Appendix: Crucial results due to Eisenhart

In the first part of this appendix we give the proof of Eisenhart's theorem and its first corollary.

If $A = \Sigma_{i,j} \mathcal{A}^{ij} p_i p_j$ is a quadratic first integral of a Hamiltonian $H = \Sigma_{i,j} g^{ij} p_i p_j$ then

$$[A, H] = 0. \tag{A.1}$$

These conditions are equivalent to the equations

$$S_{ik\ell} + S_{k\ell i} + S_{\ell ik} = 0 \tag{A.2}$$

where

$$S_{ik\ell} = \sum_i [\mathcal{A}^{ij}(\partial_j g^{k\ell}) - g^{ij}(\partial_j \mathcal{A}^{k\ell})]. \tag{A.3}$$

Furthermore, if ρ_i are the roots of the determinant equation

$$\det(\mathcal{A}^{ij} - \rho g^{ij})_{n\times n} = 0 \tag{A.4}$$

then the equations

$$\sum_j (\mathcal{A}^{ij} - \rho_h g^{ij}) \lambda_{h|j} = 0 \tag{A.5}$$

determine an orthogonal enuple of covariant vectors $\lambda_{h|j}$.

Ordinarily it is not possible to choose a set of local coordinates such that $\mathcal{A}^{ij} = \mathcal{A}^{ii}\delta^{ij}$ and $g^{ij} = H_j^{-2}\delta^{ij}$, i.e., such that the matrices $(\mathcal{A}^{ij})_{n\times n}$ $(g^{ij})_{n\times n}$ can be made simultaneously diagonal. In the case of orthogonal separable coordinates this will turn out to be the case. If we in fact assume that these matrices are diagonal in the given coordinate system $\{x^i\}$ then conditions (A.2) imply

$$\frac{\partial}{\partial x^i} \log(\mathcal{A}^{ii} H_i^2) = 0, \tag{A.6}$$

$$\frac{\partial \mathcal{A}^{ii}}{\partial x^j} - \mathcal{A}^{jj} H_j^2 \frac{\partial}{\partial x^j} H_i^{-2} = 0, \quad i \neq j. \tag{A.7}$$

(This assumption is equivalent to the condition of normality in the statement of Eisenhart's theorem in Chapter 2.)

The first of these conditions implies that

$$\mathcal{A}^{ii} = \rho_i H_i^{-2} \tag{A.8}$$

with $\partial\rho_i/\partial x^i = 0$. The second set reduces to

$$\frac{\partial}{\partial x^j} \log\left(\frac{\rho_i - \rho_j}{H_i^2}\right) = 0. \tag{A.9}$$

These equations can be written

$$\frac{\partial \rho_i}{\partial x^j} = (\rho_i - \rho_j) \frac{\partial}{\partial x^j} \log H_i^2, \qquad \frac{\partial \rho_i}{\partial x^i} = 0. \tag{A.10}$$

The integrability conditions for these conditions, regarded as a system of equations for the ρ_i, are

$$(\rho_i - \rho_j)\left(\frac{\partial^2}{\partial x^i \partial x^j} \log H_i^2 + \frac{\partial}{\partial x^j} \log H_i^2 \frac{\partial}{\partial x^i} \log H_j^2\right) = 0, \tag{A.11}$$

$$(\rho_j - \rho_k)\left(\frac{\partial^2}{\partial x^j \partial x^k} \log H_i^2 - \frac{\partial}{\partial x^j} \log H_i^2 \frac{\partial}{\partial x^k} \log H_i^2 + \frac{\partial}{\partial x^j} \log H_i^2 \frac{\partial}{\partial x^k} \log H_j^2 + \frac{\partial}{\partial x^k} \log H_i^2 \frac{\partial}{\partial x^j} \log H_k^2\right) = 0.$$

If the above system of equations admits solutions with all the ρ's different then

$$\frac{\partial^2}{\partial x^j \partial x^k} \log H_i^2 - \frac{\partial}{\partial x^j} \log H_i^2 \frac{\partial}{\partial x^k} \log H_i^2 + \frac{\partial}{\partial x^j} \log H_i^2 \frac{\partial}{\partial x^k} \log H_j^2 + \frac{\partial}{\partial x^k} \log H_i^2 \frac{\partial}{\partial x^j} \log H_k^2 = 0, \quad j \neq k. \tag{A.12}$$

It follows that if equations (A.12) are satisfied the system of equations (A.10) is completely integrable. One solution is $\rho_i = a$ (const.). We denote the other solutions by ρ_i^a , $a = 2, \ldots, n$, $(\rho_1^1 = 1)$ and assume that the determinant of the n solutions is not zero, i.e.

$$|\rho_i^a - \rho_j^a| \neq 0 \qquad (A.13)$$

for i fixed and $a = 2, \ldots, n$; $j = 1, \ldots, n$, $j \neq i$. In this case the equations of the geodesics admit $n - 1$ quadratic first integrals

$$A_a = \sum_{i,j} \mathcal{Q}^{ij}_{(a)} p_i p_j, \quad a = 2, \ldots, n \qquad (A.14)$$

where $\mathcal{Q}^{ij}_{(a)} = \rho_i^a H_i^{-2} \delta^{ij}$.

The integrability conditions (A.12) can now be shown to be equivalent to Stäckel form. To achieve this, denote by S_{ij}, $i, j=1, \ldots, n$, n^2 functions whose determinant $S \neq 0$ and denote by S^{ij} the (i, j) cofactor of S_{ij} in S. Now put $H_i^2 = S/S^{i1}$, $\rho_i^a = S^{ia}/S^{i1}$, where it is understood that $(\partial_i)\rho_i^a = 0$.

Then

$$b_{ij}^a = \frac{(\rho_i^a - \rho_j^a)}{H_i^2} = \frac{S^{j1}S^{ia} - S^{i1}S^{ja}}{SS^{j1}} \qquad (A.15)$$

where $(\partial_j)\ b_{ij}^a = 0$. We also have that

$$S^{j1}S^{ia} - S^{i1}S^{ja} = SM_{j1ia} \qquad (A.16)$$

where M_{j1ia} is the algebraic complement of $S_{j1}S_{ia} - S_{i1}S_{ja}$ in the determinant S. Consequently we have that

$$S^{j1}b_{ij}^a = M_{j1ia}, \quad i, j = 1, \ldots, n, \quad a = 2, \ldots, n. \qquad (A.17)$$

From the definition of the algebraic complement we have that

$$S^{ja} = \sum_{i \neq j} S_{i1} M_{j1ia}, \qquad (A.18)$$

consequently

$$\frac{S^{ja}}{S^{j1}} = \sum_{i \neq j} S_{i1} b_{ij}^a . \qquad (A.19)$$

Differentiating with respect to x^j, we have

$$0 = \sum_{i \neq j} \frac{\partial S_{i1}}{\partial x^j} b_{ij}^a, \quad a = 2, \ldots, n, \quad n = 1, \ldots, n. \qquad (A.20)$$

For a given j the determinant of $(b^a_{i(j)})_{n-1\times n-1}$ $(i \neq j)$ is not zero. Therefore S_{i1} is a function of x^i at most. From (A.15) we have that

$$S^{j1}b_{ij}^a = -S^{i1}b_{ji}^a . \qquad (A.21)$$

As a consequence of this result and (A.17) we have that

$$\frac{M_{j1ib}}{M_{j1i2}} = \frac{b_{ij}^b}{b_{ij}^2} = \frac{b_{ji}^b}{b_{ji}^2} = \sigma_{ijb} = \sigma_{jib}, \quad b = 3, \ldots, n, \quad i \neq j. \qquad (A.22)$$

As the second term is independent of x^j and the third is independent of x^i, it follows that σ_{ijb} does not depend on either x^i or x^j. From the identities

$$\sum_{a=2}^{n} S_{ka} M_{j1ia} = 0 \qquad (A.23)$$

we have, using (A.22),

$$S_{k2} + \sum_{b=3}^{n} S_{kb}\sigma_{ijb} = 0 . \qquad (A.24)$$

Differentiating with respect to x^i, we have

$$\frac{\partial S_{k2}}{\partial x^i} + \sum_{b=3}^{n} \frac{\partial S_{kb}}{\partial x^i}\sigma_{ijb} = 0 . \qquad (A.25)$$

For fixed i and k there are $n - 2$ quantities $S_{k2}, \ldots, S_{kn}$ and these same equations are satisfied by the derivatives of these quantities with respect to x^i. Hence we have

$$\frac{\partial S_{ka}}{\partial x^i} = \mu_{ik} S_{ka} \quad \text{or} \quad \partial_i \left(\frac{S_{ka}}{S_{kc}}\right) = 0 \qquad (A.26)$$

for $c \neq a$. These equations hold for $i = 1, \ldots, n$; $i \neq k$. Hence we have

$$S_{ia} = e^{\nu_i} \psi_{ia} \tag{A. 27}$$

where ψ_{ia} are functions of x^i at most. From (A. 17) we have

$$b^a_{j1} M_{11ia} = b^a_{i1} M_{11ja}, \qquad i, j = 2, \ldots, n \tag{A. 28}$$

and substituting from (A. 17) this condition becomes

$$e^{\nu_j} b^a_{j1} N_{1i} = e^{\nu_i} b^a_{i1} N_{ij} \tag{A. 29}$$

where N_{1i} is independent of x^1 and x^i. Differentiating with respect to x^1, we have

$$\partial_1 (\nu_i - \nu_j) = 0, \quad i, j = 2, \ldots, n. \tag{A. 30}$$

From (A. 17) and (A. 21) we have

$$e^{\nu_j} b^a_{ja} N_{ab} + e^{\nu_b} b^b_{ba} N_{aj} = 0. \tag{A. 31}$$

Differentiating with respect to x^a we have

$$\partial_a (\nu_j - \nu_b) = 0 . \tag{A. 32}$$

Combining these results, we obtain

$$\partial_k (\nu_i - \nu_j) = 0, \quad i, j, k = 1, \ldots, n, \quad i, j, k \neq \quad . \tag{A. 33}$$

Then $\nu_i - \nu_j = f_{ij}$ is at most a function of x^i and x^j.

The functions F_{ij} are also subject to the conditions

$$f_{ij} - f_{ik} + f_{jk} = 0. \tag{A. 34}$$

Differentiating with respect to x^i, we obtain

$$\frac{\partial f_{ij}}{\partial x^i} = \frac{\partial f_{ik}}{\partial x^i} . \tag{A. 35}$$

Consequently $f_{ij} = \sigma_i - \sigma_j$ where σ_i is a function of x^i alone. Hence we can write $\nu_i = \nu + \sigma_i$ and

$$S_{ia} = e^{\nu} \theta_{ia}, \quad i = 1, \ldots, n, \quad a = 2, \ldots, n, \tag{A. 36}$$

where θ_{ia} are functions of x^i alone. But if we substitute these expressions into those for H_i^{-2} and ρ^a_i, the function ν disappears and consequently we can take $\nu = 0$, and the metric tensor components H_i^{-2} are then in Stäckel form. Eisenhart's theorem and the first corollary are now proved.

In the remainder of this appendix we derive further results of Eisenhart that are necessary in the classification proofs given in Chapters 3, 4 and 5.

From conditions (A. 12) with $k = i$ we obtain

$$\frac{\partial^2}{\partial x^i \partial x^j} \log \left[\frac{H_i^2}{H_j^2}\right] = 0 \tag{A. 37}$$

from which it follows that

$$H_i^2 = \phi_{ij}^2 \theta_{ij}, \qquad H_j^2 = \phi_{ji}^2 \theta_{ij} \tag{A. 38}$$

where ϕ_{ij} is independent of x^j and ϕ_{ji} is independent of x^i. Substituting in (A. 12) with $k = i$ we find that

$$\theta_{ij} = \tau_{ij} + \tau_{ji} \tag{A. 39}$$

where τ_{ij} is independent of x^j and τ_{ji} is independent of x^i.

The Riemann curvature tensor symbol

$$R_{jiik} = \frac{1}{4} H_i^2 \Big[2 \frac{\partial^2}{\partial x^j \partial x^k} \log H_i^2 + \frac{\partial}{\partial x^j} \log H_i^2 \frac{\partial}{\partial x^k} \log H_i^2 \tag{A. 40}$$
$$- \frac{\partial}{\partial x^j} \log H_i^2 \frac{\partial}{\partial x^k} \log H_j^2 - \frac{\partial}{\partial x^k} \log H_i^2 \frac{\partial}{\partial x^j} \log H_k^2 \Big]$$

can in consequence of (A. 12) be written

$$R_{jiik} = \frac{3}{4} H_i^2 \frac{\partial^2}{\partial x^k \partial x^j} \log H_i^2 . \qquad (A. 41)$$

If we now restrict ourselves to Riemannian manifolds that admit an orthogonal coordinate system in Stäckel form and satisfy $R_{jiik} = 0$, then (A. 40) implies

$$\frac{\partial^2}{\partial x^j \partial x^k} \log H_i^2 = 0 , \qquad (A. 42)$$

$$\frac{\partial}{\partial x^j} \log H_i \frac{\partial}{\partial x^k} \log H_i - \frac{\partial}{\partial x^j} \log H_i \frac{\partial}{\partial x^k} \log H_j$$

$$- \frac{\partial}{\partial x^k} \log H_i \frac{\partial}{\partial x^j} \log H_k = 0, \quad j \neq k . \qquad (A. 43)$$

An orthogonal coordinate system on a Riemannian manifold of constant curvature satisfies these conditions. Thus coordinate systems on S_n, E_n and H_n are included.

Substituting in (A. 41) from (A. 38), we obtain

$$\frac{\partial \tau_{ji}}{\partial x^j} = (\tau_{ij} + \tau_{ji}) \psi_{ji}(x^i, x^j) , \qquad (A. 44)$$

$$\frac{\partial \tau_{ij}}{\partial x^i} = (\tau_{ij} + \tau_{ji}) \psi_{ij}(x^i, x^j) .$$

Differentiating these equations with respect to x^i, we obtain

$$\frac{\partial \psi_{ji}}{\partial x^i} + \psi_{ji} \psi_{ij} = 0, \quad \frac{\partial \psi_{ij}}{\partial x^j} + \psi_{ji} \psi_{ij} = 0 \qquad (A. 45)$$

and accordingly we have

$$\psi_{ji} = \frac{\partial}{\partial x^j} \log \alpha \qquad (A. 46)$$

and we find that $\alpha = \alpha_i + \alpha_j$ where α_i and α_j are functions of x^i and x^j respectively. It follows that

$$\tau_{ij} + \tau_{ji} = (\alpha_i + \alpha_j) \omega_{ij} \qquad (A. 47)$$

where ω_{ij} is independent of x^i and x^j. Consequently

$$H_i^2 = x_i \prod_{j \neq 1} (\sigma_{ij} + \sigma_{ji}) , \qquad i = 1, \ldots, n \qquad (A. 48)$$

where $(\partial_k) \sigma_{ij} = 0$ for $k \neq j$.

These expressions satisfy (A. 12), $i = k$. In order that (A. 43) is satisfied, we must have

$$\sigma'_{ji} \sigma'_{ki} (\sigma_{jk} + \sigma_{kj}) - \sigma'_{ji} \sigma'_{kj} (\sigma_{ki} + \sigma_{ik}) - \sigma'_{ki} \sigma'_{jk} (\sigma_{ij} + \sigma_{ji}) = 0. \quad (A.49)$$

Permuting the indices of this equation cyclically, we have

$$\sigma'_{ik} \sigma'_{ij} (\sigma_{ki} + \sigma_{ik}) - \sigma'_{kj} \sigma'_{ik} (\sigma_{ij} + \sigma_{ji}) - \sigma'_{ij} \sigma'_{ki} (\sigma_{jk} + \sigma_{kj}) = 0, \quad (A.50)$$

$$\sigma'_{ik} \sigma'_{jk} (\sigma_{ij} + \sigma_{ji}) - \sigma'_{ik} \sigma'_{ji} (\sigma_{jk} + \sigma_{kj}) - \sigma'_{jk} \sigma'_{ij} (\sigma_{ki} + \sigma_{ik}) = 0. \quad (A.51)$$

Equating to zero the determinant of these equations, we have

$$\sigma'_{ij} \sigma'_{jk} \sigma'_{ki} + \sigma'_{ji} \sigma'_{kj} \sigma'_{ik} = 0. \qquad (A. 52)$$

If none of the terms in these conditions is zero then $\sigma_{ij} / \sigma'_{ik}$ is a constant. We can therefore put

$$\sigma_{ij} = a_{ij} \sigma_i \qquad (A. 53)$$

where a_{ij} is a constant and σ_i is a function of x^i at most. These constants must satisfy

$$a_{ij} a_{jk} a_{ki} + a_{ji} a_{kj} a_{ik} = 0. \qquad (A. 54)$$

The metric coefficients are now

$$H_i^2 = x_i \prod_{j \neq i} (a_{ij} \sigma_i + a_{ji} \sigma_j). \qquad (A. 55)$$

If we put $\sigma_i = a_{jk} a_{ki} \bar{\sigma}_i$, $\sigma_j = a_{kj} a_{ik} \bar{\sigma}_j$ then, in consequence of (A. 54) we have that

$$a_{ij}\sigma_i + a_{ji}\sigma_j = a_{ij}a_{jk}a_{ki}(\bar{\sigma}_i - \bar{\sigma}_j),$$

the constant factor being absorbed in x_i. Without loss of generality we can take $a_{ij} = -a_{ji}) = 1$ and then (A. 54) becomes $a_{jk}a_{ki} - a_{kj}a_{ik} = 0$.

If we now put $a_{ki}\sigma_k = -a_{ik}\bar{\sigma}_k$ we obtain

$$a_{ki}\sigma_k + a_{ik}\bar{\sigma}_i = a_{ik}(\bar{\sigma}_i - \bar{\sigma}_k) \qquad (A.\,56)$$

so without loss of generality we may take $a_{ki} = -1 = -a_{ik}$. Then

$$a_{jk}\bar{\sigma}_j + a_{kj}\bar{\sigma}_k = a_{jk}(\bar{\sigma}_j - \bar{\sigma}_k)$$

and thus $a_{jk} = -a_{kj} = 1$ and consequently the metric coefficients have the form

$$H_i^2 = x_i \prod_{j \neq 1} (\sigma_i - \sigma_j) . \qquad (A.\,57)$$

We must now consider the case when some of the σ_{ij} functions are constant. Suppose that $\sigma_{ij} = a_{ij}$ (const) from (A. 50). It follows that either $\sigma_{ik} = a_{ik}$ or $\sigma_{kj} = a_{kj}$, the a's being constants. If $\sigma_{ik} = a_{ik}$ then (A. 51) is satisfied and (A. 49) is satisfied in the following cases.

(i) $\sigma_{ji} = a_{ji}$, $\sigma_{jk} = a_{jk}$, (A. 58)

(ii) $\sigma_{ji} = a_{ji}$, $\sigma_{ki} = a_{ki}$,

(iii) $\sigma_{ki} = a_{ki}$, $\sigma_{kj} = a_{kj}$.

The possibility (iii) follows from (i) with j and k interchanged. If σ_{ji} and σ_{ki} are not constants we can write (A. 49) in the form

$$\sigma_{jk} + \sigma_{kj} - \frac{\sigma'_{kj}}{\sigma'_{ki}}(\sigma_{ki} + \sigma_{ik}) - \frac{\sigma'_{jk}}{\sigma'_{ji}}(\sigma_{ij} + \sigma_{ji}) = 0. \qquad (A.\,59)$$

From this condition we conclude that

$$\sigma_{jk} - \frac{\sigma'_{jk}}{\sigma'_{ji}}(\sigma_{ij} + \sigma_{ji}) = c, \quad \sigma_{kj} - \frac{\sigma'_{kj}}{\sigma'_{ki}}(\sigma_{ki} + \sigma_{ik}) = -c \qquad (A.\,60)$$

for some constant c, consequently

$$\sigma_{ji} + a_{ij} = b(\sigma_{jk} - c), \quad \sigma_{ki} + a_{ik} = d(\sigma_{kj} + c) \qquad (A.\,61)$$

for constants b, d. Hence $\sigma'_{ji} = b\sigma'_{jk}$, $\sigma'_{ki} = d\sigma'_{kj}$ so that we may put

$$\sigma_{ji} = a_{ji}\sigma_j, \quad \sigma_{jk} = a_{jk}\sigma_j, \quad \sigma_{ki} = a_{ki}\sigma_k, \quad \sigma_{kj} = a_{kj}\sigma_j$$

and then from (A. 59) we have (A. 54).

There are thus three distinct cases:

I $\sigma_{ij} = a_{ij}$, $\sigma_{ji} = a_{ji}$, $\sigma_{ik} = a_{ik}$, $\sigma_{jk} = a_{jk}$, (A. 62)

II $\sigma_{ij} = a_{ij}$, $\sigma_{ji} = a_{ji}$, $\sigma_{ik} = a_{ik}$, $\sigma_{ki} = a_{ki}$,

III $\sigma_{ij} = a_{ij}$, $\sigma_{ik} = a_{ik}$, $\sigma_{ji} = a_{ji}\sigma_j$, $\sigma_{jk} = a_{jk}\sigma_j$, $\sigma_{ki} = a_{ki}\sigma_k$, $\sigma_{kj} = a_{kj}\sigma_k$.

In the first two cases the a's are arbitrary; in the last they must satisfy (A. 54). When $\sigma_{kj} = a_{kj}$ and $\sigma_{ij} = a_{ij}$ we have, from (A. 49) -(A. 51), the case (A. 58) (iii) or

$$\sigma'_{ji}(\sigma_{jk} + a_{kj}) - \sigma'_{jk}(\sigma_{ji}a_{ij}) = 0. \qquad (A.\,63)$$

If $\sigma_{ji} = a_{ji}$, $\sigma_{jk} = a_{jk}$ then we obtained (A. 62) (II) by interchanging i, j. Otherwise we have the type

IV $\sigma_{ij} = a_{ij}$, $\sigma_{kj} = a_{kj}$, $\sigma_{ji} = a_{ji}\sigma_j$, $\sigma_{jk} = a_{jk}\sigma_j$,

$a_{ji}a_{kj} - a_{jk}a_{ij} = 0.$

This completes the necessary details for the arguments in Chapter 3 to be valid.

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