0, 1, 2, \ldots,$$

with respect to the orthonormal basis $\{j_n\}$. In fact we can easily derive a generating function for these matrix elements. To do this note that

$$\mathbf{e}_s(t) = \exp(\bar{s}t) = \sum_{n=0}^{\infty} j_n(t)\overline{j_n(s)} = \sum_{n=0}^{\infty} j_n(t)\frac{\bar{s}^n}{(n!)^{1/2}},$$

and define the generating function $\mathrm{G}^{\lambda,l}(g; s, u)$ by

$$\mathrm{G}^{\lambda,l}(g; s, u) = \langle \mathbf{e}_s\,, \mathbf{U}(g)\,\mathbf{e}_{\bar{u}}\rangle = \sum_{n,m=0}^{\infty} \langle j_n\,, \mathbf{U}(g)j_m\rangle \frac{s^n u^m}{(n!\;m!)^{1/2}}. \quad (4.121)$$

Due to the delta function property of $\mathbf{e}_s$ we obtain

$$\begin{aligned}\langle \mathbf{e}_s\,, \mathbf{U}(g)\,\mathbf{e}_{\bar{u}}\rangle &= [\mathbf{U}(g)\,\mathbf{e}_{\bar{u}}](s) \\ &= \exp[i\lambda\alpha + il\delta - lr^2/2 + ue^{-i\alpha}(s + l^{1/2}re^{-i\theta}) - l^{1/2}sre^{i\theta}] \\ &= \sum_{n,m=0}^{\infty} U_{n,m}(g)\frac{s^n u^m}{(n!\;m!)^{1/2}}, \end{aligned} \quad (4.122)$$

where the last equality follows from (4.121). By equating coefficients of powers of s and u we easily find expressions for the matrix elements themselves. The result is

$$\begin{aligned}U_{n,m}(g) &= \exp[i\alpha(\lambda - m) + il\delta + i(n - m)\theta] \\ &\qquad \cdot \exp\left(-\frac{lr^2}{2}\right)\left(\frac{n!}{m!}\right)^{1/2}(rl^{1/2})^{m-n}L_n^{(m-n)}(lr^2), \end{aligned} \quad (4.123)$$

or,

$$U_{n,m}(g) = \exp[i\alpha(\lambda - m) + il\delta + i(n - m)\theta]$$
$$\cdot \exp\left(-\frac{lr^2}{2}\right)\left(\frac{m!}{n!}\right)^{1/2}(-rl^{1/2})^{n-m}L_m^{(n-m)}(lr^2),$$

where the functions $L_b^{(a)}$, a, b integers, $a + b \geqslant 0$, are the associated Laguerre polynomials. This result can be used to derive a new generating function for the associated Laguerre polynomials. Choose $\theta = \alpha = \delta = 0$, $l = 1$ and substitute (4.123) into (4.122) to obtain

$$e^{us+r(u-s)} = \sum_{m,n=0}^{\infty} \frac{r^{(m-n)}}{m!} L_n^{(m-n)}(r^2)\, s^n u^m. \tag{4.124}$$

Lemma 4.5 The representation (λ, l) is irreducible.

PROOF The method of proof is completely analogous to that of Lemma 3.2. Assume the lemma is false. Then, there exists a proper closed subspace $\mathcal{M}$ of $\mathcal{F}$ such that $\mathbf{U}(g)\, f \in \mathcal{M}$ for all $g \in S_4$, $f \in \mathcal{M}$. Let $\mathbf{P}$ be the self-adjoint projection operator on $\mathcal{M}$. Since $\mathcal{M}$ is a proper subspace of $\mathcal{F}$ it follows that $\mathbf{P} \neq \mathbf{O}, \mathbf{I}$. Moreover, $\mathbf{U}(g)\mathbf{P} = \mathbf{P}\mathbf{U}(g)$ for all $g \in S_4$. Let g_α be the element of S_4 with coordinates $\{0, \alpha, 0\}$, i.e., $re^{i\theta} = \delta = 0$. Then, $\mathbf{U}(g_\alpha)j_n = e^{i\alpha(\lambda-n)}j_n$ for all $n \geqslant 0$, which implies $\mathbf{U}(g_\alpha)\,\mathbf{P}j_n = e^{i\alpha(\lambda-n)}\mathbf{P}j_n$. Thus, $\mathbf{P}j_n = a_n j_n$, a_n a constant. Since $\mathbf{P}^2 = \mathbf{P}$ we have $a_n^2 = a_n$ for all $n \geqslant 0$; hence, $a_n = 0$ or $a_n = 1$. By hypothesis there exist integers n_0, $n_1 \geqslant 0$ such that $\mathbf{P}j_{n_0} = 0$, $\mathbf{P}j_{n_1} = j_{n_1}$. Thus, $U_{n_0,n_1}(g) = \langle j_{n_0}, \mathbf{U}(g)\,\mathbf{P}j_{n_1}\rangle = \langle \mathbf{P}j_{n_0}, \mathbf{U}(g)j_{n_1}\rangle = 0$ for all $g \in S_4$. This is a contradiction since none of the matrix elements (4.123) is identically zero for all $g \in S_4$. Q.E.D.

The generating function $G^{\lambda,l}(g; s, u)$ and the matrix elements $U_{n,m}(g)$ are functions of the group parameters r, θ, α, δ; hence, they can be integrated over the group manifold with respect to a suitable measure. As is well known from the general theory of topological groups (Naimark [1]), there is a unique (up to a constant) measure $d(g)$ on S_4 with the property

$$\int f(gg_o)\, d(g) = \int f(g)\, d(g) \tag{4.125}$$

for every integrable function f on S_4 and every $g_o \in S_4$. (The domain of integration is the entire group manifold S_4.) This measure is called the (right invariant) Haar measure. The suitably normalized Haar measure on S_4 is given by

$$d(g) = \frac{r\, dr\, d\theta\, d\alpha\, d\delta}{16\pi^3}$$

where r, θ, α, δ are the coordinates of $g \in S_4$. Thus,

$$\int f(g)\, d(g) = \frac{1}{16\pi^3} \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\infty} f(r, \theta, \alpha, \delta) r\, dr\, d\theta\, d\alpha\, d\delta.$$

The reader can verify directly that $d(g)$ has the property (4.125). The constant $(16\pi^3)^{-1}$ has been chosen for purposes of normalization.

We now compute the integral

$$\int \overline{\mathrm{G}^{\lambda,l}(g; u, s)}\, \mathrm{G}^{\lambda',l'}(g; u', s')\, d(g) = \sum_{p,q,n,m=0}^{\infty} \int \overline{U_{p,q}^{\lambda,l}(g)} \cdot U_{n,m}^{\lambda',l'}(g)\, d(g) \frac{\bar{u}^p u'^n \bar{s}^q s'^m}{(p!\, n!\, q!\, m!)^{1/2}}.$$

(The superscripts λ, l are used to denote the representation (λ, l).) The first integral can be evaluated directly if one makes use of the simple expression (4.122) for $\mathrm{G}(\cdot)$ and the properties of the vector $\mathbf{e}_b$. The result of the straightforward integration is

$$\int \overline{\mathrm{G}^{\lambda,l}(g; u, s)}\, \mathrm{G}^{\lambda',l'}(g; u', s')\, d(g) = \frac{\delta(l - l')}{4l} \delta_{\lambda,\lambda'} \exp(\bar{u}u' + \bar{s}s').$$

Here $\delta_{\lambda,\lambda'}$ is the Kronecker delta while $\delta(l - l')$ is the Dirac delta function which arises from the symbolic integration formula

$$\delta(l - l') = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{ix(l-l')}\, dx.$$

Theorem 4.3 $\displaystyle\int \overline{U_{p,q}^{\lambda,l}(g)}\, U_{n,m}^{\lambda',l'}(g)\, d(g) = \frac{\delta(l - l')}{4l} \delta_{\lambda,\lambda'}\, \delta_{p,n}\, \delta_{q,m}.$

Theorem 4.3 gives orthogonality relations for the matrix elements of the unitary representations (λ, l). If the matrix elements are expressed in terms of associated Laguerre polynomials by (4.123), these orthogonality relations reduce to the formula

$$\int_0^{\infty} \exp(-r^2)\, r^{2k+1} L_m^{(k)}(r^2) L_n^{(k)}(r^2)\, dr = \frac{(n + k)!}{2n!} \delta_{m,n} \tag{4.126}$$

valid for all integers $m, n \geqslant 0$ and all integers k such that $n + k \geqslant 0$, $m + k \geqslant 0$.

The group property $\mathbf{U}(g_1 g_2) = \mathbf{U}(g_1)\mathbf{U}(g_2)$ leads to an addition theorem

$$\langle j_n, \mathbf{U}(g_1 g_2) j_m \rangle = \sum_{k=0}^{\infty} \langle j_n, \mathbf{U}(g_1) j_k \rangle \langle j_k, \mathbf{U}(g_2) j_m \rangle$$

for the matrix elements of (λ, l). Substituting expressions (4.123) for the matrix elements and simplifying we obtain

$$(re^{i\theta})^{m-n} L_n^{(m-n)}(r^2) \exp[-r_1 r_2 e^{i(\theta_1-\theta_2)}]$$

$$= \sum_{k=0}^{\infty} (r_1 e^{i\theta_1})^{k-n} L_n^{(k-n)}(r_1^2)(r_2 e^{i\theta_2})^{m-k} L_k^{(m-k)}(r_2^2), \tag{4.127}$$

where $m, n \geqslant 0$ and $re^{i\theta} = r_1 e^{i\theta_1} + r_2 e^{i\theta_2}$. This formula is, of course, only a special case of the addition theorem (4.28) for the matrix elements of $\uparrow_{\omega,\mu}$.

We can use (4.127) to derive a simple integral formula for the product of two associated Laguerre polynomials by setting $\theta_1 = 0$ and multiplying both sides of the equation by $e^{i(p-m)\theta_2}$. Integrating with respect to θ_2 we obtain

$$r_1^{p-n} r_2^{m-p} L_n^{(p-n)}(r_1^2) L_p^{(m-p)}(r_2^2)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \exp[i(p-m)\,\theta_2 - r_1 r_2 e^{-i\theta_2}](re^{i\theta})^{m-n} L_n^{(m-n)}(r^2)\, d\theta_2, \tag{4.128}$$

where $re^{i\theta} = r_1 + r_2 e^{i\theta_2}$ and p is a nonnegative integer.

Since the operators $\mathbf{U}(g)$ are unitary and the basis vectors j_n have norm one it follows that

$$|\, U_{n,m}(g)| = |\langle j_n, \mathbf{U}(g) j_m\rangle| \leqslant 1, \qquad n, m \geqslant 0, \quad g \in S_4,$$

which implies

$$|L_n^{(m-n)}(r^2)| \leqslant (m!/n!)^{1/2}\, r^{n-m} \exp(r^2/2).$$

4-16 The Unitary Representations $(\lambda, -l)$

In exact analogy with the work of Section (4-14) we can construct unitary irreducible representations $(\lambda, -l)$ of S_4 on the Hilbert space $\mathscr{F}$ which are induced by the representations $\downarrow_{\lambda,l}$ of $\mathscr{G}(0, 1)$ (λ an integer, $l > 0$). Rather than repeat the motivation for this construction we merely present the results.

The construction of the representations $(\lambda, -l)$ of S_4 on $\mathscr{F}$ is very similar to that for (λ, l). Define the linear operator $\mathbf{V}(g)$, on $\mathscr{F}$ by

$$[\mathbf{V}(g)f](t) = \exp[l^{1/2} t r e^{-i\theta} - lr^2/2 + i\alpha(\lambda + 1) - il\delta]$$
$$\cdot f(te^{i\alpha} - l^{1/2} r e^{i\alpha + i\theta}), \qquad f \in \mathscr{F},$$

where $g \in S_4$ has coordinates $re^{i\theta}$, α, δ.

Theorem 4.4 $(\lambda, -l)$ is a unitary representation of S_4. The matrix elements of (λ, l) are given by

$$V_{n,m}(g) = \langle j_n, \mathbf{V}(g) j_m\rangle, \qquad g \in S_4 .$$

In analogy with (4.121) and (4.122), we obtain the generating function

$$G^{\lambda,-l}(g; s, u) = \sum_{m,n=0}^{\infty} V_{n,m}(g) \frac{s^n u^m}{(n!\, m!)^{1/2}}$$

$$= \exp[i\alpha(\lambda+1) - il\delta - \tfrac{1}{2}lr^2 + ue^{i\alpha}(s - l^{1/2}re^{i\theta}) + l^{1/2}sre^{-i\theta}], \quad (4.129)$$

with the result

$$V_{n,m}(g) = \exp[i\alpha(\lambda+m+1) - il\delta + i(m-n)\theta]$$

$$\cdot \exp(-lr^2/2)(n!/m!)^{1/2}(-l^{1/2}r)^{m-n} L_n^{(m-n)}(lr^2). \quad (4.130)$$

Note that the matrix elements (4.123) and (4.130) have the same r dependence. They differ only in their dependence on α, δ, and θ.

Lemma 4.6 The representation (λ, l) is irreducible.

Theorem 4.5 $\int \overline{V_{p,q}^{\lambda,-l}(g)}\, V_{n,m}^{\lambda',-l'}(g)\, d(g) = \dfrac{\delta(l'-l)}{4l}\, \delta_{\lambda,\lambda'}\, \delta_{p,n}\, \delta_{q,m}$.

Note that

$$\int \overline{U_{p,q}^{\lambda,l}(g)}\, V_{n,m}^{\lambda',-l'}(g)\, d(g) \equiv 0,$$

because of the integration over the variable δ.

The infinitesimal operators of the representation $(\lambda, -l)$ corresponding to the elements $\mathscr{J}_1, \mathscr{J}_2, \mathscr{J}_3, \mathscr{Q}$ of $\mathscr{S}_4$ are given by

$$J_1 = \frac{il^{1/2}}{2}\left(-t - \frac{d}{dt}\right), \qquad J_2 = \frac{l^{1/2}}{2}\left(t - \frac{d}{dt}\right),$$

$$J_3 = i\left(\lambda + 1 + t\frac{d}{dt}\right), \qquad Q = -il.$$

Hence,

$$J^+ = -J_2 + iJ_1 = l^{1/2}\frac{d}{dt}, \qquad J^- = J_2 + iJ_1 = l^{1/2}t,$$

$$J^3 = iJ_3 = -\lambda - 1 - t\frac{d}{dt}, \qquad E = -iQ = -l$$

and the action of these operators on the basis vectors $j_n(t) = t^n/(n!)^{1/2}$ is

$$J^3 j_n = (-\lambda - 1 - n) j_n, \qquad E j_n = -l j_n,$$
$$J^+ j_n = (ln)^{1/2} j_{n-1}, \qquad J^- j_n = [l(n+1)]^{1/2} j_{n+1}$$

for all $n \geqslant 0$. These relations are identical with Eqs. (4.107) ($\omega = \lambda$, $\mu = l$) and show that the representation $(\lambda, -l)$ of S_4 induces the representation $\downarrow_{\lambda, l}$ of $\mathscr{G}(0, 1)$.

4-17 The Tensor Products $(\lambda, l) \otimes (\lambda', l')$

We begin a study of tensor products of representations of S_4 by considering the representation $(\lambda, l) \otimes (\lambda', l')$ where λ, λ' are integers and $l, l' > 0$. This representation is defined on the space $\mathscr{F}_2 \simeq \mathscr{F} \otimes \mathscr{F}$ of entire functions $f(t, p)$ of the complex variables t and p such that

$$\int_t \int_p |f(t, p)|^2 \, d\xi(t) \, d\xi(p) < \infty,$$

where $d\xi(t) = \pi^{-1} \exp(-t\bar{t}) \, dx \, dy$, $t = x + iy$, and the domain of integration is $\mathscr{C}^2$. Clearly $\mathscr{F}_2$ is a Hilbert space with inner product

$$\langle f, h \rangle' = \iint \overline{f(t, p)} \, h(t, p) \, d\xi(t) \, d\xi(p), \qquad f, h \in \mathscr{F}_2 .$$

The action of $g \in S_4$ on $f \in \mathscr{F}_2$ is given by

$$\begin{aligned}
&[\mathbf{U}^{\lambda,l}(g) \otimes \mathbf{U}^{\lambda',l'}(g) f](t, p) \\
&\quad = \exp[-(l^{1/2} t + l'^{1/2} p) r e^{i\theta} - \tfrac{1}{2}(l + l') r^2 + i\alpha(\lambda + \lambda') + i\delta(l + l')] \\
&\qquad \cdot f(t e^{-i\alpha} + l^{1/2} r e^{-i\theta - i\alpha}, p e^{-i\alpha} + l'^{1/2} r e^{-i\theta - i\alpha})
\end{aligned} \tag{4.131}$$

where g has the coordinates $r, \theta, \alpha, \delta$. It is simple to check that the operators $\mathbf{U}^{\lambda,l}(g) \otimes \mathbf{U}^{\lambda',l'}(g)$ define a unitary representation $(\lambda, l) \otimes (\lambda', l')$ of S_4 on $\mathscr{F}_2$. However, this representation is not irreducible. The following paragraphs will be devoted to decomposing $(\lambda, l) \otimes (\lambda', l')$ into irreducible representations. Since S_4 is not a compact group, general theorems on locally compact groups tell us only that $(\lambda, l) \otimes (\lambda', l')$ is unitarily equivalent to a direct integral of unitary irreducible representations of S_4. However, we shall show that in fact it is unitarily equivalent to a direct sum of unitary irreducible representations of S_4.

The vectors $j_{m,n} = t^m p^n/(m!\,n!)^{1/2}$, $m, n = 0, 1, 2, \ldots$, form an orthonormal basis for $\mathscr{F}_2$. By means of a change of variable a new basis can be introduced in $\mathscr{F}_2$ on which the action of $(\lambda, l) \otimes (\lambda', l')$ becomes more transparent. Define new variables u and v by

$$\begin{aligned} u &= (l^{1/2}p - l'^{1/2}t)/(l + l')^{1/2}, \\ v &= (l'^{1/2}p + l^{1/2}t)/(l + l')^{1/2}. \end{aligned} \tag{4.132}$$

These equations can be inverted to give

$$\begin{aligned} t &= (l^{1/2}v - l'^{1/2}u)/(l + l')^{1/2}, \\ p &= (l'^{1/2}v + l^{1/2}u)/(l + l')^{1/2}. \end{aligned}$$

Furthermore, by evaluating the appropriate Jacobian it is easy to show

$$d\xi(t)\,d\xi(p) = d\xi(u)\,d\xi(v). \tag{4.133}$$

A polynomial in t, p can be written as a polynomial in u, v, and conversely. Indeed, if $f \in \mathscr{F}_2$, then $f(t, p) \equiv h(u, v)$, where t, p, u, and v are related by (4.132) and h is an entire function of u, v. Again the converse holds. From these remarks and (4.133) we see that the functions $u^n v^m/(m!\,n!)^{1/2}$, $n, m = 0, 1, 2, \ldots$, form an orthonormal basis for $\mathscr{F}_2$. We now apply the operator $\mathbf{U}^{\lambda,l}(g) \otimes \mathbf{U}^{\lambda',l'}(g)$ defined by (4.131) to the function $h(u, v) = u^n k(v) \in \mathscr{F}_2$, where n is a nonnegative integer and k is an entire function of v. The result is

$$\begin{aligned} [\mathbf{U}^{\lambda,l}(g) \otimes \mathbf{U}^{\lambda',l'}(g)h](u, v) = \exp[&-(l + l')^{1/2}\, vre^{i\theta} - \tfrac{1}{2}(l + l')r^2 \\ &+ i\alpha(\lambda + \lambda' - n) + i\delta(l + l')]u^n k \\ &\cdot (ve^{-i\alpha} + r(l + l')^{1/2}\, e^{-i\theta - i\alpha}). \end{aligned} \tag{4.134}$$

Thus, the functions $u^n k(v) \in \mathscr{F}_2$ for fixed n form a basis space for the irreducible representation $(\lambda + \lambda' - n, l + l')$ (compare with (4.119)). This proves:

Theorem 4.6 $(\lambda, l) \otimes (\lambda', l') \cong \sum_{n=0}^{\infty} \oplus (\lambda + \lambda' - n, l + l')$.

To study this decomposition in greater detail we adopt a new notation. Set

$$\begin{aligned} j_{m,n}(t, p) &= \frac{t^m p^n}{(m!\,n!)^{1/2}}, \quad h_m^{(\lambda+\lambda'-n,\, l+l')}(u, v) \\ &= \frac{u^n v^m}{(m!\,n!)^{1/2}}, \qquad n, m = 0, 1, 2, \ldots . \end{aligned}$$

Here, the superscripts on h denote the representation to which the corresponding basis vectors belong. From the last theorem, both the collection of vectors $\{j_{m,n}\}$ and the vectors $\{h_m^{(\lambda+\lambda'-n, l+l')}\}$ form orthonormal bases for $\mathscr{F}_2$. Thus the j basis vectors and the h basis vectors can be expressed as linear combinations of one another, the coefficients of the expansion being known as Clebsch-Gordan coefficients. We now compute these coefficients.

Clearly,

$$h_m^{(\lambda+\lambda'-n, l+l')} = \sum_{q,s=0}^{\infty} \langle j_{q,s}, h_m^{(\lambda+\lambda'-n, l+l')}\rangle' j_{q,s}.$$

Thus,

$$\frac{u^n v^m}{(n!\,m!)^{1/2}} = \frac{(l^{1/2}p - l'^{1/2}t)^n (l'^{1/2}p + l^{1/2}t)^m}{[n!\,m!\,(l+l')^{m+n}]^{1/2}}$$

$$= \sum_{q,s=0}^{\infty} \langle j_{q,s}, h_m^{(\lambda+\lambda'-n, l+l')}\rangle' \frac{t^q p^s}{(q!\,s!)^{1/2}}. \tag{4.135}$$

Define the Clebsch–Gordan coefficients $\mathrm{K}[l, q; l', s \mid n, m]$, zero unless $q + s = n + m$, by

$$\mathrm{K}[l, q; l', s \mid n, m] = \langle j_{q,s}, h_m^{(\lambda+\lambda'-n, l+l')}\rangle'.$$

From (4.135) we obtain

$$\mathrm{K}[l, q; l', s \mid n, m]$$

$$= \begin{cases} 0 & \text{if} \quad m+n \neq q+s \\ \left[\dfrac{(n)!\,(m)!\,(m+n-q)!\,(q)!\,(l/l')^{n-q}}{(1+l/l')^{m+n}}\right]^{1/2} & \end{cases}$$

$$\cdot \sum_a \frac{(-1)^{q-a}(l/l')^a}{a!\,(q-a)!\,(m-a)!\,(n-q+a)!} \qquad \text{if} \quad m+n = q+s. \tag{4.136}$$

Here the summation is taken over all integer values of a such that the summand is defined. Clearly, the Clebsch–Gordan coefficients are independent of λ, λ' and depend only on the ratio l/l' of l and l'.

It is easy to prove from (4.136) that

$$\mathrm{K}[l, q; l', s \mid n, m] = (-1)^n\, \mathrm{K}[l', s; l, q \mid n, m].$$

It is also easy to find generating functions for the coefficients K. From (4.135),

$$\exp\left[\frac{l^{1/2}(zp+wt)+l'^{1/2}(wp-zt)}{(l+l')^{1/2}}\right]=e^{zu+wv}$$

$$=\sum_{n,m,q,s=0}^{\infty}\mathrm{K}[l,q;l',s\mid n,m]\,\frac{z^n w^m t^q p^s}{(n!\,m!\,q!\,s!)^{1/2}}. \tag{4.137}$$

Since the left-hand side of this expression is invariant under the simultaneous transpositions $w\leftrightarrow p$, $t\leftrightarrow z$,

$$\mathrm{K}[l,q;l',s\mid n,m]=\mathrm{K}[l,n;l',m\mid q,s].$$

The decomposition of $(\lambda,l)\otimes(\lambda',l')$ into irreducible parts as given by Theorem 4.6 can be considered as a special case of the decomposition of the representation $\uparrow_{\omega_1,\mu_1}\otimes\uparrow_{\omega_2,\mu_2}$ of $\mathscr{G}(0,1)$ studied in Section 4-5. In fact, if $\omega_1=\lambda$, $\omega_2=\lambda'$, $\mu_1=l$, $\mu_1=l'$, the Clebsch–Gordan coefficients H, Eq. (4.58) in Section 4-5, are related to the coefficients K by the equation

$$\mathrm{H}(l,q;l',n+m-q\mid n,m)$$

$$=(-1)^n\left[\frac{(m)!\,(n!)(l'/l)^{m-q}}{(m+n-q)!\,(q)!\,(1+l'/l)^{m-n}}\right]^{1/2}\mathrm{K}[l,q;l',n+m-q\mid n,m]$$

$$=\frac{(-1)^{n+m-q}n!}{(n+m-q)!\,(1+l'/l)^m}\,\frac{F(-m,q-n-m;q-m+1;-l'/l)}{\Gamma(q-m+1)}. \tag{4.138}$$

The H and K coefficients differ by multiplicative factors because they have been defined with respect to different basis vectors.

In analogy with Eq. (4.62), we can derive the formula

$$U^{\lambda,l}_{m,s}(g)\,U^{\lambda',l'}_{n,q}(g)=\sum_{k=0}^{\infty}\mathrm{K}[l,s;l',q\mid k,s+q-k]$$

$$\cdot\,\mathrm{K}[l,m;l',n\mid k,m+n-k]\,U^{\lambda+\lambda'-k,\,l+l'}_{m+n-k,\,s+q-k}(g),$$

$$g\in S_4, \tag{4.139}$$

expressing the product of two matrix elements as a sum of matrix elements. The sum over k is finite because the coefficients K are nonzero for only a finite number of values of k.

By definition the Clebsch–Gordan coefficients satisfy the unitarity conditions

$$\sum_{n,m=0}^{\infty} \mathrm{K}[l, q_1 ; l', s_1 \mid n, m] \overline{\mathrm{K}[l\ q_2 ; l', s_2 \mid n, m]} = \delta_{q_1, q_2} \delta_{s_1, s_2},$$
$$\sum_{q,s=0}^{\infty} \mathrm{K}[l, q; l', s \mid n_1, m_1] \overline{\mathrm{K}[l, q; l', s \mid n_2, m_2]} = \delta_{n_1, n_2} \delta_{m_1, m_2}. \tag{4.140}$$

These conditions can be used to derive identities for the Jacobi polynomials.

The method needed to decompose the tensor product $(\lambda, -l) \otimes (\lambda', -l')$ (λ, λ' integers, $l, l' > 0$) is almost identical to that for $(\lambda, l) \otimes (\lambda', l')$. We define the unitary representation $(\lambda, -l) \otimes (\lambda', -l')$ of S_4 on $\mathscr{F}_2$ by means of operators $\mathbf{V}^{\lambda,-l}(g) \otimes \mathbf{V}^{\lambda',-l'}(g)$, $g \in S_4$, given by

$$\begin{aligned}[\mathbf{V}^{\lambda,-l}(g) \otimes \mathbf{V}^{\lambda',-l'}(g) f](t, p) &\\ = \exp[(l^{1/2}t + l'^{1/2}p)\, re^{-i\theta} - \tfrac{1}{2}(l + l')r^2 + i\alpha(\lambda + \lambda' + 2) - i(l + l')\delta] &\\ \cdot f(te^{i\alpha} - e^{i\alpha}l^{1/2}re^{i\theta}, pe^{i\alpha} - e^{i\alpha}l'^{1/2}re^{i\theta}). &\end{aligned} \tag{4.141}$$

Introducing the variables u, v defined by (4.132) we can easily decompose this representation into its irreducible parts. The result is

Theorem 4.7 $(\lambda, -l) \otimes (\lambda', -l') \cong \sum_{n=0}^{\infty} \oplus\, (\lambda + \lambda' + n + 1, -l - l')$.
The Clebsch–Gordan coefficients for this decomposition are

$$\langle j_{q,s}, h_m^{(\lambda+\lambda'+n+1,-l-l')} \rangle' = \mathrm{K}[l, q; l', s \mid n, m], \tag{4.142}$$

where the coefficients K are defined by (4.136).

4-18 The Representations $(\lambda, l) \otimes (\lambda', -l')$

The problem of decomposing the unitary representation $(\lambda, l) \otimes (\lambda', -l')$ where λ, λ' are integers and $l, l' > 0$ is more difficult than the problems considered in the last section. It naturally divides into three cases: (1) $l > l'$, (2) $l' > l$, and (3) $l = l'$. In each case we define the representation $(\lambda, l) \otimes (\lambda', -l')$ on $\mathscr{F}_2$ as follows: For each $g \in S_4$, designate by $\mathbf{U}^{\lambda,l}(g) \otimes \mathbf{V}^{\lambda',-l'}(g) = \mathbf{W}(g)$ the operator

$$\begin{aligned}[\mathbf{W}(g)f(]t, p) = \exp[&-l^{1/2}tre^{i\theta} + l'^{1/2}pre^{-i\theta}\\ &- \tfrac{1}{2}(l + l')r^2 + i\alpha(\lambda + \lambda' + 1) + i(l - l')\delta]\\ &\cdot f(te^{-i\alpha} + l^{1/2}re^{-i\theta-i\alpha}, pe^{i\alpha} - l'^{1/2}re^{i\theta+i\alpha})\end{aligned} \tag{4.143}$$

for all $f \in \mathscr{F}_2$. It is easy to verify that these operators yield a unitary representation of S_4 on $\mathscr{F}_2$.

CASE 1. $l > l'$. We could treat this case by Lie algebraic methods in analogy with the work of Section 4-5. Indeed, we could compute the possible eigenvectors f_o in $\mathscr{F}_2$ of the operator $J^3 = -\lambda - \lambda' - 1 + t\,\partial/\partial t - p\,\partial/\partial p$ with the property $J^- f_o = 0$, where $J^- = l^{1/2}\,\partial/\partial t + l'^{1/2} p$. For each such f_o we could then show that the vectors $(J^+)^n f_o$, $n = 0, 1, 2,\ldots$, form a basis for an irreducible representation of S_4. Here, $J^+ = l^{1/2} t + l'^{1/2}\,\partial/\partial p$. Rather than work out the details of this construction it will be more convenient to present the results and verify that they are correct. We define the function $G(u, v, t, p)$, u, v, t, p complex, by

$$G(u, v, t, p) = \exp[(ut + vp)(1 - l'/l)^{1/2} + (uv - tp)(l'/l)^{1/2}]\,(1 - l'/l)^{1/2}$$
$$= \sum_{k,n=0}^{\infty} \frac{u^k v^n}{(k!\,n!)^{1/2}} h_k^{(n)}(t, p). \tag{4.144}$$

$G(\cdot)$ is a generating function for the vectors $h_k^{(n)} \in \mathscr{F}_2$ which are defined by (4.144). Clearly, $G(\cdot)$ is an entire analytic function of each of its variables. Also it is square integrable with respect to either of the measures $d\xi(u)\,d\xi(v)$ or $d\xi(t)\,d\xi(p)$ taken over the domain $\mathcal{C}^2$.

It will be shown that the vectors $h_k^{(n)}$ form an orthonormal basis for $\mathscr{F}_2$, and the matrix elements of $\mathbf{W}(g)$ with respect to this basis will be computed. To this end we evaluate the inner product

$$\langle G(u, v, \cdot, \cdot), [\mathbf{W}(g)]\, G(w, z, \cdot, \cdot)\rangle'$$
$$= \frac{l}{l - l'} \sum_{k,n,q,m=0}^{\infty} \frac{u^k v^n w^q z^m}{(k!\,n!\,q!\,m!)^{1/2}} \langle h_k^{(n)}, \mathbf{W}(g)\, h_q^{(m)}\rangle', \qquad g \in S_4. \tag{4.145}$$

The integration is carried out over the variables p, t. The inner product on the left-hand side of (4.145) can be evaluated directly by making use of the definitions (4.143), (4.144). This integration is elementary and can be performed with no other tool than the delta function property of the vectors $\mathbf{e}_b$. The result is

$$\exp[vze^{i\alpha} - uwe^{-i\alpha} - u(l - l')^{1/2}\, re^{i\theta} - \tfrac{1}{2}(l - l')r^2 + i\alpha(\lambda + \lambda' + 1)$$
$$+ i\,\delta(l - l') + w(l - l')^{1/2}\, re^{-i\theta - i\alpha}]$$
$$= \sum_{k,n,q,m=0}^{\infty} \frac{u^k v^n w^q z^m}{(k!\,n!\,q!\,m!)^{1/2}} \langle h_k^{(n)}, \mathbf{W}(g)\, h_q^{(m)}\rangle', \tag{4.146}$$

where $g \in S_4$ has coordinates r, θ, α, δ. We can draw several immediate conclusions from this equation.

Lemma 4.7 $\langle h_k^{(n)}, \mathbf{W}(g)\, h_q^{(m)}\rangle' = 0$ unless $m = n$.

Lemma 4.8 For fixed n,

$$\begin{aligned}\exp[uwe^{-i\alpha} &- ure^{i\theta}(l-l')^{1/2} - \tfrac{1}{2}(l-l')r^2 + i\alpha(\lambda+\lambda'+n+1)\\ &+ i\,\delta(l-l') + w(l-l')^{1/2}\, re^{-i\theta-i\alpha}]\\ &= \sum_{k,q=0}^{\infty} \frac{u^k w^q}{(k!\,q!)^{1/2}} \langle h_k^{(n)}, \mathbf{W}(g)\, h_q^{(n)}\rangle'. \qquad (4.147)\end{aligned}$$

PROOF Equate the coefficients of $(vz)^n$ on both sides of (4.146).

Comparing the generating function (4.147) with (4.122) we see that the vectors $h_k^{(n)}$, $k = 0, 1, 2,\ldots$, form an orthonormal basis for the irreducible representation $(\lambda + \lambda' + n + 1, l - l')$ of S_4. Furthermore, if g is the identity element in S_4 $(r = \theta = \alpha = \delta = 0)$, then according to Eq. (4.146) the vectors $\{h_k^{(n)}\}$, $n, k = 0, 1, 2,\ldots$, form an orthonormal set. We must still show that the $\{h_k^{(n)}\}$ form a basis for $\mathscr{F}_2$. To do this we evaluate the inner product

$$\begin{aligned}&\langle G(.,., u, v), G(.,., t, p)\rangle'\\ &\quad = \frac{l}{l-l'} \sum_{k,n,q,m=0}^{\infty} \int \frac{\bar{w}^k w^q \bar{z}^n z^m}{(k!\,q!\,n!\,m!)^{1/2}}\, d\xi(w)\, d\xi(z)\, \overline{h_k^{(n)}(u, v)}\, h_q^{(m)}(t, p)\\ &\quad = \frac{l}{l-l'} \sum_{k,n=0}^{\infty} \overline{h_k^{(n)}(u, v)}\, h_k^{(n)}(t, p), \qquad u, v, t, p \in \mathscr{C}.\end{aligned}$$

This inner product can be computed directly by using the vectors $\mathbf{e}_b$. The final result is

$$\exp(\bar{u}t + \bar{v}p) = \sum_{k,n=0}^{\infty} \overline{h_k^{(n)}(u, v)}\, h_k^{(n)}(t, p). \qquad (4.148)$$

We have proved only pointwise convergence in (4.148), but convergence in the mean $(d\xi(t)\, d\xi(p))$ follows from the orthonormality of the $h_k^{(n)}$. Set $\mathbf{e}_{u,v}(t, p) = \exp(\bar{u}t + \bar{v}p)$. From I, Section 4-13, we find $\langle \mathbf{e}_{u,v}, f\rangle' = f(u, v)$ for all $f \in \mathscr{F}_2$. Thus,

$$f(u, v) = \langle \mathbf{e}_{u,v}, f\rangle' = \sum_{k,n=0}^{\infty} \langle h_k^{(n)}, f\rangle'\, h_k^{(n)}(u, v),$$

proving that the $h_k^{(n)}$ form an orthonormal basis for $\mathscr{F}_2$.

Theorem 4.8 If $l > l' > 0$ then

$$(\lambda, l) \otimes (\lambda', -l') \cong \sum_{n=0}^{\infty} \oplus (\lambda + \lambda' + n + 1, l - l').$$

The basis vectors $h_k^{(n)}$ are easily computed from the generating function $G(\cdot)$:

$$h_k^{(n)}(t, p) = \exp[-(l'/l)^{1/2} tp] \sum_a (k!\, n!)^{1/2} (l'/l)^{a/2} (1 - l'/l)^{[(k+n+1)/2 - a]}$$
$$\cdot \frac{t^{k-a} p^{n-a}}{a!\,(k-a)!\,(n-a)!}, \qquad k, n = 0, 1, 2, \ldots .$$

The summation is carried out over all values of a such that the summand is defined. Using this expression we can calculate the Clebsch–Gordan coefficients

$$\langle j_{q,s}, h_k^{(n)} \rangle' \equiv G[l, q; l', s \mid n, k],$$

where $j_{q,s}(t, p) = t^q p^s / (q!\, s!)^{1/2}$, $q, s \geqslant 0$. The result is

$$G[l, q; l', s \mid n, k]$$
$$= \begin{cases} 0 & \text{if } \; s + k \neq q + n \\ \left[\dfrac{q!\,(q+n-k)!\,k!\,n!\,(l'/l)^{q-k}}{(1 - l'/l)^{-n-k-1}}\right]^{1/2} \displaystyle\sum_a \frac{(-1)^{q-k} (-l'/l)^a (1 - l'/l)^{-a}}{(q-k+a)!\,a!\,(k-a)!\,(n-a)!} & \\ \qquad \text{if } \; s + k = q + n. & \end{cases} \tag{4.149}$$

From (4.144) one readily deduces the generating function

$$\exp[(ut + vp)(1 - l'/l)^{1/2} + (uv - tp)(l'/l)^{1/2}](1 - l'/l)^{1/2}$$
$$= \sum_{k,n,q,s=0}^{\infty} \frac{u^k v^n t^q p^s}{(k!\, n!\, q!\, s!)^{1/2}} G[l, q; l', s \mid n, k]. \tag{4.150}$$

This function is invariant under the simultaneous transpositions $t \leftrightarrow p$ and $u \leftrightarrow v$ which implies

$$G[l, q; l', s \mid n, k] = G[l, s; l', q \mid k, n]$$

for all $q, s, n, k \geqslant 0$. Another symmetry is $u \leftrightarrow t$, $v \leftrightarrow p$, which implies

$$G[l, q; l', s \mid n, k] = (-1)^{n+s}\, G[l, k; l', n \mid s, q].$$

Comparing the generating functions (4.137) and (4.150) we find

$$G[l, q; l', s \mid n, k] = (1 - l'/l)^{1/2}\, K[l - l', q; l', n \mid s, k].$$

As a simple consequence of the definition of Clebsch–Gordan coefficients one obtains the formula

$$U^{\lambda,l}_{m,q}(g)\, V^{\lambda',-l'}_{n,s}(g) = \sum_{k=0}^{\infty} \mathrm{G}[l, q; l', s \mid k, q + k - s]$$
$$\cdot\, \mathrm{G}[l, m; l', n \mid k, m + k - n]\, U^{\lambda+\lambda'+k+1,\, l-l'}_{m-n+k,\, q-s+k}(g)$$

expressing the product of two matrix elements as a sum of matrix elements. In terms of the associated Laguerre polynomials these relations become

$$\exp(-l'r^2)(-lr^2)^{s-n} L^{(q-m)}_{m}(lr^2)\, L^{(s-n)}_{n}(l'r^2)$$
$$= \frac{(1 - l'/l)^{n+q+1}}{m!\, n!} \sum_{k=0}^{\infty} k!\, (k + m - n)!\, (l'/l)^{k-s}$$
$$\cdot \frac{F(-m, -n; k - n + 1; -l'/(l - l'))}{\Gamma(k - n + 1)} \frac{F(-q, -s; k - s + 1; -l'/(l - l'))}{\Gamma(k - s + 1)}$$
$$\cdot\, L^{(q-s-m+n)}_{m-n+k}[(l - l')r^2]. \tag{4.151}$$

CASE 2. $l' > l$. In this case the functions $h^{(n)}_k$ are no longer elements of $\mathscr{F}_2$ (since $l'/l > 1$) and we must modify our procedure. In terms of our Lie algebraic method we would look for eigenvectors f_o of J^3 such that $\mathrm{J}^+ f_o = 0$ and use these vectors to generate a new orthonormal basis for $\mathscr{F}_2$. However, we omit this and merely verify the validity of the results.

The functions $f^{(n)}_k(t, p) \in \mathscr{F}_2$, $n, k = 0, 1, 2, \ldots$, are defined by means of the generating functions $\mathrm{F}(u, v, t, p)$:

$$\mathrm{F}(u, v, t, p) = \exp[(up + vt)(1 - l/l')^{1/2} + (uv - tp)(l/l')^{1/2}](1 - l/l')^{1/2}$$
$$= \sum_{k,n=0}^{\infty} \frac{u^k v^n}{(k!\, n!)^{1/2}} f^{(n)}_k(t, p), \qquad u, v, t, p \in \mathcal{C}.$$

As in case 1 we have

$$(1 - l/l')\langle \mathrm{F}(u, v, .,.), [\mathbf{W}(g)]\, \mathrm{F}(w, z, .,.)\rangle'$$
$$= \sum_{k,n,q,m=0}^{\infty} \frac{u^k v^n w^q z^m}{(k!\, n!\, q!\, m!)^{1/2}} \langle f^{(n)}_k, \mathbf{W}(g) f^{(m)}_q\rangle'$$
$$= \exp[-\tfrac{1}{2}(l' - l)r^2 + i\alpha(\lambda + \lambda' + 1) - i\,\delta(l' - l)$$
$$- wr(l' - l)^{1/2} e^{i\alpha+i\theta} + u(l' - l)^{1/2} re^{-i\theta} + vze^{-i\alpha} + uwe^{i\alpha}],$$

where $g \in S_4$ has coordinates r, θ, α, δ.

Lemma 4.9 $\langle f_k^{(n)}, \mathbf{W}(g) f_q^{(m)}\rangle' = 0$ unless $m = n$.

Lemma 4.10 For fixed n,

$$\exp[uwe^{i\alpha} + u(l'-l)^{1/2} re^{-i\theta} - \tfrac{1}{2}(l'-l)r^2 + i\alpha(\lambda+\lambda'+1-n) - i\,\delta(l'-l) - wr(l'-l)^{1/2} e^{i\alpha+i\theta}] = \sum_{k,q=0}^{\infty} \frac{u^k w^q}{(k!\,q!)^{1/2}} \langle f_k^{(n)}, \mathbf{W}(g) f_q^{(n)}\rangle'.$$

Comparing this expression with the generating function (4.129), we see that the vectors $\{f_k^{(n)}\}$, $k = 0, 1, 2, \ldots$, form an orthonormal basis for the representation $(\lambda+\lambda'-n, -(l'-l))$ of S_4. Finally, in analogy with case 1 we can show

$$\mathbf{e}_{u,q}(t,p) = \exp(t\bar{u} + p\bar{q}) = \sum_{k,n=0}^{\infty} f_k^{(n)}(t,p)\overline{f_k^{(n)}(u,q)},$$

where the convergence is in the mean. This is the completeness relation for the set $\{f_k^{(n)}\}$ and proves:

Theorem 4.9 If $l' > l > 0$ then

$$(\lambda, l) \otimes (\lambda', -l') \cong \sum_{n=0}^{\infty} \oplus (\lambda+\lambda'-n, -(l'-l)).$$

The Clebsch–Gordan coefficients for this decomposition are given by

$$\langle j_{q,s}, f_k^{(n)}\rangle' = \mathrm{G}[l, s; l', q \mid n, k]$$

where $\mathrm{G}[\cdot]$ is defined by (4.149). Note the interchange of s and q in the above expression.

4-19 The Representations $(\lambda, l) \otimes (\lambda', -l)$

According to (4.143) the representation $(\lambda, l) \otimes (\lambda', -l)$ acts on $\mathscr{F}_2$ as follows: Let $g \in S_4$ with coordinates $r, \theta, \alpha, \delta$. Then,

$$[\mathbf{W}(g)f](t,p) = \exp[l^{1/2}(pre^{-i\theta} - tre^{i\theta}) - lr^2 + i\alpha(\lambda+\lambda'+1)] \cdot f(te^{-i\alpha} + rl^{1/2}e^{-i\theta-i\alpha}, pe^{i\alpha} - rl^{1/2}e^{i\theta+i\alpha}) \tag{4.152}$$

for every $f \in \mathscr{F}_2$. This representation of S_4 is independent of the parameter δ; hence, every element of the normal subgroup D of S_4,

$$D = \{g \in S_4 : g = \exp \delta\mathscr{Q},\ \delta \text{ real}\},$$

is mapped into the identity operator on $\mathscr{F}_2$. Thus, this representation can be considered as a representation of the (global) factor group $S_4/D \cong E_3$. In fact, if $g_1, g_2 \in S_4$ have coordinates r_1, θ_1, α_1 and r_2, θ_2, α_2, respectively, then the element $g_1 g_2 \in S_4$ has coordinates r, θ, α given by

$$re^{i\theta} = r_1 e^{i\theta_1} + r_2 e^{i(\theta_2 - \alpha_1)}, \qquad \alpha = \alpha_1 + \alpha_2$$

(we ignore the parameter δ). For $\varphi = -\alpha$ these relations become

$$re^{i\theta} = r_1 e^{i\theta_1} + r_2 e^{i(\theta_2 + \varphi_1)}, \qquad \varphi = \varphi_1 + \varphi_2,$$

which are identical with expression (3.43) giving the coordinate transformation for the product of two group elements in E_3.

From (4.152) it is easy to show that the infinitesimal operators J_k, $k = 1, 2, 3$, and Q of the representation $(\lambda, l) \otimes (\lambda', -l)$ are given by

$$\mathrm{J}_1 = -\frac{il^{1/2}}{2}\left(t + \frac{\partial}{\partial t} + p + \frac{\partial}{\partial p}\right), \qquad \mathrm{J}_2 = \frac{l^{1/2}}{2}\left(-t + \frac{\partial}{\partial t} + p - \frac{\partial}{\partial p}\right),$$

$$\mathrm{J}_3 = i\left(\lambda + \lambda' + 1 - t\frac{\partial}{\partial t} + p\frac{\partial}{\partial p}\right), \qquad \mathrm{Q} = 0$$

with commutation relations

$$[\mathrm{J}_1, \mathrm{J}_2] = 0, \qquad [\mathrm{J}_3, \mathrm{J}_1] = \mathrm{J}_2, \qquad [\mathrm{J}_3, \mathrm{J}_2] = -\mathrm{J}_1.$$

Thus the infinitesimal operators generate a real Lie algebra isomorphic to $\mathscr{E}_3$.

According to the above remarks, $(\lambda, l) \otimes (\lambda', -l)$ can be considered as a unitary representation of E_3. From the general theory of representations of locally compact groups we know that this representation can be decomposed into a direct integral of irreducible representations of E_3 (Naimark [1], Chapter 8). In terms of special function theory this would yield an expression for the product of two associated Laguerre polynomials as an integral over Bessel functions. We will determine this decomposition explicitly.

Our realization of $(\lambda, l) \otimes (\lambda', -l)$ on the Hilbert space $\mathscr{F}_2$ is not a convenient one to use for this decomposition. However, there is another realization which is more convenient for this purpose. Bargmann has studied $\mathscr{F}$ and $\mathscr{F}_2$ in connection with the relationship between the usual representation of the canonical commutation relations in quantum mechanics and the Fock representation (Bargmann [3]). In the course of this study, he established a useful unitary mapping **A** of the Hilbert space $L_2(R)$ onto $\mathscr{F}$. The $L_2(R)$ is the space of all complex functions on the real line, square integrable with respect to Lebesgue measure, and with inner product

$$\langle\psi_1, \psi_2\rangle = \int_{-\infty}^{\infty} \overline{\psi_1(q)}\, \psi_2(q)\, dq, \qquad \psi_1, \psi_2 \in L_2(R).$$

The unitary map $\mathbf{A}$ of $L_2(R)$ onto $\mathscr{F}$ is defined by $f = \mathbf{A}\psi$ where

$$f(t) = (\mathbf{A}\psi)(t) = \int_{-\infty}^{\infty} A(t, q)\, \psi(q)\, dq, \qquad f \in \mathscr{F}, \quad \psi \in L_2(R), \tag{4.153}$$

$$A(t, q) = \pi^{-1/4} \exp[-(t^2 + q^2)/2 + \sqrt{2}tq]. \tag{4.154}$$

Since $A(t, .) \in L_2(R)$ for each $t \in \mathcal{C}$, the above integral is always defined. To understand the origin of the operator $\mathbf{A}$ and to verify its unitarity, we compare (4.154) with the generating function (4.77) for the Hermite polynomials. Clearly,

$$A(t, q) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^{1/2}} \varphi_n(q) = \sum_{n=0}^{\infty} j_n(t)\, \varphi_n(q) \tag{4.155}$$

where

$$\varphi_n(q) = \frac{\exp(-q^2/2)\, H_n(q)}{(\sqrt{\pi}\, n!\, 2^n)^{1/2}}.$$

It is elementary to verify the integral formula

$$\int_{-\infty}^{\infty} \overline{A(z, q)}\, A(t, q)\, dq = \exp(t\bar{z}) = \mathbf{e}_z(t).$$

On the other hand, using property III, Section 4-13, we have

$$\int_{-\infty}^{\infty} \overline{A(z, q)}\, A(t, q)\, dq = \sum_{n,k=0}^{\infty} \frac{\bar{z}^n t^k}{(n!\, k!)^{1/2}} \int_{-\infty}^{\infty} \varphi_n(q)\, \varphi_k(q)\, dq.$$

Thus,

$$\int_{-\infty}^{\infty} \varphi_n(q)\, \varphi_k(q)\, dq = \delta_{n,k}, \qquad n, k \geqslant 0. \tag{4.156}$$

These are the orthogonality relations for the Hermite polynomials, and they show that the functions $\{\varphi_n\}$, $n = 0, 1, 2, \ldots$, form an orthonormal set in $L_2(R)$. In fact, it is well known that the $\{\varphi_n\}$ form an orthonormal basis for $L_2(R)$. We shall not demonstrate this here since group theoretic methods do not appear to simplify the proof. However, see Courant and Hilbert [1], and Bargmann [3].

If $\psi \in L_2(R)$, then ψ can be written uniquely in the form, $\psi = \sum_{n=0}^{\infty} c_n \varphi_n$, $c_n \in \mathcal{C}$, where convergence is in the mean. From (4.153) and (4.155) we have

$$f = \mathbf{A}\psi = \sum_{n=0}^{\infty} c_n j_n \in \mathcal{F}$$

where the $\{j_n\}$ form an orthonormal basis for $\mathcal{F}$. Thus,

$$\| \mathbf{A}\psi \| = \| \psi \| = \left(\sum_{n=0}^{\infty} | c_n |^2 \right)^{1/2}$$

where the first norm is in $\mathcal{F}$ while the second is in $L_2(R)$. Therefore, the linear mapping $\mathbf{A}$ is one-to-one, onto, and norm preserving, i.e., $\mathbf{A}$ is a unitary mapping from $L_2(R)$ onto $\mathcal{F}$.

By construction we have

$$j_n = \mathbf{A}\varphi_n, \qquad n = 0, 1, 2, \ldots,$$

so the inverse map $\mathbf{A}^{-1}$ of $\mathcal{F}$ onto $L_2(R)$ must satisfy the relations $\varphi_n = \mathbf{A}^{-1} j_n$. This suggests that the unitary mapping $\mathbf{A}^{-1}$ is given by the integral

$$(\mathbf{A}^{-1}f)(q) = \int_{\mathcal{C}} \overline{A(t, q)} f(t)\, d\xi(t) \tag{4.157}$$

for all $f \in \mathcal{F}$. However, this integral is not properly defined because $A(., q)$ is not an element of $\mathcal{F}$ for $q \in R$. (Recall $d\xi(t) = \pi^{-1} \exp(-t\bar{t})\, dx\, dy$ where $t = x + iy$.) To get around this difficulty note that

$$| A(t, q)| \leqslant C \exp(| t |^2/2)$$

for all $t \in \mathcal{C}$ where C is a positive constant depending on q. If $0 < \mu < 1$ we have $| A(\mu t, q)| \leqslant C \exp[(\mu^2 | t |^2)/2]$. The $A(\mu t, q)$ is square integrable with respect to the measure $d\xi$ for all $q \in R$; hence, $A(\mu t, q)$ is in $\mathcal{F}$. Indeed direct computation (using property III, Section 4-13) yields

$$\begin{aligned} &\int \overline{A(\mu t, q)}\, A(\mu t, q')\, d\xi(t) \\ &\qquad = [\pi(1 - \mu^4)]^{-1/2} \exp \left[\frac{2\mu^2 q q' - (1 + \mu^4)(q^2 + q'^2)}{2(1 - \mu^4)} \right] \end{aligned}$$

where $0 < \mu < 1$ (Bargmann [3]). However,

$$\begin{aligned} \int \overline{A(\mu t, q)}\, A(\mu t, q')\, d\xi(t) &= \sum_{m,k=0}^{\infty} \mu^{n+k} \varphi_n(q)\, \varphi_k(q') \int \overline{j_n(t)}\, j_k(t)\, d\xi(t) \\ &= \sum_{n=0}^{\infty} \mu^{2n} \varphi_n(q)\, \varphi_n(q'). \end{aligned}$$

Equating these expressions we obtain

$$[\pi(1-\rho^2)]^{-1/2}\exp\left[\frac{2\rho qq' - (1+\rho^2)(q^2+q'^2)}{2(1-\rho^2)}\right] = \sum_{n=0}^{\infty} \rho^n \varphi_n(q)\,\varphi_n(q'),$$
$$0<\rho<1, \quad (4.158)$$

which is equivalent to Mehler's formula. (This is a special case of a generating function for Hermite functions which will be derived in Chapter 9.)

Returning to the problem of justifying (4.157) we see that if $f = \sum_{n=0}^{\infty} c_n j_n \in \mathscr{F}$ and $0<\mu<1$ then $\psi_\mu \in L_2(R)$ where

$$\psi_\mu(q) = \int \overline{A(\mu t, q)} f(t)\, d\xi(t) = \sum_{n=0}^{\infty} \mu^n c_n \varphi_n(q).$$

If we define $\psi \in L_2(R)$ by $\psi = \mathbf{A}^{-1} f = \sum c_n \varphi_n$ we have

$$\| \psi - \psi_\mu \|^2 = \sum_{n=0}^{\infty} | c_n |^2 (1-\mu^n)^2 \leqslant \| \psi \|^2.$$

A simple limit argument shows

$$\lim_{\mu\to 1} \| \psi - \psi_\mu \|^2 = 0$$

so $\psi_\mu \to \psi$ in the mean as $\mu \to 1$. Therefore, the unitary map $\mathbf{A}^{-1}$ of $\mathscr{F}$ onto $L_2(R)$ can be defined by the expression

$$(\mathbf{A}^{-1} f)(q) = \lim_{\substack{\mu\to 1\\ \mu<1}} \int \overline{A(\mu t, q)} f(t)\, d\xi(t) \tag{4.159}$$

for all $f \in \mathscr{F}$. Clearly, $f = \mathbf{A}\psi$ if and only if $\psi = \mathbf{A}^{-1} f$ for $\psi \in L_2(R)$, $f \in \mathscr{F}$.

Using the unitary maps $\mathbf{A}$ and $\mathbf{A}^{-1}$ we can define operators $\mathbf{T}^{\lambda,l}(g)$, $\mathbf{T}^{\lambda,-l}(g)$, on $L_2(R)$ such that

$$\mathbf{T}^{\lambda,l}(g) = \mathbf{A}^{-1}\mathbf{U}^{\lambda,l}(g)\mathbf{A}, \qquad \mathbf{T}^{\lambda,-l}(g) = \mathbf{A}^{-1}\mathbf{V}^{\lambda,-l}(g)\mathbf{A}.$$

The operators $\mathbf{T}^{\lambda,\pm l}(g)$ for all $g \in S_4$ define unitary representations of S_4 on $L_2(R)$ unitarily equivalent to the representations $(\lambda, \pm l)$ of S_4 on $\mathscr{F}$.

These operators can be computed explicitly. If $[z]$ is the element of S_4 with parameters $(r, \theta, \alpha, \delta) = (r, \theta, 0, 0)$, where $z = x + iy = 2l^{1/2} r e^{i\theta}$, then

$$[\mathbf{T}^{\lambda,l}[z]\psi](q) = [\mathbf{A}^{-1}\mathbf{U}^{\lambda,l}[z]\mathbf{A}\psi](q). \tag{4.160}$$

Following Bargmann [3], we set

$$f = \mathbf{A}\psi, \qquad f_1 = \mathbf{U}^{\lambda,l}[z]f, \qquad \mathbf{T}^{\lambda,l}[z]\psi = \psi_1 .$$

Then,

$$f_1(t) = \int_{-\infty}^{\infty} \exp[-tz/2 - z\bar{z}/8]\, A(t + \bar{z}/2, q)\, \psi(q)\, dq,$$

and from the identity

$$\exp[-tz/2 - z\bar{z}/8]\, A(t + \bar{z}/2, q) = \exp[\tfrac{1}{2}iy(-\sqrt{2}q + x/2)]\, A(t, q - x/\sqrt{2})$$

we obtain

$$f_1(t) = \int_{-\infty}^{\infty} A(t, q) \exp[-\tfrac{1}{2}iy(\sqrt{2}q + x/2)]\, \psi(q + x/\sqrt{2})\, dq.$$

Thus,

$$[\mathbf{T}^{\lambda,l}[z]\psi](q) = \psi_1(q) = \exp[-\tfrac{1}{2}iy(\sqrt{2}q + x/2)]\, \psi(q + x/\sqrt{2}) \qquad (4.161)$$

for all $\psi \in L_2(R)$. Similarly, if $\langle\delta\rangle$ is the element of S_4 with coordinates $(r, \theta, \alpha, \delta) = (0, 0, 0, \delta)$, we easily obtain

$$[\mathbf{T}^{\lambda,l}\langle\delta\rangle\psi](q) = e^{il\delta}\psi(q), \qquad \psi \in L_2(R). \qquad (4.162)$$

Let (α) be the element of S_4 with coordinates $(0, 0, \alpha, 0)$. Then,

$$[\mathbf{T}^{\lambda,l}(\alpha)\psi](q) = \lim_{\mu\to 1} e^{i\alpha\lambda} \int \overline{A(\mu t, q)}\, A(e^{-i\alpha}t, q')\, \psi(q')\, d\xi(t)\, dq'.$$

If $e^{-i\alpha} = 1$ then $\mathbf{T}^{\lambda,l}(\alpha) = \mathbf{I}$, while if $e^{-i\alpha} = -1$ we have $[\mathbf{T}^{\lambda,l}(\alpha)\psi]\,(q) = \psi(-q)$. However, if $e^{-i\alpha} \neq \pm 1$ it follows from Fubini's theorem that

$$[\mathbf{T}^{\lambda,l}(\alpha)\psi](q) = e^{i\alpha\lambda} \lim_{n\to\infty} \int_{-n}^{n} \sigma(e^{-i\alpha}, q', q)\, \psi(q')\, dq' \qquad (4.163)$$

where

$$\sigma(e^{-i\alpha}, q', q) = \int \overline{A(t, q)}\, A(e^{-i\alpha}t, q')\, d\xi(t).$$

An explicit computation yields

$$\sigma(e^{-i\alpha}, q', q) = \frac{e^{-i\epsilon(\pi/4 - \beta/2)}}{(2\pi \mid \sin\alpha \mid)^{1/2}} \exp\left[i \frac{(q^2 + q'^2)}{2} \cot\alpha - \frac{iqq'}{\sin\alpha}\right]. \qquad (4.164)$$

Here, $\alpha = 2k\pi + \epsilon\beta$, k an integer, $\epsilon = \pm 1$, $0 < \beta < \pi$. (See Bargmann [3] for details. Equation (4.164) is presented for completeness. It will not be needed in the proofs of the various identities for special functions to follow.) When $\alpha = \pi/2$, $\mathbf{T}^{\lambda,l}(\pi/2)\psi$ is the Fourier transform of ψ.

In a similar manner we can compute the operators $\mathbf{T}^{\lambda,-l}(g)$ on $L_2(R)$, induced by the operators $\mathbf{V}^{\lambda,-l}(g)$ on $\mathscr{F}$. The results are

$$\begin{aligned}
[\mathbf{T}^{\lambda,-l}[z]\psi](q) &= \exp[-\tfrac{1}{2}iy(\sqrt{2}q - x/2)]\,\psi(q - x/\sqrt{2}),\\
[\mathbf{T}^{\lambda,-l}\langle\delta\rangle\psi](q) &= e^{-il\delta}\psi(q), \qquad\qquad (4.165)\\
[\mathbf{T}^{\lambda,-l}(\alpha)\psi](q) &= e^{i\alpha(\lambda+1)} \lim_{n\to\infty} \int_{-n}^{n} \sigma(e^{+i\alpha}, q', q)\,\psi(q')\,dq'
\end{aligned}$$

for all $\psi \in L_2(R)$, where the function σ is given by (4.164).

From the relation $U^{\lambda,l}_{n,m}(g) = \langle j_n , \mathbf{U}^{\lambda,l}(g) j_m\rangle = \langle \varphi_n , \mathbf{T}^{\lambda,l}(g)\varphi_m\rangle$ and Eq. (4.161), we obtain the identities

$$\begin{aligned}
&\exp(x^2/8)(x/2)^{m-n} L_n^{(m-n)}(x^2/4)\\
&\quad = [\pi^{1/2}2^{(n+m)/2}n!]^{-1} \int_{-\infty}^{\infty} \exp[-q^2 - qx/\sqrt{2}]\, H_n(q)\, H_m(q + x/\sqrt{2})\, dq,\\
&\qquad\qquad (4.166)\\
&\exp(-y^2/8)(y/2)^{m-n} L_n^{(m-n)}(y^2/4)\\
&\quad = [\pi^{1/2}2^{(n+m)/2}n!]^{-1} \int_{-\infty}^{\infty} \exp[-q^2 - iqy/\sqrt{2}]\, H_n(q)\, H_m(q)\, dq
\end{aligned}$$

where m, n are nonnegative integers and $x, y \geqslant 0$.

Equation (4.152) defines the representation $(\lambda, l) \otimes (\lambda', -l)$ acting on $\mathscr{F}_2 \cong \mathscr{F} \otimes \mathscr{F}$. We shall find it more convenient to consider this representation of S_4 as acting on the Hilbert space $L_2(R^2) \cong L_2(R) \otimes L_2(R)$. The appropriate transformation is easily carried out through the use of Eqs. (4.161)–(4.165).

The elements of $L_2(R^2)$ are complex functions square integrable with respect to Lebesgue measure in the real plane R^2. The $L_2(R^2)$ is a Hilbert space with inner product

$$\langle \Psi, \Phi\rangle = \int_{R^2} \overline{\Psi(q_1, q_2)}\, \Phi(q_1, q_2)\, dq_1\, dq_2, \qquad \Psi, \Phi \in L_2(R^2).$$

The operators

$$\mathbf{T}^{\lambda,l}(g) \otimes \mathbf{T}^{\lambda',l}(g) = \mathbf{A}^{-1} \otimes \mathbf{A}^{-1}(\mathbf{U}^{\lambda,l}(g) \otimes \mathbf{V}^{\lambda',-l}(g))\,\mathbf{A} \otimes \mathbf{A} = \mathbf{M}(g)$$

are defined on $L_2(R^2)$ as follows:

$$\begin{aligned}
[\mathbf{M}[z]\Psi](q_1, q_2) &= \exp\left[-\frac{iy}{\sqrt{2}}(q_1+q_2)\right]\Psi\left(q_1+\frac{x}{\sqrt{2}}, q_2-\frac{x}{\sqrt{2}}\right), \\
&\qquad\qquad z = x+iy, \\
[\mathbf{M}\langle\delta\rangle\Psi](q_1, q_2) &= \Psi(q_1, q_2), \\
[\mathbf{M}(\alpha)\Psi](q_1, q_2) &= \lim_{n\to\infty}\int_{-n}^{n}\int_{-n}^{n} e^{i\alpha(\lambda+\lambda'+1)} \\
&\quad\cdot\exp\left[i\left(\frac{q_1^2+q_1'^2-q_2^2-q_2'^2}{2}\right)\cot\alpha\right. \\
&\quad\left.+i\left(\frac{q_2q_2'-q_1q_1'}{\sin\alpha}\right)\right]\frac{\Psi(q_1', q_2')}{2\pi\,|\sin\alpha|}\,dq_1'\,dq_2'
\end{aligned} \tag{4.167}$$

for all $\Psi\in L_2(R^2)$. For notational brevity we have suppressed the dependence of $\mathbf{M}$ on λ, λ', l. The operators $\mathbf{M}(g)$ generate a unitary representation of S_4 such that $\mathbf{M}\langle\delta\rangle = \mathbf{I}$ for all $\langle\delta\rangle \in S_4$. As remarked above this representation can be considered as a unitary representation of E_3. As such it can be decomposed into a direct integral of irreducible unitary representations (ρ) of E_3. We will carry out this decomposition.

Given $\Psi \in L_2(R^2)$ we define the function $\Phi \in L_2(R^2)$ by

$$\Phi(s_1, s_2) \equiv \Psi\left(\frac{-s_1+s_2}{\sqrt{2}}, \frac{-s_1-s_2}{\sqrt{2}}\right), \qquad s_1, s_2 \in R. \tag{4.168}$$

Introducing the change of coordinates $s_1 = -(q_1+q_2)/\sqrt{2}$, $s_2 = (q_1-q_2)/\sqrt{2}$, we obtain

$$\Phi\left(\frac{-q_1-q_2}{\sqrt{2}}, \frac{q_1-q_2}{\sqrt{2}}\right) \equiv \Psi(q_1, q_2).$$

Note that $dq_1\,dq_2 = ds_1\,ds_2$. The operator $\mathbf{M}[z]$ acting on Ψ induces an operator, which we will also call $\mathbf{M}[z]$, acting on Φ,

$$[\mathbf{M}[z]\Phi](s_1, s_2) = e^{is_1y}\Phi(s_1, s_2+x). \tag{4.169}$$

Let the function $\tilde{\Phi}$ be defined by

$$\tilde{\Phi}(s_1, u_2) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-iu_2s_2}\Phi(s_1, s_2)\,ds_2, \qquad u_2 \in R. \tag{4.170}$$

(To be more precise we should write $\lim_{n\to\infty}\int_{-n}^{n}$ in place of the integral in (4.170). However, this will be assumed to be understood in the following paragraphs.)

It is easy to show that $\mathbf{M}[z]$ acting on Φ induces an operator, which will also be called $\mathbf{M}[z]$, acting on $\tilde{\Phi}$ and given by

$$[\mathbf{M}[z]\tilde{\Phi}](s_1, u_2) = e^{i(s_1 y + u_2 x)}\tilde{\Phi}(s_1, u_2). \tag{4.171}$$

We now pass to polar coordinates and set

$$x = \tau \cos\theta, \qquad y = \tau \sin\theta, \qquad u_2 = \rho\cos\gamma, \qquad s_1 = \rho \sin\gamma,$$

where from (4.160) we have $\tau = 2l^{1/2}r$. Note that $ds_1\, du_2 = \rho\, d\rho\, d\gamma$. Define $\hat{\Phi}$ by

$$\hat{\Phi}(\rho, \gamma) \equiv \tilde{\Phi}(\rho \sin\gamma, \rho\cos\gamma).$$

The action of $\mathbf{M}[z]$ induced on $\hat{\Phi}$ then becomes

$$[\mathbf{M}[z]\hat{\Phi}](\rho, \gamma) = e^{i\tau\rho\cos(\gamma-\theta)}\hat{\Phi}(\rho, \gamma) \tag{4.172}$$

where $\hat{\Phi}$ is square integrable with respect to the measure $\rho\, d\rho\, d\gamma$. A comparison of Eqs. (4.172) and (3.56) gives the motivation for the above manipulations.

We collect our results and use them to compute the matrix element $\langle \Psi_1, \mathbf{M}[z]\Psi_2\rangle$, $\Psi_1, \Psi_2 \in L_2(R^2)$. The result is (by use of the Plancherel theorem for the Fourier transform)

$$\langle \Psi_1, \mathbf{M}[z]\Psi_2\rangle = \int_0^\infty \int_0^{2\pi} e^{i\rho\tau\cos(\gamma-\theta)}\overline{\hat{\Phi}_1(\rho, \gamma)}\, \hat{\Phi}_2(\rho, \gamma)\rho\, d\rho\, d\gamma, \tag{4.173}$$

where the $\hat{\Phi}_j$, $j = 1, 2$, are defined by

$$\hat{\Phi}_j(\rho, \gamma) = \tilde{\Phi}_j(s_1, u_2) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-iu_2 s_2}\Psi_j\left(\frac{-s_1 + s_2}{\sqrt{2}}, \frac{-s_1 - s_2}{\sqrt{2}}\right) ds_2\,. \tag{4.174}$$

Using the Fourier integral theorem, we can invert (4.174):

$$\Psi_j(q_1, q_2) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \exp[iu_2(q_1 - q_2)/\sqrt{2}]\, \tilde{\Phi}_j\left(\frac{q_1 - q_2}{2}, u_2\right) du_2\,. \tag{4.175}$$

Finally, a straightforward computation gives

$$\langle \Psi_1, \mathbf{M}(\alpha)\, \Psi_2\rangle = \exp[-i\alpha(\lambda + \lambda' + 1)] \cdot \int_0^\infty \int_0^{2\pi} \hat{\Phi}_1(\rho, \gamma)\, \hat{\Phi}_2(\rho, \gamma - \alpha)\rho\, d\rho\, d\gamma. \tag{4.176}$$

Comparing these results with the formulas for the irreducible representations (ρ) of E_3 derived in Section 3-8, we obtain

Theorem 4.10 $(\lambda, l) \otimes (\lambda', -l) \cong \oplus \int_0^\infty (\rho)\rho\, d\rho.$

Theorem 4.10 is merely a symbolic way of denoting the relationship between $(\lambda, l) \otimes (\lambda', -l)$ and (ρ) given by Eqs. (4.173) and (4.176). Explicitly, we can use our decomposition to relate matrix elements for the representations on the left- and right-hand sides of (4.173). The vectors $\Psi_{n,m}(q_1, q_2) = \varphi_n(q_1)\,\varphi_m(q_2)$, $0 \leqslant n, m < \infty$, form an orthonormal basis for $L_2(R^2)$. From (4.174) the transformed vectors $\hat{\Phi}_{n,m}(\rho, \gamma)$, $\rho e^{i\gamma} = u_2 + is_1$, are given by

$$\begin{aligned}\hat{\Phi}_{n,m}(\rho, \gamma) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-iu_2 s_2} \varphi_n\left(\frac{-s_1 + s_2}{\sqrt{2}}\right) \varphi_m\left(\frac{-s_1 - s_2}{\sqrt{2}}\right) ds_2 \\ &= i^{-n-m}\pi^{-1/2} \exp(-\rho^2/2)(m!/n!)^{1/2}(\rho e^{-i\gamma})^{n-m} L_m^{(n-m)}(\rho^2) \qquad (4.177)\end{aligned}$$

where the last equality follows from (4.161) and a change of variable. Equation (4.175) then yields the relation

$$\begin{aligned}H_n(q_1)\, H_m(q_2) = \exp\left[\frac{(q_1 - q_2)^2}{4}\right] m!\, (2)^{(m+n-1)/2}(i)^{-n-m}\pi^{-1/2} \\ \cdot \int_{-\infty}^{\infty} \exp\left[\left(iu_2 \frac{q_1 - q_2}{\sqrt{2}}\right) - \frac{u_2^2}{2}\right]\left[u_2 + i\left(\frac{q_1 + q_2}{\sqrt{2}}\right)\right]^{n-m} \\ \cdot L_m^{(n-m)}\left[u_2^2 + \left(\frac{q_1 + q_2}{\sqrt{2}}\right)^2\right] du_2\end{aligned}$$

between associated Laguerre polynomials and Hermite polynomials.

The matrix elements of $\mathbf{M}[z]$ acting on $L_2(R^2)$ are given by

$$\langle \Psi_{n,k}, \mathbf{M}[z]\, \Psi_{m,j} \rangle = U_{n,m}^{\lambda,l}([z])\, V_{k,j}^{\lambda',-l}([z]).$$

On the other hand, from (4.173), (4.177), and (3.57), we have

$$\begin{aligned}U_{n,m}^{\lambda,l}([z])\, V_{k,j}^{\lambda',-l}([z]) = (i)^{n+k-m-j} \int_0^{\infty}\int_0^{2\pi} e^{i\rho\tau\cos(\gamma-\theta)} e^{i(n-k+j-m)\gamma} \\ \cdot \exp(-\rho^2)\left(\frac{k!\,j!}{n!\,m!}\right)^{1/2} (\rho)^{n-k+m-j+1} L_k^{(n-k)} \\ \cdot (\rho^2)\, L_j^{(m-j)}(\rho^2)\, d\rho\, d\gamma,\end{aligned}$$

so

$$\begin{aligned}\exp(-\tau^2/4)(\tau/2)^{m+j-n-k} L_n^{(m-n)}(\tau^2/4)\, L_k^{(j-k)}(\tau^2/4) \\ = 2(j!/n!) \int_0^{\infty} \exp(-\rho^2)(\rho)^{n-k+m-j}\, J_{k+m-n-j}(\rho\tau) \\ \cdot L_k^{(n-k)}(\rho^2)\, L_j^{(m-j)}(\rho^2)\rho\, d\rho, \qquad (4.178)\end{aligned}$$

where m, j, n, k, are nonnegative integers. This equation is the desired identity between associated Laguerre polynomials and Bessel functions whose existence is suggested by Theorem 4.10.

As a special case of (4.178) set $k = j = 0$ to obtain

$$\exp(-\tau^2/4)(\tau/2)^{m-n} L_n^{(m-n)}(\tau^2/4) = (2/n!) \int_0^\infty \exp(-\rho^2)\, \rho^{n+m} J_{m-n}(\rho\tau)\rho\, d\rho.$$

Also, if $k = 0$, $n = j$, we have

$$\exp(-\tau^2/4)(\tau/2)^m L_n^{(m-n)}(\tau^2/4) = 2 \int_0^\infty \exp(-\rho^2)\, J_{m-2n}(\rho\tau)\rho^m L_n^{(m-n)}(\rho^2)\rho\, d\rho.$$

4-20 The Representations $(\rho) \otimes (\lambda, l)$

As we have seen, the representation (ρ), $\rho > 0$, of E_3 on the Hilbert space $\mathscr{H}$, constructed in Section 3-8, can be considered as a representation of S_4. Thus, it makes sense to define a representation $(\rho) \otimes (\lambda, l)$ of S_4 on the Hilbert space $\mathscr{H} \otimes \mathscr{F}$. We will decompose $(\rho) \otimes (\lambda, l)$ into a direct sum of irreducible representations.

The elements of $\mathscr{H} \otimes \mathscr{F}$ are formal series

$$f(\gamma, t) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} c_n^k e^{in\gamma} t^k/(k!)^{1/2}$$

such that

$$\sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} | c_n^k |^2 < \infty; \qquad t, c_n^k \in \mathscr{C}, \quad 0 \leqslant \gamma < 2\pi, \quad (\text{mod } 2\pi).$$

Clearly $f(\gamma, .) \in \mathscr{F}$ for almost every γ and $f(., t) \in \mathscr{H}$ for fixed t. In general $f(\gamma, t)$ does not converge pointwise. $\mathscr{H} \otimes \mathscr{F}$ is a Hilbert space with inner product

$$\langle f, h\rangle = \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathscr{C}} \overline{f(\gamma, t)}\, h(\gamma, t)\, d\gamma\, d\xi(t), \qquad f, h \in \mathscr{H} \otimes \mathscr{F}.$$

Thus, if $f(\gamma, t) = \sum c_n^k e^{in\gamma} t^k/(k!)^{1/2}$, $h(\gamma, t) = \sum b_n^k e^{in\gamma} t^k/(k!)^{1/2}$, we have

$$\langle f, h\rangle = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \overline{c_n^k}\, b_n^k.$$

The vectors $f_{n,k}(\gamma, t) = e^{in\gamma} t^k/(k!)^{1/2}$, n, k integers, $k \geqslant 0$, form an orthonormal basis for $\mathscr{H} \otimes \mathscr{F}$. (For a careful treatment of tensor products of Hilbert spaces see Murray and von Neumann [1].) Corresponding to every $g \in S_4$ with coordinates $(r, \theta, \alpha, \delta)$ we define the operator $\mathbf{T}^{(\rho)}(g) \otimes \mathbf{U}^{\lambda,l}(g) \equiv \mathbf{N}(g)$ on $\mathscr{H} \otimes \mathscr{F}$ as follows:

$$[\mathbf{N}(g)f](\gamma, t) = \exp[-tl^{1/2}re^{i\theta} - lr^2/2 + i\alpha\lambda + il\delta + i\rho r\cos(\theta - \gamma)]f(\gamma + \alpha, te^{-i\alpha} + l^{1/2}re^{-i(\theta+\alpha)}),$$
$$f \in \mathscr{H} \otimes \mathscr{F}, \quad (4.179)$$

where λ is an integer and $\rho, l > 0$. It is a straightforward computation to check that the operators $\mathbf{N}(g)$ yield a unitary representation $(\rho) \otimes (\lambda, l)$ of S_4. The infinitesimal operators J_k, $k = 1, 2, 3$, and Q corresponding to this representation are

$$\mathrm{J}_1 = \frac{il^{1/2}}{2}\left(-t - \frac{\partial}{\partial t}\right) + \frac{i\rho}{2}\sin\gamma, \qquad \mathrm{J}_2 = \frac{l^{1/2}}{2}\left(-t + \frac{\partial}{\partial t}\right) + \frac{i\rho}{2}\cos\gamma,$$

$$\mathrm{J}_3 = i\left(\lambda - t\frac{\partial}{\partial t}\right) + \frac{\partial}{\partial\gamma}, \qquad \mathrm{Q} = il.$$

These operators satisfy the commutation relations

$$[\mathrm{J}_1, \mathrm{J}_2] = \tfrac{1}{2}\mathrm{Q}, \qquad [\mathrm{J}_3, \mathrm{J}_1] = \mathrm{J}_2, \qquad [\mathrm{J}_3, \mathrm{J}_2] = -\mathrm{J}_1, \qquad [\mathrm{J}_k, \mathrm{Q}] = 0, \quad k = 1, 2, 3,$$

so they generate a Lie algebra isomorphic to $\mathscr{S}_4$. As in Section 4-5, we can use the operators

$$\mathrm{J}^+ = -\mathrm{J}_2 + i\mathrm{J}_1 = l^{1/2}t - i\frac{\rho}{2}e^{-i\gamma}, \qquad \mathrm{J}^- = +\mathrm{J}_2 + i\mathrm{J}_1 = l^{1/2}\frac{\partial}{\partial t} + \frac{i\rho}{2}e^{i\gamma},$$
$$\mathrm{J}^3 = i\mathrm{J}_3 = -\lambda + t\frac{\partial}{\partial t} + i\frac{\partial}{\partial\gamma}, \qquad \mathrm{E} = -i\mathrm{Q} = l$$

to decompose $\mathscr{H} \otimes \mathscr{F}$ into a direct sum of subspaces, each subspace irreducible under the action of $\mathbf{N}(g)$. Rather than repeat this analysis, however, we will merely present the results and verify that they yield the desired decomposition.

Let n be an integer, $u \in \mathscr{C}$, and $\mathrm{N}(n, u; \gamma, t)$ the function

$$\mathrm{N}(n, u; \gamma, t) = \exp[-\rho^2/8l + in\gamma + ut + u\rho e^{-i\gamma}/2l^{1/2} - t\rho e^{i\gamma}/2l^{1/2}]$$
$$= \sum_{k=0}^{\infty} \frac{u^k}{(k!)^{1/2}} h_n^{(k)}(\gamma, t). \quad (4.180)$$

$\mathrm{N}(\cdot)$ is a generating function for the vectors $h_n^{(k)} \in \mathscr{H} \otimes \mathscr{F}$ defined by (4.180). Clearly, $\mathrm{N}(\cdot)$ is an entire function of the variables u, t and is an element of $\mathscr{H} \otimes \mathscr{F}$ for all values of n and u.

We will show that the vectors $h_n^{(k)}$, $k = 0, 1, 2, \ldots$, $n = 0, \pm 1, \pm 2, \ldots$, form an orthonormal basis for $\mathscr{H} \otimes \mathscr{F}$ and will calculate the matrix elements of $\mathbf{N}(g)$ with respect to this basis. To do this we compute the inner product (in $\mathscr{H} \otimes \mathscr{F}$)

$$\langle \mathrm{N}(m, \bar{v}; \cdot, \cdot), [\mathbf{N}(g)]\, \mathrm{N}(n, u; \cdot, \cdot)\rangle = \sum_{q,k=0}^{\infty} \frac{v^q u^k}{(q!\, k!)^{1/2}} \langle h_m^{(q)}, \mathbf{N}(g)\, h_n^{(k)}\rangle \quad (4.181)$$

where the integration is taken over the variables γ, t. (The validity of the expansion on the right-hand side of (4.181) follows from property III, Section 4-13.) The integral on the left-hand side of this expression can be evaluated directly using the delta function property of the vector $\mathbf{e}_b$ and the fact that

$$(2\pi)^{-1} \int_0^{2\pi} e^{i(m-n)\gamma}\, d\gamma = \delta_{m,n}.$$

The result of the integration is

$$\begin{aligned} &\delta_{m,n} \exp[i\alpha(\lambda + n) + il\delta - lr^2/2 + ue^{-i\alpha}(s + l^{1/2} re^{-i\theta}) - sl^{1/2} re^{i\theta}] \\ &= \sum_{q,k=0}^{\infty} \frac{s^q u^k}{(q!\, k!)^{1/2}} \langle h_m^{(q)}, \mathbf{N}(g)\, h_n^{(k)}\rangle. \end{aligned} \quad (4.182)$$

The following are immediate consequences of (4.182):

Lemma 4.11 If $m \neq n$, then $\langle h_m^{(q)}, \mathbf{N}(g) h_n^{(k)}\rangle = 0$ for all $g \in S_4$.

Lemma 4.12 For fixed n the vectors $\{h_n^{(k)}\}$, $k = 0, 1, 2, \ldots$, form an orthonormal basis for the irreducible representation $(\lambda + n, l)$ of S_4.

PROOF If $m = n$, Eq. (4.182) is identical with the generating function for the matrix elements of the representation $(\lambda + n, l)$.

Equation (4.182) shows that the totality of vectors $\{h_n^{(k)}\}$, $k = 0, 1, 2, \ldots$, $n = 0, \pm 1, \pm 2, \ldots$, form an orthonormal set in $\mathscr{H} \otimes \mathscr{F}$. We will sketch the proof that they span $\mathscr{H} \otimes \mathscr{F}$.

Suppose the vectors $\{h_n^{(k)}\}$ do not span. Then, there exists $f \in \mathscr{H} \otimes \mathscr{F}$, $f \neq 0$, such that $\langle f, h_n^{(k)}\rangle = 0$, $k, \pm n = 0, 1, 2, \ldots$. From (4.180)

$$\begin{aligned} 0 &= \langle \mathrm{N}(n, \bar{u}; \cdot, \cdot), f\rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathcal{C}} f(\gamma, t) \exp[-\rho^2/8 - in\gamma + u\bar{t} + u\rho e^{i\gamma}/2l^{1/2} - \bar{t}\rho e^{-i\gamma}/2l^{1/2}]\, d\gamma\, d\xi(t) \end{aligned}$$

for all integers n and complex numbers u. Using the delta function property of the vectors $\mathbf{e}_b$ where $b = u - \rho e^{-i\gamma}/2l^{1/2}$, we obtain

$$0 \equiv \int_0^{2\pi} f\left(\gamma, u - \frac{\rho e^{-i\gamma}}{2l^{1/2}}\right) \exp\left[-in\gamma + \frac{u\rho e^{i\gamma}}{2l^{1/2}}\right] d\gamma. \quad (4.183)$$

This equation forces us to conclude that

$$f\left(\gamma, u - \frac{\rho e^{-i\gamma}}{2l^{1/2}}\right) \exp\left(\frac{u\rho e^{i\gamma}}{2l^{1/2}}\right) = 0$$

for all γ, u; hence, $f = 0$. This contradiction proves that the vectors $h_n^{(k)}$ form an orthonormal basis for $\mathscr{H} \otimes \mathscr{F}$.

Theorem 4.11 $(\rho) \otimes (\lambda, l) \cong \sum_{n=-\infty}^{\infty} \oplus (\lambda + n, l)$.

The basis vectors $h_n^{(k)}$ are easily computed from the generating function (4.180):

$$h_n^{(k)}(\gamma, t) = (k!)^{-1/2}(t + (\rho e^{-i\gamma})/2l^{1/2})^k \exp[-\rho^2/8l + in\gamma - (t\rho e^{i\gamma})/2l^{1/2}]$$

$$= \exp\left(-\frac{\rho^2}{8l}\right) \sum_{q=0}^{\infty} e^{i(n+q-k)\gamma} \frac{t^q}{(q!)^{1/2}}$$

$$\cdot \sum_a \frac{(k!\, q!)^{1/2}(\rho/2l^{1/2})^{q+2a-k}}{a!\,(k-a)!\,(q-k+a)!}(-1)^{q+a-k}, \tag{4.184}$$

where the last summation is taken over all integer values of a such that the summand is defined.

The Clebsch–Gordan coefficients for the decomposition given by Theorem 4.11 are obtained directly from (4.184). Recall that the vectors $f_{w,q}(\gamma, t) = e^{iw\gamma}t^q/(q!)^{1/2}$, $q, \pm w = 0, 1, 2, \ldots$, form an orthonormal basis for $\mathscr{H} \otimes \mathscr{F}$. The Clebsch–Gordan coefficients $\mathrm{E}(w, q; n, k; \rho^2/l)$ are given by

$$\mathrm{E}(w, q; n, k; \rho^2/l) = \langle f_{w,q}, h_n^{(k)} \rangle$$

$$= \begin{cases} 0 \quad \text{if} \quad w + k \neq n + q, \\ \exp\left(-\dfrac{\rho^2}{8l}\right)[(n+q-w)!\, q!]^{1/2} \displaystyle\sum_a \frac{(\rho^2/4l)^{(w-n+2a)/2}(-1)^{w-n+a}}{a!\,(q+n-w-a)!\,(w-n+a)!} \\ \qquad\qquad \text{if} \quad w + k = n + q. \end{cases} \tag{4.185}$$

Here q, k are nonnegative integers and n, w are integers. From (4.180) there follow the generating function

$$\mathrm{N}(n, u; \gamma, t) = \exp[-\rho^2/8l + in\gamma + ut + (u\rho e^{-i\gamma} - t\rho e^{i\gamma})/2l^{1/2}]$$

$$= \sum_{w=-\infty}^{\infty} \sum_{q,k=0}^{\infty} \frac{u^k t^q e^{iw\gamma}}{(k!\, q!)^{1/2}} \mathrm{E}(w, q; n, k; \rho^2/l) \tag{4.186}$$

and the symmetry relations

$$\begin{aligned}E(w, q; n, k; \rho^2/l) &= E(w + m - n, q; m, k; \rho^2/l) = E(w - n, q; 0, k; \rho^2/l),\\ E(w, q; 0, k; \rho^2/l) &= (-1)^{k+q}E(-w, k; 0, q; \rho^2/l).\end{aligned} \tag{4.187}$$

Comparison of (4.186) with the generating function (4.124) for associated Laguerre polynomials yields

$$(q!/k!)^{1/2} \exp\left(-\frac{\rho^2}{8l}\right)(\rho/2l^{1/2})^{k-q} L_q^{(k-q)}(\rho/2l^{1/2}) = E(q - k, q; 0, k; \rho^2/l). \tag{4.188}$$

In terms of matrix elements with respect to the two sets of basis vectors $\{f_{w,q}\}$ and $\{h_n^{(k)}\}$, the decomposition given by Theorem 4.11 becomes

$$\begin{aligned}U_{w,s}^{(\rho)}(g)\, U_{q,k}^{(\lambda,l)}(g) = \sum_{n=-\infty}^{\infty} &E(w, q; n, n + q - w; \rho^2/l)\\ &\cdot E(s, k; n, n + k - s; \rho^2/l)\, U_{n+q-w,n+k-s}^{(\lambda+n,l)}(g).\end{aligned} \tag{4.189}$$

Here, w and s are integers while q and k are nonnegative integers. The matrix elements $U_{w,s}^{(\rho)}(g)$ are expressed in terms of Bessel functions by Eq. (3.57). Making use of (4.188) we can reduce (4.189) to the identity

$$\begin{aligned}&\exp(\rho^2)(r)^{s-w} J_{w-s}(2\rho r)\, L_q^{(k-q)}(r^2)\\ &\quad = \sum_n \frac{i^{s-w}k!}{(n + k - s)!} L_q^{(n-w)}(\rho)\, L_k^{(n-s)}(\rho)\, L_{n+q-w}^{(k-q+w-s)}(r^2)(\rho)^{2n-w-s},\end{aligned} \tag{4.190}$$

where the sum is taken over all integers n such that $n \geqslant s - k$ and $n \geqslant w - q$. If $q = w = s = k = 0$ this formula simplifies to the well-known relation

$$\exp(\rho^2)\, J_0(2\rho r) = \sum_{n=0}^{\infty} \frac{(\rho^2)^n}{n!} L_n^{(0)}(r^2).$$

The decomposition of the representation $(\rho) \otimes (\lambda, -l)$ of S_4 is very similar to that presented above and leads to no new relations for special functions. We quote only

Theorem 4.12 $(\rho) \otimes (\lambda, -l) \cong \sum_{n=-\infty}^{\infty} \oplus (\lambda + n, -l)$.

4-21 A Contraction of $\mathscr{G}(0, 1)$

In Chapter 2 it was shown that the Lie algebra $\mathscr{G}(0, 0) \cong \mathscr{T}_3 \oplus (\mathscr{E})$ was a contraction of $\mathscr{G}(0, 1)$. Here, we shall use this fact to obtain relations between associated Laguerre polynomials and Bessel functions.

The infinitesimal operators J_k, $k = 1, 2, 3$, Q corresponding to the representation (λ, l) of S_4 on $\mathscr{F}$ are given by

$$J_1 = -\frac{il^{1/2}}{2}\left(t + \frac{d}{dt}\right), \qquad J_2 = \frac{l^{1/2}}{2}\left(-t + \frac{d}{dt}\right),$$

$$J_3 = i\left(\lambda - t\frac{d}{dt}\right), \qquad Q = il$$

and obey the commutation relations

$$[J_3, J_1] = J_2, \quad [J_3, J_2] = -J_1, \quad [J_1, J_2] = \tfrac{1}{2}Q, \quad [J_k, Q] = 0, \qquad k = 1, 2, 3. \tag{4.191}$$

We relabel the orthonormal basis vectors $j_k(t) = t^k/(k!)^{1/2}$ of $\mathscr{F}$ as follows: $f_n = j_{\lambda+n}$, $n = -\lambda, -\lambda+1, -\lambda+2, \ldots$. In terms of this basis the matrix elements of the operator J_3 are given by

$$(J_3^{\lambda,l})_{n,n'} = \langle f_n, J_3 f_{n'}\rangle = -in\,\delta_{n,n'} \tag{4.192}$$

where the inner product is taken in $\mathscr{F}$. Similarly

$$\begin{aligned}
(J_1^{\lambda,l})_{n,n'} &= \langle f_n, J_1 f_{n'}\rangle = -\tfrac{1}{2}i[l(\lambda+n+1)]^{1/2}\,\delta_{n+1,n'} \\
&\qquad - \tfrac{1}{2}i[l(\lambda+n)]^{1/2}\,\delta_{n-1,n'}, \\
(J_2^{\lambda,l})_{n,n'} &= \langle f_n, J_2 f'_{n'}\rangle = \tfrac{1}{2}[l(\lambda+n+1)]^{1/2}\,\delta_{n+1,n'} \\
&\qquad - \tfrac{1}{2}[l(\lambda+n)]^{1/2}\,\delta_{n-1,n'}, \\
(Q^{\lambda,l})_{n,n'} &= \langle f_n, Qf_{n'}\rangle = il\,\delta_{n,n'}, \\
&\qquad n, n' = -\lambda, -\lambda+1, -\lambda+2, \ldots.
\end{aligned} \tag{4.193}$$

Moreover it follows from Chapter 3, Eq. (3.64), that the operators K_k, $k = 1, 2, 3$, given by

$$K_1 = -i\rho\cos\alpha, \qquad K_2 = -i\rho\sin\alpha, \qquad K_3 = -\frac{\partial}{\partial\alpha}$$

are infinitesimal operators induced by the representation (ρ) of E_3 on the Hilbert space $\mathscr{H}$, and obey the commutation relations

$$[K_3, K_1] = K_2, \qquad [K_3, K_2] = -K_1, \qquad [K_1, K_2] = 0. \tag{4.194}$$

Clearly, K_1, K_2, K_3 generate a Lie algebra isomorphic to $\mathscr{E}_3$. In terms of the orthonormal basis $h_n(\alpha) = e^{in\alpha}$, $\pm n = 0, 1, 2, \ldots$, for $\mathscr{H}$ the matrix elements of these operators are

$$\begin{aligned}
(K_3^\rho)_{n,n'} &= \langle h_n , K_3 h_{n'} \rangle^* = - in\, \delta_{n,n'} , \\
(K_1^\rho)_{n,n'} &= \langle h_n , K_1 h_{n'} \rangle^* = - \tfrac{1}{2} i\rho(\delta_{n+1,n'} + \delta_{n-1,n'}), \qquad (4.195) \\
(K_2^\rho)_{n,n'} &= \langle h_n , K_2 h_{n'} \rangle^* = \tfrac{1}{2}\rho(\delta_{n+1,n'} - \delta_{n-1,n'}), \\
&\qquad n, n' = 0, \pm 1, \pm 2, \ldots .
\end{aligned}$$

Here, the inner product $\langle ., . \rangle^*$ corresponds to the Hilbert space $\mathscr{H}$.

Choosing a parameter $\epsilon > 0$ we define a new set of generators for the Lie algebra $\mathscr{S}_4$. The new operators are

$$G_1 = \epsilon J_1 , \qquad G_2 = \epsilon J_2 , \qquad G_3 = J_3 , \qquad G_0 = \epsilon Q. \qquad (4.196)$$

The structure constants for the commutation relations of the G operators are functions of ϵ. In fact,

$$[G_3 , G_1] = G_2 , \quad [G_3 , G_2] = -G_1 , \quad [G_1 , G_2] = \tfrac{1}{2}\epsilon G_0 , \quad [G_k , G_0] = 0, \quad k = 1, 2, 3.$$

As $\epsilon \to 0$ the structure constants approach limits which are the structure constants for a new 4-dimensional Lie algebra. In the limit we have

$$[G_3 , G_1] = G_2 , \quad [G_3 , G_2] = -G_1 , \quad [G_1 , G_2] = 0, \quad [G_k , G_0] = 0, \quad k = 1, 2, 3.$$

This Lie algebra is isomorphic to $\mathscr{E}_3 \oplus \{G_0\}$, where the commutation relations for a basis of $\mathscr{E}_3$ are given by (4.194) and $\{G_0\}$ is the 1-dimensional real Lie algebra generated by G_0. Thus, $\mathscr{E}_3 \oplus \{G_0\}$ is a contraction of $\mathscr{S}_4$ (see Section 2-5).

We show how an irreducible representation of the contracted algebra $\mathscr{E}_3 \oplus \{G_0\}$ can be obtained from a sequence of representations of $\mathscr{S}_4$. Consider the representation (λ, ρ^2) of S_4 (λ an integer, $\rho > 0$). The matrix elements of the G operators for this representation can easily be computed from (4.193) and (4.196). We take the limit of these matrix elements as $\epsilon \to 0$ and $\lambda \to +\infty$ in such a manner that $\epsilon^2\lambda \to 1$. For G_1 the result is

$$\begin{aligned}
\lim_{\epsilon\to 0} (G_1^{\epsilon^{-2},\rho^2})_{n,n'} &= \lim_{\substack{\epsilon\to 0 \\ \epsilon^2\lambda\to 1}} \langle f_n , \epsilon J_1 f_{n'} \rangle \\
&= - \tfrac{1}{2} i\rho(\delta_{n-1,n'} + \delta_{n+1,n'}) \\
&= (K_1^\rho)_{n,n'} \qquad (4.197)
\end{aligned}$$

(n and n' are held fixed). Similarly, we find

$$\lim_{\epsilon\to 0}(G_2^{\epsilon^{-2},\rho^2})_{n,n'} = \tfrac{1}{2}\rho(-\delta_{n-1,n'} + \delta_{n+1,n'}) = (K_2^\rho)_{n,n'},$$

$$\lim_{\epsilon\to 0}(G_3^{\epsilon^{-2},\rho^2})_{n,n'} = -\,in\,\delta_{n,n'} = (K_3^\rho)_{n,n'}, \tag{4.198}$$

$$\lim_{\epsilon\to 0}(G_0^{\epsilon^{-2},\rho^2})_{n,n'} = 0, \qquad n, n' = 0, \pm 1, \pm 2, \ldots .$$

Thus, we have derived an irreducible representation of the contracted algebra $\mathscr{E}_3 \oplus \{G_0\}$ as a limit of irreducible representations of $\mathscr{S}_4$. The effect of this limiting procedure on the matrix elements of the representation (λ, ρ^2) of the group S_4 is easily computed from the above considerations. For example, from (4.123) we know that the matrix elements of the unitary operator $e^{-yJ_1} = \mathbf{U}^{\lambda,\rho^2}(g)$, $y > 0$ (g has coordinates $\{re^{i\theta}, \alpha, \delta\} = \{-\tfrac{1}{2}iy, 0, 0\}$), are given by

$$(e^{-yJ_1})_{n,n'} = \exp\left(-\frac{\rho^2 y^2}{8}\right)\left[\frac{(n+\lambda)!}{(n'+\lambda)!}\right]^{1/2}\left(\frac{i\rho y}{2}\right)^{n'-n} L_{n+\lambda}^{(n'-n)}\left(\frac{\rho^2 y^2}{4}\right). \tag{4.199}$$

Similarly from (3.57) we see that for the representation (ρ) of E_3 the matrix elements of e^{-yK_1} are given by

$$e^{-yK_1})_{n,n'} = i^{n-n'} J_{n-n'}(\rho y).$$

Thus, (4.197) implies

$$\lim_{\substack{\epsilon\to 0\\ \epsilon^2\lambda\to 1}} (e^{-yG_1})_{n,n'} = (e^{-yK_1})_{n,n'} \tag{4.200}$$

or,

$$\lim_{\epsilon\to 0} \exp\left(-\frac{\rho^2\epsilon^2 y^2}{8}\right)\left[\frac{(n+\epsilon^{-2})!}{(n'+\epsilon^{-2})!}\right]^{1/2} (\tfrac{1}{2}i\epsilon\rho y)^{n'-n}$$

$$\cdot\, L_{n+\epsilon^{-2}}^{(n'-n)}(\rho^2\epsilon^2 y^2/4) = i^{n-n'} J_{n-n'}(\rho y),$$

which simplifies to the relation

$$\lim_{m\to\infty} m^{-n} y^n L_m^{(n)}(y^2/m) = J_n(2y). \tag{4.201}$$

This proof of the limit (4.201) is merely formal; we have not verified the validity of (4.200). However, now that this limit relation has been motivated it is easy to prove it directly from the power series expansions for Laguerre and Bessel functions.

The preceding discussion was concerned with the relationship between matrix elements of unitary representations of S_4 and E_3. The restriction to unitary representations is not essential, however, and similar arguments can be used to relate matrix elements of irreducible representations of $G(0, 1)$ and matrix elements of irreducible representations of T_3. We omit this.