2 Historical outline of the separation of variables (principal results)

The history of variable separation dates back to the work of Liouville [4] who considered a dynamical system with kinetic energy $2T=\lambda[(\dot{x}^1)^2+(\dot{x}^2)^2]$ and potential $V(x^1, x^2)$ and showed that if the Hamilton-Jacobi equation

$$\frac{1}{2\lambda}\left[\left(\frac{\partial W}{\partial x^1}\right)^2 + \left(\frac{\partial W}{\partial x^2}\right)^2\right] + V(x^1, x^2) = E \tag{2.1}$$

admits a **complete integral** of the form

$$W = W_1(x^1;\ c_1,\ c_2) + W_2(x^2;\ c_1,\ c_2) \tag{2.2}$$

then

$$\lambda = \sigma_1(x^1) + \sigma_2(x^2), \quad V = \frac{\mu_1(x^1) + \mu_2(x^2)}{\sigma_1(x^1) + \sigma_2(x^2)} .$$

Dynamical systems of this type are said to be in Liouville form. These coordinate systems readily generalize to n-dimensional Liouville systems in which the kinetic energy is given by

$$2T = \left[\sum_{i=1}^{n} \sigma_i(x^i)\right]\left[\sum_{j=1}^{n} (\dot{x}^j)^2\right] \tag{2.3}$$

and the potential is

$$V = \left[\sum_{j=1}^{n} \mu_i(x^i)\right] / \left[\sum_{i=1}^{n} \sigma_i(x^i)\right]. \tag{2.3}$$

The associated Hamilton-Jacobi equation has the form

$$\frac{1}{2\sigma}\sum_{j=1}^{n}\left\{\left(\frac{\partial W}{\partial x^j}\right)^2 + 2\mu_j(x^j)\right\} = E \tag{2.4}$$

where $\sigma = \Sigma_{i=1}^{n}\sigma_i(x^i)$. The complete integral of this equation can be obtained by looking for a separable solution $W = \Sigma_{i=1}^{n} W_i(x^i)$; then

$$W_i(x^i) = \int\sqrt{(2(-\mu_i(x^i) + E\sigma_i(x^i) + \alpha_i))}\,dx^i \tag{2.5}$$

where $\Sigma_{i=1}^{n}\alpha_i = 0$. The motion can then be solved from the equations

$$\frac{\partial W}{\partial \alpha_j} - \frac{\partial W}{\partial \alpha_n} = -\beta_j \qquad (j = 1, \ldots, n-1) \tag{2.6}$$

$$t - t_0 + \frac{\partial W}{\partial E}, \qquad \frac{\partial W}{\partial x^i} = p_i \qquad (i = 1, \ldots, n).$$

Writing $F_i(x^i) = \sqrt{(2(-\mu_i(x^i) + E\sigma_i(x^i) + \alpha_i))}$, $i = 1, \ldots, n-1$, the solution is obtained from

$$\int\frac{dx^1}{\sqrt{(F_1(x^1))}} + \beta_1 = \int\frac{dx^2}{\sqrt{(F_2(x^2))}} + \beta_2 = \ldots = \int\frac{dx^{n-1}}{\sqrt{(F_{n-1}(x^{n-1}))}} + \beta_{n-1}$$

$$= \int\frac{dx^n}{\sqrt{(F_n(x^n))}} \tag{2.7}$$

$$t - t_0 = \int\frac{\sigma_1(x^1)\,dx^1}{\sqrt{(F_1(x^1))}} + \ldots + \int\frac{\sigma_n(x^n)\,dx^n}{\sqrt{(F_n(x^n))}} .$$

Consequently, if the dynamical system is in Liouville form, the solution for the motion can be obtained by the 'method of separation of variables' and reduced to quadratures.

The complete solution of the separation of variables problem (III) (Chapter 1) in two dimensions for the Hamilton-Jacobi equation has been obtained by Stäckel [5]. This is, of course, a special case of the general problem for arbitrary n. Stäckel obtained a classification which listed three types of possible Riemannian metrics:

I (Liouville forms)

$$ds^2 = (\sigma_1(x^1) + \sigma_2(x^2))[(dx^1)^2 + (dx^2)^2]$$
$$V = (\mu_1(x^1) + \mu_2(x^2))/(\sigma_1(x^1) + \sigma_2(x^2)). \tag{2.8}$$

II

$$ds^2 = g_{11}(x^1)(dx^1)^2 + 2g_{12}(x^1)\,dx^1dx^2 + g_{22}(x^1)(dx^2)^2$$
$$V = V(x^1).$$

III $$ds^2 = (dx^1)^2 - 2\cos(\sigma_1(x^1) + \sigma_2(x^2))\,dx^1dx^2 + (dx^2)^2$$
$$V = 0.$$

Some crucial observations greatly simplify this list. The reader will probably already have realized that if the Hamilton-Jacobi equation admits a solution via the separation of variables ansatz

$$W = \sum_{i=1}^{n} W_i(x^i; \underline{c}), \quad \underline{c} = (c_1, \ldots, c_n) \tag{2.9}$$

in some set of variables x^i, then we can just as well choose a set of coordinates y^i, where $y^i = f_i(x^i)$ $(i = 1, \ldots, n)$ and $f_i(x^i)$ are a suitable set of real analytic functions and all considerations are of course local (i.e., we are working on a coordinate patch). Any such coordinate systems which are related in this way will be considered to be equivalent, in the sense of problem (III) of Chapter 1. Metrics of type (2.8) II contain a variable x^2 which corresponds to an ignorable variable. (Recall that in classical mechanics [2] a variable is ignorable if it does not appear explicitly in the metric components g_{ij}.) It is then possible to find a solution of the Hamilton-Jacobi equation

$$g^{11}(x^1)\left(\frac{\partial W}{\partial x^1}\right)^2 + 2g^{12}(x^1)\frac{\partial W}{\partial x^1}\frac{\partial W}{\partial x^2} + g^{22}(x^1)\left(\frac{\partial W}{\partial x^2}\right)^2 = E \tag{2.10}$$

by looking for a solution of the form

$$W = W_1(x^1; c_1, c_2) + c_2 x^2 \tag{2.11}$$

If we now define new variables y^i $(i = 1, 2)$ by

$$y^1 = \int(\sqrt{g}/g_{11})\,dx^1, \quad y^2 = x^2 + \int(g_{12}/g_{22})\,dx^1 \tag{2.12}$$

where $g = g_{11}g_{22} - g_{12}^2$, then the metric II assumes the form

$$ds^2 = g(y^1)\left[(dy^1)^2 + (dy^2)^2\right]. \tag{2.13}$$

This change of variables does not affect variable separation, as the original solution would have the form

$$W = \overline{W}_1(y^1; c_1, c_2) + c_2 y^2 . \tag{2.14}$$

For this reason we can regard variables which are related in the manner (2.12) as 'equivalent' (in the sense of problem (III)), in that they give rise to variable separation for the Hamilton-Jacobi equation which gives basically the same solutions. The metrics for type (2.8) III correspond to locally flat spaces for which cartesian coordinates can be chosen as

$$x = \int\cos\sigma_1(x^1)\,dx^1 - \int\cos\sigma_2(x^2)\,dx^2 \tag{2.15}$$
$$y = \int\sin\sigma_1(x^1)\,dx^1 + \int\sin\sigma_2(x^2)\,dx^2 .$$

The separable solutions of the corresponding Hamilton-Jacobi equation are $W = c_1 x + c_2 y$, $(c_1^2 + c_2^2 = E)$. These coordinates are a canonical form for separable systems which can be obtained from cartesian coordinates via the transformation

$$x = F(x^1) + G(x^2), \quad y = H(x^1) + J(x^2). \tag{2.16}$$

Again, we do not regard coordinate systems related in this way as being essentially different and we extend our notion of 'equivalence' to include coordinate systems related via equations of type (2.16). Given this equivalence of coordinate systems, we see that for $n = 2$ any coordinate system for which the Hamilton-Jacobi equation admits solution via separation of variables is 'equivalent' to a coordinate system in which the Riemannian metric is in Liouville form.

The most significant development due to Stäckel [6] was to give the general solution to the separation of variables problem for the Hamilton-Jacobi equation for an orthogonal coordinate system.

Stäckel's Theorem: The necessary and sufficient conditions that the Hamilton-Jacobi equation

$$H = \sum_{i=1}^{n} H_i^{-2}\left(\frac{\partial W}{\partial x^i}\right)^2 + V(\underline{x}) = E \qquad (E \neq 0) \tag{2.17}$$

admits a complete integral via separation of variables (i.e. a solution $W = \Sigma_{i=1}^{n} W_i(x^i, \underline{c})$ for which $\Delta = \det(\partial^2 W/\partial x^i \partial c_j)_{n\times n} \neq 0$, $\underline{c} = (c_1, \ldots, c_n)$) are:

(i) that there exist a Stäckel matrix $\tilde{S} = (S_{ij}(x^i))_{n\times n}$ such that $H_i^{-2} = S^{i1}/S$ $(i = 1, \ldots, n)$ where $S = \det\tilde{S}$ and S^{i1} is the $(i, 1)$ cofactor of $\tilde{S}$. The elements of the Stäckel matrix are such that $\partial S_{ij}/\partial x^j = 0$ if $j \neq i$;

(ii) that there are functions $v_i(x^i)$ such that

$$V = \sum_{i=1}^{n} v_i \frac{S^{i1}}{S} \, .$$

<u>Proof:</u> Let $W = \Sigma_{i=1}^{n} W_i(x^i, c_1, \ldots, c_n)$ be a complete integral of (2.17) and choose $c_1 = E$. If we substitute this form of W into the Hamilton-Jacobi equation and differentiate with respect to c_k then

$$\sum_{i=1}^{n} \frac{1}{H_i^2} \frac{\partial}{\partial c_k}\left(\frac{\partial W_i}{\partial x^i}\right)^2 = \delta_{k1}, \qquad k = 1, \ldots, n. \tag{2.18}$$

As W is a complete integral, $\Delta = \det(\partial^2 W/\partial x^i \partial c_j)_{n\times n} \neq 0$; consequently, if we write

$$S_{ij}(x^i) = \frac{\partial}{\partial c_j}\left(\frac{\partial W_i}{\partial x^i}\right)^2$$

we see that $S = \det(S_{ij}(x^i)) = 2^n \Pi_{i=1}^{n} (\partial W_i/\partial x^i)\,\Delta \neq 0$ and the system (2.18) can be solved for H_i^{-2}, $(i = 1, \ldots, n)$ to give $H_i^{-2} = S^{i1}/S$ with the Stäckel matrix $\tilde{S} = (S_{ij})_{n\times n}$. Substituting this form for the coefficients back into the original equation, we see that

$$V = \sum_{i=1}^{n} \frac{S^{i1}}{S} v_i(x^i)$$

where $v_i = S_{i1}(x^i) - (\partial W_i/\partial x^i)^2$. To complete the proof we need only establish sufficiency. If there exist a Stäckel matrix $\tilde{S}$ and functions v_i such that conditions (i) and (ii) hold then the Hamilton-Jacobi equation can be written

$$\sum_{i=1}^{n} \frac{S^{i1}}{S}\left[\left(\frac{\partial W}{\partial x^i}\right)^2 + v_i(x^i)\right] = E. \tag{2.20}$$

The separation equations are

$$\left(\frac{dW_i}{dx^i}\right)^2 + v_i(x^i) = \sum_{j=1}^{n} c_j S_{ij}(x^i), \quad i = 1, \ldots, n. \tag{2.21}$$

which have the solution

$$W_i = \int \left[\sum_{j=1}^{n} c_j S_{ij}(x^i) - v_i(x^i)\right]^{\frac{1}{2}} dx^i \tag{2.22}$$

The sufficiency of conditions (i) and (ii) has been proved by Stäckel [6]. It was also observed by Stäckel [7] that a Riemannian space which satisfies condition (i) (i.e. is in Stäckel form) admits n-quadratic first integrals of the geodesics

$$A_i = \sum_{j=1}^{n} \frac{S^{ji}}{S} p_j^2 \qquad i = 1, \ldots, n \tag{2.23}$$

where of course $A_1 = H$, the Hamiltonian, and $p_j = \partial W/\partial x^j$ is the canonical momentum. Furthermore these first integrals are independent and in involution

$$[A_i, A_j] = 0 \qquad i, j = 1, \ldots, n;\ i \neq j. \tag{2.24}$$

(Here $[\ ,\ ]$ is the Poisson bracket [2].) In other articles [8]-[9] Stäckel studies the solutions to the Hamilton-Jacobi equation when the kinetic energy is in Stäckel form. Stäckel's theorem is a basic result in the study of variable separation. Stäckel matrices are a recurring phenomenon.

Stäckel [10] obtained an extension of his first theorem which we now give without proof.

<u>Stäckel's (second) theorem:</u> Let $P = \{P_1, P_2, \ldots, P_N\}$ be a partition of the integers $\{1, \ldots, n\}$ into mutually exclusive non empty sets. Further, let $\tilde{S} = (S_{IJ}(x^I))_{N\times N}$, be a Stäckel matrix, i.e. $x^I = \{x^i; i \in P_I\}$; then if the Hamilton-Jacobi equation has the form

$$H = \sum_{I=1}^{N} \sum_{i,i' \in P_I} \frac{S^{I1}}{S} A_I^{ii'}(x^I)\, p_i p_{i'} + \sum_{I=1}^{N} B_I(x^I) \frac{S^{I1}}{S} = E \tag{2.25}$$

then there exist $N - 1$ orthogonal quadratic first integrals of the motion

$$A_J = \sum_{I=1}^{N} \sum_{i,i' \in P_I} \frac{S^{IJ}}{S} A_I^{ii'}(x^I)\, p_i p_{i'} + \sum_{I=1}^{N} B_I(x^I) \frac{S^{IJ}}{S}, \tag{2.26}$$

$J = 2, \ldots, N$. Furthermore this form of the Hamilton-Jacobi equation permits a partial separation of variables. If we look for a solution of the form

$$W = \sum_{I=1}^{N} W_I(x^I; C_1, \ldots, C_N) \tag{2.27}$$

then each W_I satisfies the 'partial separability' equations

$$\sum_{i,i' \in P_I} A_I^{i,i'}(x^I) \frac{\partial W_I}{\partial x^i} \frac{\partial W_I}{\partial x^{i'}} + B_I(x^I) = \sum_{J=1}^{N} C_J S_{IJ}. \tag{2.28}$$

The problem of central interest for Stäckel (and other authors) was to try to find all force free dynamical systems which admit first integrals of the motion that are homogeneous quadratic forms in the canonical momenta of a given dynamical system [10].

With the appearance of the class of Hamiltonians in Stäckel form it was natural to ask: what are the necessary and sufficient conditions that the Hamilton-Jacobi equation admits a complete integral $W = \Sigma_{i=1}^{n} W_i(x^i, \underline{c})$ in a given set of coordinates x^i $(i = 1, \ldots, n)$ (not necessarily orthogonal)? Recall that a complete integral is a solution $W(\underline{x}, \underline{c})$ of (1.1) such that

$$\det \left[\frac{\partial^2 W}{\partial x^i \partial c_j}\right]_{n \times n} \neq 0 .$$

Levi Civita [11] provided the answer to this question with the following Theorem.

Theorem (Levi Civita): The necessary and sufficient condition for the Hamilton-Jacobi equation

$$H(x^1, \ldots, x^n; p_1, \ldots, p_n) = E, \quad p_i = \frac{\partial W}{\partial x^i} \quad i = 1, \ldots, n. \tag{2.29}$$

to admit a complete integral of the form

$$W = \sum_{i=1}^{n} W_i(x^i, \underline{c})$$

is that H satisfy the $\frac{1}{2}n(n-1)$ equations

$$\frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial x^i \partial x^j} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial x^j} \frac{\partial^2 H}{\partial x^i \partial p_j} \tag{2.30}$$

$$- \frac{\partial H}{\partial x^i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial x^j} + \frac{\partial H}{\partial x^i} \frac{\partial H}{\partial x^j} \frac{\partial^2 H}{\partial p_i \partial p_j} = 0,$$

$$i \neq j, \quad i, j = 1, \ldots, n.$$

Proof: For a solution in this form it is necessary and sufficient that $dp_i/dx^j = 0$, $(i \neq j)$ where each p_i is considered a function of $x^1, \ldots, x^n$. Differentiating $H = E$ with respect to x^i, we obtain

$$\frac{\partial H}{\partial x^i} + \frac{\partial H}{\partial p_i} \frac{\partial p_i}{\partial x^i} = 0 \qquad i = 1, \ldots, n \tag{2.31}$$

and consequently

$$\frac{\partial p_i}{\partial x^i} = -\left(\frac{\partial H/\partial x^i}{\partial H/\partial p_i}\right) \tag{2.32}$$

The condition for separation of variables is then

$$\frac{d}{dx^j}\left(\frac{\partial H/\partial x^i}{\partial H/\partial p_i}\right) = 0 \qquad (i \neq j) \tag{2.33}$$

where

$$\frac{d}{dx^j} = \frac{\partial}{\partial x^j} + \frac{\partial p_j}{\partial x^j} \frac{\partial}{\partial p_j} .$$

These are just the conditions of the theorem. If the Hamiltonian can be written in the form $H = Q(p_1, \ldots, p_n; x^1, \ldots, x^n) + V(x^1, \ldots, x^n)$, these conditions become

$$\frac{\partial^2 Q}{\partial x^i \partial x^j} \frac{\partial Q}{\partial p_i} \frac{\partial Q}{\partial p_j} - \frac{\partial Q}{\partial p_i} \frac{\partial Q}{\partial x^j} \frac{\partial^2 Q}{\partial x^i \partial p_j} \tag{2.34}$$

$$- \frac{\partial Q}{\partial x^i} \frac{\partial Q}{\partial p_j} \frac{\partial^2 Q}{\partial p_i \partial x^j} + \frac{\partial Q}{\partial x^i} \frac{\partial Q}{\partial x^j} \frac{\partial^2 Q}{\partial p_i \partial p_j} = 0$$

$$\frac{\partial Q}{\partial p_i} \frac{\partial Q}{\partial p_j} \frac{\partial^2 V}{\partial x^i \partial x^j} - \frac{\partial Q}{\partial p_i} \frac{\partial^2 Q}{\partial x^i \partial p_j} \frac{\partial V}{\partial x^j}$$

$$- \frac{\partial Q}{\partial p_j} \frac{\partial^2 Q}{\partial x^j \partial p_i} \frac{\partial V}{\partial x^i} + \frac{\partial^2 Q}{\partial p_i \partial p_j} \left(\frac{\partial Q}{\partial x^i} \frac{\partial V}{\partial x^j} + \frac{\partial Q}{\partial x^j} \frac{\partial V}{\partial x^i} \right) = 0$$

$$\frac{\partial^2 Q}{\partial p_i \partial p_j} \frac{\partial V}{\partial x^i} \frac{\partial V}{\partial x^j} = 0 \qquad i \neq j, \quad i, j = 1, \ldots, n,$$

from which we note that if the Hamilton-Jacobi equation is separable for a Hamiltonian of the form $H = \frac{1}{2} g^{ij} p_i p_j + l^i p_i + V$ then the same holds for the 'geodesic' Hamiltonian $H = g^{ij} p_i p_j$. The solution of the separation of variables problem for the 'geodesic' Hamilton-Jacobi equation

$$H = \tfrac{1}{2} g^{ij} p_i p_j = E \tag{2.35}$$

then becomes the crucial problem.

In order to analyse separable solutions of (1.3) where $(g^{ij})_{n \times n}$ is positive definite, Levi Civita distinguishes two types of coordinates. He does this as follows; putting

$$\sigma_{ij} = - \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial x^j} + g^{ij} \frac{\partial H}{\partial x^j} \tag{2.36}$$

then conditions (2.30) become (for i fixed)

$$\frac{\partial H}{\partial p_i} \left(\frac{\partial H}{\partial p_i} \frac{\partial^2 H}{\partial x^i \partial x^j} - \frac{\partial H}{\partial x^j} \frac{\partial^2 H}{\partial x^i \partial p_j} \right) + \frac{\partial H}{\partial x^i} \sigma_{ij} = 0. \tag{2.37}$$

Here $\partial H / \partial p_i$ is a linear form in the p_i and $\partial H / \partial x^i$ and σ_{ij} are quadratic forms. In order that (2.37) be an identity, one of the two latter expressions must be divisible by $\partial H / \partial p_i$. If $\partial H / \partial x^i$ is divisible by $\partial H / \partial p_i$ then

$$\frac{\partial g_{jl}}{\partial x^i} = 0, \qquad j, l \neq i. \tag{2.38}$$

If σ_{ij} is divisible by $\partial H / \partial p_i$ then

$$g^{ij} \frac{\partial g_{rs}}{\partial x^j} = 0, \tag{2.39}$$

$$\sum_{l=1}^{n} g^{il} \frac{\partial g_{rl}}{\partial x^j} - g^{ij} \frac{\partial g_{rj}}{\partial x^j} = 0,$$

$$\sum_{l=1}^{n} g^{il} \frac{\partial g_{jl}}{\partial x^j} - \tfrac{1}{2} g^{ij} \frac{\partial g_{jj}}{\partial x^j} = 0.$$

The two possibilities for divisibility by $\partial H / \partial p_i$ form the basis of Levi-Civita's classification of types of coordinate. The coordinate x^i is called a first class coordinate if $\partial H / \partial x^i$ is divisible by $\partial H / \partial p_i$, otherwise it is said to be a second class coordinate. We adopt the convention of denoting first class variables by Greek indices $(\sigma, \beta, \ldots)$ and second class variables by $(a, b, \ldots)$. Levi-Civita dealt with two cases:

(i) $n = 2$, in which he showed that one obtained the list due to Stäckel;

(ii) the case in which all coordinates are first class. In this case the space can be shown to be Euclidean.

Proof: If each coordinate is of first kind then

$$\frac{\partial g_{jl}}{\partial x^i} = 0, \qquad j, l \neq i, \tag{2.40}$$

and we have

$$\left\{ {ij \atop s} \right\} = \sum_{r=1}^{n} g^{sr} [ij; r] \tag{2.41}$$

which relates Christoffel symbols of the first kind $[ij, r]$ to symbols of the second kind. As

$$[ij, r] = \tfrac{1}{2}\left[\frac{\partial g_{ir}}{\partial x^j} + \frac{\partial g_{jr}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^r} \right], \qquad (2.42)$$

we see from (2.41) that $\{{}^{i\,j}_{s}\} = 0$ if $i \neq j$. Using (2.31),

$$-\frac{(\partial H/\partial x^i)}{(\partial H/\partial p_i)} = \frac{dp_i}{dx^i} = \sum_{s=1}^{n} \{{}^{i\,i}_{s}\} p_s \qquad (2.43)$$

in addition to $dp_i/dx^j = 0$. If we differentiate, we deduce that

$$R_{isij} = \frac{\partial}{\partial x^j}\{{}^{i\,i}_{s}\} + \{{}^{i\,i}_{j}\}\{{}^{j\,j}_{s}\} = 0. \qquad (2.45)$$

Here R_{isij} is the only component of the Riemannian curvature tensor which is not already zero. Consequently R_{ijkl} has all components zero and the underlying space is Euclidean. Cartesian coordinates $y^i(x^1, \ldots, x^n)$ can be obtained by solving the equations

$$y^{(r)}_{;ij} = \frac{\partial^2 y^{(r)}}{\partial x^i \partial x^i} - \sum_{s=1}^{n} \{{}^{i\,j}_{s}\} \frac{\partial y^{(r)}}{\partial x^s} = 0 \qquad i \neq j;\ r = 1, \ldots, n. \qquad (2.45)$$

These equations are clearly equivalent to

$$\frac{\partial^2 y^{(r)}}{\partial x^i \partial x^j} = 0 \qquad (2.46)$$

and consequently

$$y^r = \sum_{i=1}^{n} X_i^{(r)}(x^i), \qquad (2.47)$$

where each of the $X_i^{(r)}$ functions depends on the x^i coordinate only. The corresponding infinitesimal distance is $ds^2 = \Sigma_{r=1}^{n}(dy^r)^2$.

We see from our earlier discussions that this type of coordinate system is the natural generalization of systems of type (2.8) III in Stäckel's list for $n = 2$. Again, if we were to extend the notion of 'equivalence' of separable systems we would not really wish to distinguish this system from cartesian coordinates. Dall'Acqua [12]

extended the application of Levi Civita's integrability conditions to three dimensions. If we distinguish coordinate types by the indices $(n_1, n-n_1)$, where n_1 is the number of first class coordinates, then the Dall'Acqua solution produced a list of four metric types:

$$\text{I} \qquad (3,0), \quad ds^2 = \sum_{r,s=1}^{3} (a_r a_s + b_r b_s + c_r c_s)\, dx^r dx^s, \qquad (2.48)$$

$$\partial a_i / \partial x^j = 0;\ i \neq j;\ i, j = 1, 2, 3.$$

This is the maximal (or geodesic) case treated by Levi Civita and is accordingly 'equivalent' to the choice of cartesian coordinates

$$y^j = \sum_{i=1}^{3} \int \lambda_i dx^i$$

where $\lambda = a, b, c$ when $j = 1, 2, 3$ respectively. The corresponding metric is

$$ds^2 = \sum_{i=1}^{3} (dy^i)^2 .$$

$$\begin{aligned} \text{II} \qquad (2,1) \qquad ds^2 &= (a_3 + l_1 e_3 + l_1^2 b_3)(dx^1)^2 \\ &+ (m_2^2 a_3 + 2m_2 e_3 + b_3)(dx^2)^2 + (dx^3)^2 \\ &+ 2(m_2 a_3 + l_1 b_3 + (1 + l_1 m_2) e_3)\, dx^1 dx^2 \\ &+ 2(c_3 + m_2 s_3)\, dx^2 dx^3 + 2(l_1 c_3 + s_3)\, dx^1 dx^3. \end{aligned} \qquad (2.49)$$

In this expression the subscripts on the functions denote variable dependence, e.g. $\partial a_1/\partial x^j = 0$ unless $j = 1$ etc.

This metric can be put into a much more transparent form. If we change variables according to

$$y^1 = x^1 + \int m_2 dx^2, \qquad y^2 = x^2 + \int l_1 dx^1, \qquad y^3 = x^3$$

then

$$ds^2 = a_3(dy^1)^2 + b_3(dy^2)^2 + 2e_3 dy^1 dy^2 + 2c_3 dy^2 + 2s_3 dy^1 dy^3 .$$

This change of variables relates to two 'equivalent' coordinate systems as it did in the two-dimensional case for type II coordinates in Stäckel's list.

This can be seen from the observation that solutions of the Hamilton-Jacobi equation in the coordinates y^i are of the form $W = c_1 y^1 + c_2 y^2 + W_3(y^3)$. Consequently any new set of coordinates given by

$$y^i = \sum_{j=1}^{3} X_j^{(i)}(x^j), \qquad (i = 1, 2), \qquad y^3 = F(x^3)$$

would give rise to essentially the same separable solutions of the Hamilton-Jacobi equation. We shall therefore regard coordinate systems related in this way as 'equivalent'. We observe here that first class coordinates relate to the existence of an equivalent set of ignorable variables. (Recall that a variable x^1 is ignorable if p_1 is a linear first integral of the geodesic equations, i.e. $[H, p_1] = 0$.) In Chapter 3, this relationship is made precise in a theorem due to Benenti, who showed that the first class coordinates are always equivalent to a choice of equivalent ignorable variables.

III $\quad (1,2) \quad$ $$ds^2 = \frac{a_1 - b_2}{c_1 - d_2}\Big[(\ell_1^2 + c_1 - d_2)(dx^1)^2 + (m_2^2 + c_1 - d_2)(dx^2)^2 + (dx^3)^2 + 2\ell_1 m_2 dx^1 dx^2 + 2m_2 dx^2 dx^3 + 2\ell_1 dx^1 dx^3\Big], \qquad (2.50)$$

subscripts on the functions having the same significance as in type II coordinates.

This metric is 'equivalent', via the change of variables $y^i = x^i$ $(i = 1, 2)$, $y^3 = x^3 + \int m_2 dx^2 + \int \ell_1 dx^1$, to the orthogonal metric

$$ds^2 = (a_1 - b_2)\Big[(dy^1)^2 + (dy^2)^2 + \frac{1}{(c_1 - d_2)}(dy^3)^2\Big]$$

which is seen to be in Stäckel form with Stäckel matrix

$$\tilde{S} = \begin{bmatrix} a_1 & -c_1 & 1 \\ -b_2 & d_2 & -1 \\ 0 & 1 & 0 \end{bmatrix} .$$

IV $\quad (0,3) \quad$ $$ds^2 = Q\Big[\frac{(dx^1)^2}{(q_2 - q_3)} + \frac{(dx^2)^2}{(q_3 - q_1)} + \frac{(dx^3)^2}{(q_1 - q_2)}\Big] \qquad (2.51)$$

where $Q = r_1(q_2 - q_3) + r_2(q_2 - q_1) + r_3(q_1 - q_2)$. This metric is orthogonal and already in Stäckel form with Stäckel matrix

$$\tilde{S} = \begin{bmatrix} r_1 & 1 & q_1 \\ r_2 & 1 & q_2 \\ r_3 & 1 & q_3 \end{bmatrix}$$

For product separation of the Helmholtz equation

$$(\Delta_n + V)\psi = \sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\Big(\sqrt{(g)}\, g^{ij} \frac{\partial}{\partial x^j}\Big)\psi + V\psi = \lambda\psi ,$$

Robertson [13] obtained the first definitive result concerning the conditions under which this equation admits a separation of variables by means of a solution of the form $\psi = \Pi_{i=1}^{n} \psi_i(x^i, \underline{c})$.

<u>Theorem</u> (Robertson condition). The Helmholtz equation $\Delta_n \psi + V\psi = E\psi$ is separable in an orthogonal coordinate system x^i if and only if the components g^{ii} and potential V satisfy the requirements of Stäckel's theorem and the additional 'Robertson condition'

$$\frac{\Pi_{i=1}^{n} S^{i1}}{S^{n-2}} = \prod_{i=1}^{n} f_i(x^i) . \qquad (2.52)$$

<u>Proof:</u> To achieve separation, the quotient of the coefficients of $(\partial/\partial x^i)^2$ and $\partial/\partial x^i$ should be a function of x^i alone, i.e.

$$\frac{\partial}{\partial x^i} \log(\sqrt{(g)}\, g^{ii}) = F_i(x^i), \qquad i = 1, \ldots, n . \qquad (2.53)$$

These conditions are equivalent to the Robertson condition. The remainder of the argument is directly analogous to that used to prove Stäckel's theorem, as the equation can now be written

$$\sum_{i=1}^{n} g^{ii} H_i(\psi_i, x^i; \underline{c}) = E - V \qquad (2.54)$$

where

$$H_i(\psi_i, x^i; \underline{c}) = \psi_i^{-1}\left[\left(\frac{\partial}{\partial x^i}\right)^2 + F_i(x^i)\frac{\partial}{\partial x^i}\right]\psi_i . \tag{2.55}$$

In one of the key papers on the subject Eisenhart [14] took up the question of orthogonal coordinates for which the Hamilton-Jacobi equation separates and investigated the geometric significance of the Robertson condition. We summarize and discuss his results below. The proofs are given in detail in the appendix.

Theorem (Eisenhart). Let $H = \Sigma^n_{i,j=1} g^{ij} p_i p_j$ be the fundamental quadratic form on a Riemannian manifold M. The necessary and sufficient conditions that there exists a local coordinate system $\{y^i\}$ such that H is in Stäckel form $H = \Sigma^n_{i=1} \frac{S^{i1}}{S} p_i^2$ are:

(i) The equations of the geodesics admit $n - 1$ independent (linearly) quadratic first integrals $A_a = \mathfrak{A}^{ij}_{(a)} p_i p_j$, $a = 1, \ldots, n-1$, which together with H form a complete involutive set satisfying

$$[A_a, A_b] = 0, \quad [A_a, H] = 0, \quad a, b = 1, \ldots, n. \tag{2.56}$$

(ii) The roots ρ^j_a, $j = 1, 2, \ldots, n$, $a = 2, \ldots, n$, of the characteristic equations

$$\det(\mathfrak{A}^{ij}_{(a)} - \rho_b g^{ij}) = 0, \quad a = 2, \ldots, n, \tag{2.57}$$

of these first integrals are simple and satisfy

$$\det|\rho^{\alpha}_i - \rho^{\alpha}_j| \neq 0 \tag{2.58}$$

where i is fixed and $\alpha = 2,\ldots,n$; $j = 1, \ldots, n$, $j \neq 1$.

(iii) The vector fields $\lambda_{(h)i}$, $h = 1, \ldots, n$, determined from these first integrals via

$$(\mathfrak{A}^{ij}_{(a)} - \rho_b g^{ij})\lambda_{(b)j} = 0 \tag{2.59}$$

should be normal and be the same vector fields for the first integrals

A_a. Furthermore, the hypersurfaces defined by these vector fields may be taken as parametric. The coordinate system thus defined is such that the matrices $(g^{ij})_{n\times n}$, $(\mathfrak{A}^{ij}_{(a)})_{n\times n}$ can all be taken to be diagonal.

What Eisenhart's theorem gives is a geometric characterization of Stäckel form. Given a suitable involutive family $\{H, A_1, \ldots, A_{n-1}\}$, from purely algebraic criteria one can determine whether there are separable coordinates. Implicit in this result is the determination of a suitable set of separable coordinates $\{y^i\}$. The condition of normality is crucial, for this is a requirement that each of the quadratic forms $A_a = \mathfrak{A}^{ij}_{(a)} p_i p_j$, $a = 1, \ldots, n-1$, can be simultaneously diagonalized in the given coordinate system. This result and its subsequent development provided the successful development of the solution of problem II of Chapter 1. As a useful corollary to this theorem, Eisenhart showed:

Corollary 1 (Eisenhart): The necessary and sufficient conditions that $H = \Sigma^n_{i=1} H_i^{-2} p_i^2$ is in Stäckel form are

$$\frac{\partial^2}{\partial x^k \partial x^j}\log H_i^2 - \frac{\partial}{\partial x^j}\log H_i^2 \frac{\partial}{\partial x^k}\log H_i^2 \tag{2.60}$$

$$+ \frac{\partial}{\partial x^j}\log H_i^2 \frac{\partial}{\partial x^k}\log H_j^2 + \frac{\partial}{\partial x^k}\log H_i^2 \frac{\partial}{\partial x^j}\log H_k^2 = 0,$$

$$k, j \neq i; \quad i, j, k = 1, \ldots, n.$$

An additional corollary enabled Eisenhart to characterize geometrically the Robertson condition:

Corollary 2 (Eisenhart): The necessary and sufficient conditions that the Robertson condition (2.49) holds for a given orthogonal coordinate system $\{x^i\}$ for which $H = \Sigma^n_{i=1} H_i^{-2} p_i^2$ is in Stäckel form are that $R_{ij} = 0$, $i \neq j$, i.e., the Ricci tensor R_{ij} is diagonal.

In addition to these results, Eisenhart was able to give a complete classification of all inequivalent orthogonal separable coordinates on E_3 and S_3. These coordinate systems can be found in many standard reference works [1].

We note that, because of Corollary 2, any orthogonal separable system which provides an additive separation of variables in a space of constant curvature for the Hamilton-Jacobi equation also allows a product separation of variables for the corresponding Helmholtz equation.

Two good reviews on the subject of separation of variables from a historical point of view are those of Prange [15] and Haux [16].

3 Separation of variables on the n–sphere S_n

1. MATHEMATICAL PRELIMINARIES

In Chapter 1 we gave a suitable definition of the n sphere S_n. As we have seen in Chapter 2, coordinates that occur in the separation of variables for Riemannian manifolds which are positive definite are of two types. Benenti [17] has made a complete analysis of problem III of Chapter 1 for such manifolds and proved the following theorem.

Theorem 3.1 (Benenti): Let M be a positive definite Riemannian manifold of dimension n for which the Hamilton-Jacobi equation

$$H = \sum_{i,j=1}^{n} g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j} = E \qquad \text{I}$$

admits an additive separation of variables in a system of coordinates $\{y^i\}$. Then there exists a system of coordinates $\{x^i\}$ 'equivalent' to $\{y^i\}$ such that the contravariant metric tensor has the form

$$(g^{ij})_{n\times n} = \begin{array}{c c} & \begin{array}{cc} \longleftarrow n_1 \longrightarrow & \longleftarrow n_2 \longrightarrow \end{array} \\ \begin{array}{c} \updownarrow n_1 \\ \updownarrow n_2 \end{array} & \left[\begin{array}{c|c} \delta^{ab} H_a^{-2} & \bigcirc \\ \hline \bigcirc & g^{\alpha\beta} \end{array}\right] \end{array} \qquad (3.1)$$

where the functions H_a^{-2} and $g^{\alpha\beta}$ can be expressed as

$$H_a^{-2} = \frac{S^{a1}}{S}, \quad g^{\alpha\beta} = \sum_b A_b^{\alpha\beta}(x^b) \frac{S^{b1}}{S}, \qquad (3.2)$$

i.e., there exists a Stäckel matrix $\tilde{S} = (S_{ab}(x^b))_{n_1\times n_1}$ depending only on the variables $\{x^a\}$ such that the H_a^{-2} are in Stäckel form. Here $n_1 = \dim\{x^a\}$ is the number of second class coordinates, in the nomenclature of Levi Civita.

The variables x^α are such that $\partial g^{ij}/\partial x^\alpha = 0$ for all i, j, and they correspond to first class coordinates.

A few comments on this theorem are in order:

(i) The coordinates $\{y^i\}$ and $\{x^i\}$ are in general related by equations of the type

$$x^a = f(y^a)$$

$$x^\alpha = \sum_\beta X^\alpha_\beta(y^\beta) + \sum_b a^\alpha_b(y^b). \tag{3.3}$$

(ii) Clearly, the coordinates $\{x^i\}$ are not chosen to be unique, in order that the contravariant components of the metric tensor may have the form (3.1). If $\{x^{i'}\}$ is another such system then, in general,

$$x^{a'} = h(x^a)$$

$$x^{\alpha'} = \sum_\beta a^\alpha_\beta x^\beta, \quad \det(a^\alpha_\beta) \neq 0. \tag{3.4}$$

Coordinate systems related in this way will of course be regarded as 'equivalent'.

(iii) The Hamilton-Jacobi equation I admits a separable solution of the form

$$W = \sum_a W_a(x^a) + \sum_\alpha c_\alpha x^\alpha$$

with the separation equations

$$\left(\frac{dW_a}{dx^a}\right)^2 + \sum_{\alpha,\beta} A^{\alpha\beta}_a c_\alpha c_\beta = \sum \lambda_b S_{ab} \tag{3.5}$$

with $\lambda_1 = E$.

(iv) The variables x^α are the ignorable variables one encounters in classical mechanics [2], [18]. In fact, $[p_\alpha, H] = 0$. Each p_α corresponds to a linear first integral (Lie symmetry) of the geodesics. Furthermore, $[p_\alpha, p_\beta] = 0$. This is an important observation. For the general form of contravariant metric tensor (3.1), this implies that the underlying Riemannian manifold M_n admits an abelian algebra (under the Poisson bracket) of first order Lie symmetries of dimension at least $n_2 = \dim\{x^\alpha\}$, $n_1 + n_2 = n$. In the case of the n-sphere S_n, the algebra of Lie symmetries has dimension $\frac{1}{2}n(n+1)$ and basis

$$I_{ab} = s_a p_{s_b} - s_b p_{s_a}, \quad a > b, \ a, b = 1, \ldots, n+1, \tag{3.6}$$

which satisfy the commutation relations

$$[I_{ab}, I_{cd}] = \delta_{bc} I_{ad} + \delta_{ad} I_{bc} + \delta_{bd} I_{ca} + \delta_{ac} I_{db}. \tag{3.7}$$

The Lie algebra of the Lie symmetries described by the commutation relations (3.7) is that of SO(n + 1), the orthogonal group in n dimensions. The global action of these symmetries is via real orthogonal $(n+1) \times (n+1)$ matrices 0 acting on the projective coordinates $(s_1, \ldots, s_{n+1})$ via $\underset{\sim}{s} \to 0\underset{\sim}{s}$.

In our discussions of the notion of equivalence thus far, we have observed several ways in which this can occur. However, if the manifold in question admits a group of first order Lie symmetries then an additional concept of 'equivalence' must be introduced. This is essentially the notion that two coordinate systems $\{x^i\}$ and $\{y^i\}$ that are related by a group motion are not essentially different. To make this more precise, in the case of S_n consider that we have a system of coordinates $\{x^i\}$ such that the defining projective coordinates s_i are well defined functions of the $\{x^i\}$, i.e., $s = s(x^j)$. If we rotate the vector $\underset{\sim}{s}$ via an orthogonal matrix 0 then $\underset{\sim}{s}' = 0\underset{\sim}{s}$ and

$$ds^2 = d\underset{\sim}{s}' \cdot d\underset{\sim}{s}' = d\underset{\sim}{s} \cdot d\underset{\sim}{s} = g_{ij} dx^i dx^j. \tag{3.8}$$

Therefore both the choices of vectors $\underset{\sim}{s}$ in terms of the coordinates $\{x^i\}$ are indistinguishable when it comes to a discussion of their separability properties.

Chosen coordinates which are related in this way are then regarded as being 'equivalent'. This is equivalence between the specification of the projective coordinates s_i $(i = 1, \ldots, n+1)$ in terms of the separable coordinates $\{x^i\}$, there being no essential distinction made between coordinates specified by the vectors $\underset{\sim}{s}(x^i)$ and $0\underset{\sim}{s}(x^i) = \underset{\sim}{s}'(x^i)$ with 0 an orthogonal matrix.

From the form of the contravariant metric components (3.1) and the separation equations we see that we can write

$$\lambda_c = \sum_a \frac{S^{ac}}{S} p_a^2 + \sum_{\alpha,\beta} \left(\sum_b A_b^{\alpha\beta} \frac{S^{bc}}{S}\right) p_\alpha p_\beta = \sum_{j,k=1}^{n} \mathcal{A}^{jk}_{(c)} p_j p_k = A_c ,$$

$$c = 1, \ldots, n_1, \qquad (3.9)$$

where $\mathcal{A}^{jk}_{(i)} = \mathcal{A}^{kj}_{(i)}$. These quadratic functions λ_c together with the ignorable momenta p_α form a complete involutive set of constants of the motion. There is consequently no relation of the form

$$\sum_c \mu^c \lambda_c + \sum_{\alpha,\beta} \nu^{\alpha\beta} p_\alpha p_\beta = 0 \qquad (3.10)$$

$$\sum_\alpha \eta^\alpha p_\alpha = 0$$

for non-zero coefficients $\mu^c, \nu^{\alpha\beta}, \eta^\alpha$.

We also have the relations

$$[\lambda_c, \lambda_b] = 0, \quad [\lambda_c, p_\alpha] = 0, \quad [p_\alpha, p_\beta] = 0, \qquad (3.11)$$

with $\lambda_1 = H$. The quadratic forms λ_c are quadratic first integrals of the geodesic equations. (The coefficients $\mathcal{A}^{jk}_{(c)}$ are also referred to as Killing tensors [18].) We note that as the metric tensor is not orthogonal in this case, the complete integral which this coordinate system specifies has associated with it n_1 quadratic first integrals A_c and $n - n_1$ linear first integrals of the geodesics (first order Lie symmetries) p_α. Only when $n_1 = n - 1$ can all the constants of the motion be characterized by quadratic first integrals (since then the metric tensor is necessarily orthogonal).

As we wish to deal also with product separable solutions of the Helmholtz equation II, we can ask the question: how are the separation constants appearing in the separation equations to be characterized? The relevant concept here is that of symmetry operators of the Laplace

operator Δ_n. First order Lie symmetry operators are defined as partial differential operators of the form $L = \Sigma_i a^i(\partial/\partial x^i)$, for which

$$\{L, \Delta_n\}\psi = L(\Delta_n \psi) - \Delta_n(L\psi) = 0$$

for all suitably differentiable functions ψ; consequently $\{L, \Delta_n\} = 0$ is an operator identity. The $\{\ ,\ \}$ bracket is the operator commutator bracket.

On S_n the first order Lie symmetry operators form a vector space having a basis

$$\hat{I}_{ij} = s_i \frac{\partial}{\partial s_j} - s_j \frac{\partial}{\partial s_i}, \qquad i, j = 1, \ldots, n+1 . \qquad (3.12)$$

Second order symmetry operators of the Helmholtz equation are correspondingly defined as operators

$$m = \sum_{i,j} a^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_k b^k \frac{\partial}{\partial x^k}$$

for which

$$\{m, \Delta_n\} = 0 \qquad (3.13)$$

is an operator identity. If the coordinate system $\{x^i\}$ is also a separable coordinate system for the Helmholtz equation, then further restrictions must be placed on the metric coefficients.

<u>Theorem 3.2</u>: Let M be a positive definite Riemannian manifold of dimension n, for which the Helmholtz equation II admits a separation of variables in a coordinate system $\{y^i\}$. Then there exists a system of coordinates $\{x^i\}$ 'equivalent' to $\{y^i\}$ such that the contravariant metric tensor has the form (3.1) given in Benenti's theorem and

$$R_{ab} = 0 \qquad \forall a, b \neq \qquad (3.14)$$

in the coordinate system $\{x^i\}$.

This theorem follows from Benenti's theorem and the Robertson condition. We make a few pertinent comments on this result.

(i) The condition $R_{ab} = 0$ is the analogue of the Robertson condition. It is readily proved from the conditions

$$\frac{\partial}{\partial x^a} \log(\sqrt{(g)}\, g^{aa}) = F_a(x^a), \quad \forall a. \tag{3.15}$$

(ii) The use of 'equivalent' is meant in the same sense as it is for the Hamilton-Jacobi equation. This can readily be seen as follows. In the coordinate system $\{x^i\}$ the coordinates x^α are ignorable and the Helmholtz equation can be written as

$$\sum_{i,j=1}^{n} G^{ij}(x^a) \frac{\partial^2 \psi}{\partial x^i \partial x^j} + \sum_{j=1}^{n} B^j(x^a) \frac{\partial \psi}{\partial x^j} = \lambda \psi \tag{3.16}$$

Consequently we can always choose separable solutions of the form

$$\psi = \prod_a \psi_a(x^a) \exp(\sum_\alpha \nu_\alpha x^\alpha) . \tag{3.17}$$

We thus see that if a coordinate system $\{x^i\}$ is separable for both the Hamilton-Jacobi and Helmholtz equations then the same notion of equivalence applies to both these equations.

(iii) The product separable solutions of the Helmholtz equation satisfy the separation equations

$$[(\frac{d}{dx^a})^2 + F_a(x^a) \frac{d}{dx^a}]\psi_a + (\sum_{\alpha,\beta} A_a^{\alpha\beta} \nu_\alpha \nu_\beta - \sum_b \lambda_b S_{ab})\psi_a = 0, \ \forall a \tag{3.18}$$

where $\lambda_1 = E$ and

$$\frac{d\psi_\alpha}{dx^\alpha} = \nu_\alpha \psi_\alpha \tag{3.19}$$

where $\psi_\alpha = \exp(\nu_\alpha x^\alpha)$.

From these equations we see that product separable solutions of the Helmholtz equation are characterized by n_1 second order symmetry operators

$$A_b = \sum_a \frac{S^{ab}}{S} [(\frac{\partial}{\partial x^a})^2 + F_a(x^a)(\frac{\partial}{\partial x^a})]$$

$$+ \sum_{\alpha,\beta} (\sum_c A_c^{\alpha\beta} \frac{S^{cb}}{S}) \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \tag{3.20}$$

and $n - n_1$ first order Lie symmetry operators

$$L_\alpha = \frac{\partial}{\partial x^\alpha}, \tag{3.21}$$

in that the product solutions are simultaneous eigenfunctions of these operators with eigenvalues λ_b and ν_α, respectively.

For each theorem relating to separability of the Hamilton-Jacobi equation there is a corresponding result concerning the separability of the Helmholtz equation. For the Riemannian manifolds S_n, H_n and E_n, every separable coordinate system will provide a separation of variables for both the Hamilton-Jacobi equation and the corresponding Helmholtz equation. This follows from the condition

$$R_{hijk} = \varepsilon(g_{hk} g_{ij} - g_{hj} g_{ik}) \tag{3.22}$$

where $\varepsilon = 1, -1$ or 0 according to whether the manifold is S_n, H_n or E_n. From (3.22) and (3.1) it follows that $R_{ab} = 0$ for $a \neq b$.

2. SEPARATION OF VARIABLES ON S_n

The complete solution of problem I for S_n depends in a critical way on the underlying Lie algebra of $SO(n+1)$.

The following is a crucial result in the classification of separable coordinate systems on S_n:

Theorem 3.3: Let $\{x^i\}$ be a coordinate system on S_n for which the Hamilton-Jacobi equation admits a separation of variables. Then, by passing to an 'equivalent' system of coordinates if necessary, we have

$g^{ij} = \delta^{ij} H_i^{-2}$, i.e., separation of variables occurs only in orthogonal coordinates. Furthermore in terms of the standard coordinates on the sphere $s_1, \ldots, s_{n+1}$, the ignorable variables can be chosen such that

$$p_{\alpha_1} = I_{12}, \quad p_{\alpha_2} = I_{34}, \ldots, p_{\alpha_q} = I_{2q+1, 2q+2} \tag{3.23}$$

where the number of ignorable variables is q.

<u>Proof</u>: This is based on the general block diagonal form (3.1) of the contravariant metric tensor for a separable coordinate system. Any Lie symmetry of S_n is conjugate, under the action of $SO(n+1)$, to a Lie symmetry element of the form [19]

$$L = I_{12} + b_2 I_{34} + \ldots + b_\nu I_{2\nu-1, 2\nu}. \tag{3.24}$$

If this element corresponds to the ignorable variable x^{α_1}, i.e., $L = p_{\alpha_1}$, then by local Lie theory the standard coordinates on the n-sphere can be taken as

$$\begin{aligned}(s_1, \ldots, s_{n+1}) = (&\rho_1 \cos(x^{\alpha_1} + w_1),\ \rho_1 \sin(x^{\alpha_1} + w_1),\\ &\rho_2 \cos(b_2 x^{\alpha_2} + w_2),\ \rho_2 \sin(b_2 x^{\alpha_2} + w_2), \ldots, \rho_\nu \cos(b_\nu x^{\alpha_1} + w_\nu),\\ &\rho_\nu \sin(b_\nu x^{\alpha_1} + w_\nu),\ s_{2\nu+1}, \ldots, s_{n+1})\end{aligned} \tag{3.25}$$

where $\rho_1^2 + \ldots + \rho_\nu^2 + s_{2\nu+1}^2 + \ldots + s_{n+1}^2 = 1$. The infinitesimal distance then has the form

$$\begin{aligned}ds^2 = d\rho_1^2 + \ldots + d\rho_\nu^2 + \rho_1^2 (dx^{\alpha_1} + dw_1)^2 + \ldots + \rho_\nu^2 (b_\nu dx^{\alpha_1} + dw_\nu)^2\\ + ds_{2\nu+1}^2 + \ldots + ds_{n+1}^2.\end{aligned} \tag{3.26}$$

If there is only one ignorable variable then the coordinate system must be orthogonal and this is only possible if $b_2 = \ldots = b_\nu = 0$, i.e., $p_{\alpha_1} = I_{12}$. Indeed, the requirement that the contravariant metric have the form (3.1) (orthogonal in this case) is that

$$-dw_1 = \sum_{j=2}^{\nu} \frac{\rho_j^2}{\rho_1^2} b_j dw_j. \tag{3.27}$$

Since the differentials $d\rho_j$, dw_j $(j \geq 2)$, must be independent and the only condition on ρ_1^2 is $\Sigma_{i=1}^{\nu} \rho_i^2 + s_{2v+1}^2 + \ldots + s_{n+1}^2 = 1$, the condition $d^2 w_1 = 0$ implies $b_j = 0$, $j = 2, \ldots, \nu$, and $dw_1 = 0$. We can then take the constant $w_1 = 0$ by suitably redefining α_1. The theorem is proved in this case.

Now suppose there are $q > 1$ ignorable variables. The Lie symmetries p_{α_i}, $i = 1, \ldots, q$, must form an involutive set. It follows from the spectral theorem [20] for commuting skew adjoint matrices that, for each i, p_{α_i} has a representation of the form

$$p_{\alpha_i} = b_1^i I_{12} + b_2^i I_{34} + \ldots + b_{\nu_i}^i I_{2\nu_i - 1, 2\nu_i} \tag{3.28}$$

for $i = 2, \ldots, q$. In fact we can assume

$$p_{\alpha_i} = I_{2i-1, 2i} + \sum_{l=q+1}^{N} b_l^i I_{2l-1, 2l}, \qquad i = 1, \ldots, q, \tag{3.29}$$

for some $N \geq q + 1$. The projective coordinates on the sphere then have the form

$$\begin{aligned}(s_1, \ldots, s_{n+1}) = (&\rho_1 \cos(x^{\alpha_1} + w_1),\ \rho_1 \sin(x^{\alpha_1} + w_1),\\ &\rho_q \cos(x^{\alpha_q} + w_q),\ \rho_q \sin(x^{\alpha_q} + w_q),\\ &\rho_{q+1} \cos\Big(\sum_{i=1}^{q} b_{q+1}^i x^{\alpha_1} + w_{q+1}\Big),\\ &\rho_{q+1} \sin\Big(\sum_{i=1}^{q} b_{q+1}^i x^{\alpha_i} + w_{q+1}\Big),\\ &\rho_N \sin\Big(\sum_{i=1}^{q} b_N^i x^{\alpha_i} + w_N\Big),\ s_{2N+1}, \ldots, s_{n+1}).\end{aligned} \tag{3.30}$$

We now make the crucial requirement that the ignorable variables x^{α_i}, $i = 1, \ldots, q$, are part of a separable coordinate system. If we compute the covariant metric, it should be in block diagonal form with respect to the two classes of variables. Just as in the case $q = 1$, this is only possible if

$b^i_l = 0$, $i = 1, \ldots, q$, $l = q+1, \ldots, N$, and $dw_i = 0$, $1 \le i \le q$. We can therefore assume that $L_1 = I_{12}$, $L_2 = I_{34}$, $\ldots$, $L_q = I_{2q-1,2q}$; the ignorable coordinates α_i can then always be chosen such that $w_i = 0$, $1 \le i \le q$, and the system is orthogonal.

This theorem enables us to bring to bear Eisenhart's results on orthogonal systems of Stäckel type. Our problem reduces to the enumeration of all orthogonal separable coordinate systems. We use an inductive procedure such that, given all separable systems for S_j, $j < n$, we can give the rules for construction of all systems on S_n.

If $\{x^i\}$ is an orthogonal coordinate system with infinitesimal distance $ds^2 = \Sigma^n_{i=1} H^2_i (dx^i)^2$ then the conditions necessary and sufficient that the space be S_n are,

$$\text{(i)} \quad R_{ijji} = -H^2_i H^2_j, \qquad i \neq j, \tag{3.31}$$

$$\text{(ii)} \quad R_{hiik} = 0, \qquad i \neq h \neq k.$$

Furthermore, the corresponding Hamiltonian $H = \Sigma^n_{i=1} H_i^{-2} p_i^2$ must be in Stäckel form as is proved in the Appendix. Eisenhart [14] has shown that the conditions (3.31) (ii) and the requirement of Stäckel form are equivalent to the equation (A.43), i.e.

$$\frac{\partial}{\partial x^j} \log H^2_i \frac{\partial}{\partial x^k} \log H^2_i - \frac{\partial}{\partial x^j} \log H^2_i \frac{\partial}{\partial x^k} \log H^2_j$$

$$- \frac{\partial}{\partial x^k} \log H^2_i \frac{\partial}{\partial x^j} \log H^2_k = 0,$$

i, j, k pairwise distinct. The metric for a separable system can be written in the form (A.48)

$$g_{ii} = H^2_i = X_i \prod_{j\neq i} (\sigma_{ij} + \sigma_{ji}), \qquad i = 1, \ldots, n,$$

where X_i, σ_{ij} are functions of x^i at most.

There are various possibilities for the functions σ_{ij}. If all the functions σ_{ij} are such that $\sigma'_{ij} \neq 0$ then Eisenhart has shown that the metric coefficients have the form (A.57)

$$g_{ii} = H^2_i = X_i \prod_{j\neq i} (\sigma_i - \sigma_j)$$

where $\sigma_i = \sigma_i(x^i)$ and $\sigma'_i \neq 0$. This metric will be the basic building block on which we can formulate our inductive construction. Without loss of generality we can redefine variables $\{x^i\}$ in such a way that $\sigma_i = x^i$, i.e.,

$$H^2_i = X_i \prod_{j\neq i} (x^i - x^j). \tag{3.32}$$

The conditions (3.31) (i) then amount to

$$\Big[\prod_{l\neq j} (x^j - x^l)\Big]^{-1} \Big\{\frac{-2}{(x^i - x^j)^2}\Big(\frac{1}{X_j}\Big) + \frac{-1}{(x^i - x^j)}\Big(\frac{1}{X_j}\Big)'\Big\} \tag{3.33}$$

$$+ \Big[\prod_{l\neq i} (x^i - x^l)\Big]^{-1} \Big\{\frac{-2}{(x^i - x^j)^2}\Big(\frac{1}{X_i}\Big) + \frac{-1}{(x^j - x^i)}\Big(\frac{1}{X_i}\Big)'\Big\}$$

$$+ \sum_{l\neq i,j} \frac{1}{X_l (x^l - x^i)(x^l - x^j) \prod_{k\neq l} (x^l - x^k)} = -4.$$

These equations have the solution

$$\Big(\frac{1}{X_i}\Big)^{(n+1)} + 4(n+1)! = 0, \qquad i = 1, \ldots, n, \tag{3.34}$$

i.e.,

$$\frac{1}{X_i} = -4(x^i)^{n+1} + \sum_{l=0}^{n} a_l (x^i)^{n-l} = f(x^i).$$

The function $f(x)$ can also be written

$$f(x) = -4 \prod_{j=1}^{n+1} (x - e_j). \tag{3.35}$$

There are two requirements to determine which metrics of this type occur on S_n: (i) the metric must be positive definite; (ii) the variables x^i should vary in such a way that they correspond to a coordinate patch which is compact. There is a unique solution to these requirements: the x^i, e_i should satisfy

$$e_1 < x^1 < e_2 < \ldots < e_n < x^n < e_{n+1}. \tag{3.36}$$

These are ellipsoidal coordinates on the n-sphere S_n. They can be related to the coordinates $\{s_j\}$ via a standardized choice

$$s_j^2 = \frac{\Pi_{i=1}^n (x^i - e_j)}{\Pi_{j\neq i}(e_i - e_j)}, \quad j = 1, \ldots, n+1. \tag{3.37}$$

These systems are the basic building blocks for separable coordinate systems on real spheres. To complete the analysis of possible orthogonal separable systems we need to consider the case when some of the σ_{ij} functions are constants. If $\sigma_{ij} = a_{ij}$ (const) there are four possibilities:

$$\begin{aligned}
&\text{(i)}\ \sigma_{ij} = a_{ij},\quad \sigma_{ji} = a_{ji},\quad \sigma_{ik} = a_{ik},\quad \sigma_{jk} = a_{jk}\\
&\text{(ii)}\ \sigma_{ij} = a_{ij},\quad \sigma_{ji} = a_{ji},\quad \sigma_{ik} = a_{ik},\quad \sigma_{ki} = a_{ki}\\
&\text{(iii)}\ \sigma_{ij} = a_{ij},\quad \sigma_{ik} = a_{ik},\quad \sigma_{ji} = a_{ji}\sigma_j,\ \sigma_{jk} = a_{jk}\sigma_j\\
&\qquad \sigma_{ki} = a_{ki}\sigma_k,\ \sigma_{kj} = a_{kj}\sigma_k\\
&\text{(iv)}\ \sigma_{ij} = a_{ij},\quad \sigma_{kj} = a_{kj},\quad \sigma_{ji} = a_{ji}\sigma_j,\quad \sigma_{jk} = a_{jk}\sigma_j\\
&\qquad a_{ji}a_{kj} - a_{jk}a_{ij} = 0
\end{aligned} \tag{3.38}$$

where σ_j is a function of x^j only and i, j, k are pairwise distinct. If we fix i and j then for k values corresponding to cases (i)-(iii), $\sigma_{ik} = a_{ik}$. To examine how the inductive process works, let us take $\sigma_{1\ell} = a_{1\ell}$ for $\ell = k+1, \ldots, n$ and $\sigma'_{1j} \neq 0$ for $j = 2, \ldots, k$. Then we have

$$\sigma_{j\ell} = a_{j\ell},\quad \sigma_{\ell 1} = a_{\ell 1}\sigma_\ell,\quad \sigma_{\ell j} = a_{\ell j}\sigma_\ell$$

$$a_{\ell 1}a_{j\ell} - a_{\ell j}a_{1\ell} = 0 \quad \text{for } \ell = k+1, \ldots, n,\quad j = 2, \ldots, k.$$

Assuming that $a_{\ell j} \neq 0$ for $\ell = k+1, \ldots, n$, $j = 2, \ldots, k$, we find the

metric coefficients have the form

$$H_i^2 = [X_i \prod_{j\neq i} (\sigma_{ij} + \sigma_{ji})][\prod_{\substack{\ell=k+1\\ 1\le j\le k}}^{n} (a_{i\ell} + a_{\ell i}\sigma_\ell)],\quad i = 1, \ldots, k \tag{3.39}$$

$$H_\ell^2 = X_\ell \prod_{\substack{m\neq\ell\\ m\geq k+1}} (\sigma_{\ell m} + \sigma_{m\ell}) \qquad \ell = k+1, \ldots, n.$$

Let us assume that no further functions σ_{ij}, $\sigma_{\ell m}$ are constants. Then we can take the metric coefficients as

$$H_i^2 = [X_i \prod_{j\neq i} (x^i - x^j)](\prod_{\ell=k+1} \sigma_\ell),\quad H_\ell^2 = [X_\ell \prod_{m\neq\ell} (x^\ell - x^m)]. \tag{3.40}$$

The conditions $R_{k\ell\ell k} = -H_k^2 H_\ell^2$ are equivalent to (3.33), (3.34) with $i = k+1, \ldots, n$ and $n - k = n'$. Putting $\tilde{H}_i^2 = [X_i \Pi_{j\neq i}(x^i - x^j)]$, the conditions $R_{ijji} = -H_i^2 H_j^2$ and $R_{i\ell\ell i} = -H_i^2 H_\ell^2$ are equivalent to

$$\tilde{H}_i^{-2}\tilde{H}_j^{-2}\tilde{R}_{ijji} + (\prod_{\ell=k+1}^{n} \sigma_\ell)[\sum_{\ell'=k+1}^{n} \frac{1}{4H_{\ell'}^2}(\frac{\sigma'_\ell}{\sigma'_\ell})^2 + 1] = 0 \tag{3.41}$$

$$2\frac{\sigma''_\ell}{\sigma_\ell} - (\frac{\sigma'_\ell}{\sigma_\ell})^2 - (\frac{\sigma'_\ell}{\sigma_\ell})[\frac{\partial}{\partial x^\ell}\log H_\ell^2 + H_\ell^2 \sum_{m\neq\ell} \frac{1}{H_m^2(x^\ell - x^m)}]$$

$$= -4H_\ell^2 \tag{3.42}$$

where $\tilde{R}_{ijji}$ is the Riemann curvature tensor for the Riemannian manifold with infinitesimal distance $ds^2 = \Sigma_{i=1}^k \tilde{H}_i^2 (dx^i)^2$. These equations are satisfied if and only if

$$\frac{1}{X_\ell} = -4 \prod_{m=1}^{n-k+1} (x^\ell - f_m),\qquad \ell = k+1, \ldots, n \tag{3.43}$$

and

$$\sigma_\ell = \frac{(x^\ell - f_{n-k+1})}{(f_{\ell-k} - f_{n-k+1})},$$

where we take $f_1 < f_2 < \dots < f_{n-k+1}$. The remaining condition then is $\tilde{R}_{ijji} = -\tilde{H}_i^2 \tilde{H}_j^2$ so that

$$\frac{1}{X_i} = -4 \prod_{j=1}^{k+1} (x^i - e_j) . \tag{3.44}$$

The coordinates on S_n can be taken as

$$(s_1, \dots, s_{n+1}) = (u_1 v_1, \dots, u_1 v_{k+1}, u_2, \dots, u_{n-k+1}) \tag{3.45}$$

where $\Sigma_{i=1}^{k+1} v_i^2 = 1$, $\Sigma_{\ell=1}^{n-k+1} u_\ell^2 = 1$ and

$$v_j^2 = -\frac{\Pi_{i=1}^k (x^i - f_j)}{\Pi_{j \neq i} (f_i - f_j)} \tag{3.46}$$

$$u_m^2 = -\frac{\Pi_{\ell=k+1}^n (x^\ell - e_m)}{\Pi_{n \neq m} (e_n - e_m)} \tag{3.47}$$

The infinitesimal distance has the form

$$ds_0^2 = ds_1^2 \left[\frac{\Pi_{\ell=k+1}^n (x^\ell - f_{n-k+1})}{\Pi_{m \neq n-k+1} (f_m - f_{n-k+1})} \right] + ds_2^2 \tag{3.48}$$

where

$$ds_1^2 = -\frac{1}{4} \sum_{i=1}^k \left[\frac{\Pi_{j \neq i} (x^i - x^j)}{\Pi_{j=1}^{k+1} (x^i - e_j)} \right] (dx^i)^2 \tag{3.49}$$

$$ds_2^2 = \frac{-1}{4} \sum_{\ell=k+1}^n \left[\frac{\Pi_{m \neq \ell} (x^\ell - x^j)}{\Pi_{m=1}^{n+k+1} (x^\ell - f_m)} \right] (dx^\ell)^2 , \tag{3.50}$$

$$m = k+1, \dots, n, \qquad j = 1, \dots, k.$$

The choice of embedding of the sphere S_k in the n-sphere S_n given by (3.45) is not, of course, unique. However we will make the convention of taking this choice of imbedding (other choices would correspond to applying an orthogonal matrix to the vector $\underset{\sim}{s}$).

Now suppose one of the constants $a_{\ell j} = 0$ for some fixed ℓ and j.

Then from the relations

$$a_{\ell 1} a_{j\ell} - a_{\ell j} a_{1\ell} = 0 \tag{3.51}$$

we have $a_{\ell 1} = 0$ and consequently $a_{\ell i} = 0$ for $i = 1, \dots, k$. This implies that σ_ℓ does not appear in H_i^2 $i = 1, \dots, k$.

Referring to the curvature equation $R_{i\ell\ell i} = -H_i^2 H_\ell^2$, we see that it cannot be satisfied if $\sigma_{\ell i} = a_{\ell i} \sigma_\ell = 0$ as this would imply $-4H_\ell^2 = 0$. Thus $a_{\ell j} \neq 0$ for each ℓ, j. Recall here that we have assumed that none of the functions $\sigma_{ij}(i, j = 1, \dots, k; i \neq j)$, $\sigma_{\ell m}(\ell, m = k+1, \dots, n; \ell \neq m)$ is a constant. Let us now push this process one step further: let $\sigma_{k+1,s} = a_{k+1,s}$ for $s = p+1, \dots, n$ and $\sigma'_{k+1,s} \neq 0$ for $s = k+1, \dots, p$. Then applying the same arguments as previously, we see that the metric coefficients H_ℓ^2, $\ell = k+1, \dots, n$, can be brought to the form

$$H_\ell^2 = X_\ell \Big[\prod_{\substack{m \neq \ell \\ k+1 \le \ell \le p}} (\sigma_{\ell m} + \sigma_{m\ell}) \Big] \Big[\prod_{s=p+1}^n (a_{\ell s} + a_{s\ell} \sigma_s) \Big] \tag{3.52}$$

$$H_t^2 = X_t \Big[\prod_{\substack{s \neq t \\ s \ge p+1}} (\sigma_{st} + \sigma_{ts}) \Big] . \tag{3.53}$$

Here the indices run over the ranges

$$i, j, \dots = 1, \dots, k; \ \ell, m, \dots = k+1, \dots, p; \tag{3.54}$$

$$s, t, u, \dots = p+1, \dots, n.$$

We follow this convention unless otherwise stated. If none of the remaining σ_{ab}'s are constants there are two cases to consider:

Case (i). $\dfrac{a_{\ell s}}{a_{s\ell}} = \dfrac{a_{is}}{a_{si}}$ for $s = p+1, \dots, n$, $i = 1, \dots, k$, $\ell = k+1, \dots, p$.

Then the infinitesimal distance has the form

$$ds^2 = \left(\prod_{t=p+1}^{n} \sigma_t\right) d\omega^2 + \sum_{t=p+1}^{n} X_t \left[\prod_{u\neq t} (\sigma_{ut} + \sigma_{tu}) \right] (dx^t)^2 \tag{3.55}$$

where

$$d\omega^2 = \left(\prod_{l=k+1}^{p} \sigma_l\right) \sum_{i=1}^{k} X_i \left[\prod_{j\neq i} (\sigma_{ij} + \sigma_{ji}) \right] (dx^i)^2 \tag{3.56}$$

$$+ \sum_{l=k+1}^{p} X_l \left[\prod_{m\neq l} (\sigma_{lm} + \sigma_{ml}) \right] (dx^l)^2 .$$

The form $d\omega^2$ corresponds to the choice of metric coefficients with $l = k+1, \ldots, p < n$. If we impose the conditions $R_{abba} = -H_a^2 H_b^2$ then we see that for $a, b = 1, \ldots, k, k+1, \ldots, p$ the conditions are identical with (3.33). Hence

$$\frac{1}{X_i} = -4 \prod_{j=1}^{k+1} (x^i - e_j), \quad i = 1, \ldots, k, \tag{3.57}$$

$$\frac{1}{X_l} = -4 \prod_{m=1}^{p-k+1} (x^l - f_m), \quad l = k+1, \ldots, p, \tag{3.58}$$

and

$$\sigma_l = \frac{(x^l - f_{p-k+1})}{(f_{l-k} - f_{p-k+1})}, \quad l = k+1, \ldots, p. \tag{3.59}$$

The remaining conditions $R_{tuut} = -H_t^2 H_u^2$ and $R_{taat} = -H_a^2 H_t^2$ $(a = 1, \ldots, p)$ also imply

$$\frac{1}{X_s} = -4 \prod_{t=1}^{n-p+1} (x^s - g_t), \quad s = p+1, \ldots, n, \tag{3.60}$$

and

$$\sigma_t = \frac{(x^t - g_{n-p+1})}{(g_{t-p} - g_{n-p+1})}, \quad t = p+1, \ldots, n.$$

These coordinates on S_n can then be constructed in a standard way:

$$(s_1, \ldots, s_{n+1}) = (u_1 v_1 w_1, \ldots, u_1 v_1 w_{k+1}, u_1 v_2, \ldots, u_1 v_{p-k+1}, u_2, \ldots, u_{n-p+1}) \tag{3.61}$$

where

$$\sum_{i=1}^{k+1} w_i^2 = 1, \quad \sum_{l=1}^{p-k+1} v_l^2 = 1, \quad \sum_{i=1}^{n-p+1} u_i^2 = 1$$

and on each of the spheres defined by the u_i, v_j and w_k coordinates, elliptic coordinates are chosen, i.e.,

$$v_i^2 = \frac{-\prod_{i=1}^{k}(x^i - e_j)}{\prod_{j\neq i}^{k+1}(e_i - e_j)}, \quad j, i = 1, \ldots, k+1, \tag{3.62}$$

$$w_l^2 = \frac{-\prod_{m=k+1}^{p}(x^m - f_l)}{\prod_{m\neq l}(f_m - f_l)}, \quad m, l = 1, \ldots, p-k+1, \tag{3.63}$$

$$u_t^2 = \frac{-\prod_{s=p+1}^{n}(x^s - g_t)}{\prod_{s\neq t}(g_s - g_t)}, \quad s, t = 1, \ldots, n-p+1. \tag{3.64}$$

<u>Case (ii)</u>. $\dfrac{a_{ls}}{a_{sl}} \neq \dfrac{a_{is}}{a_{si}}$

In this case $\sigma_l = a_l$ for $l = k+1, \ldots, p$ as follows from Eisenhart's cases (3.38)(i)-(iv). The infinitesimal distance has the form

$$ds^2 = \left(\prod_{t=p+1}^{n} \sigma_t\right) d\omega_1^2 + \left(\prod_{t=p+1}^{n} (\sigma_t + \alpha)\right) d\omega_2^2 \tag{3.65}$$

$$+ \sum_{t=p+1}^{n} X_i \left[\prod_{\substack{u\neq t \\ u\geq p+1}} (\sigma_{ut} + \sigma_{tu}) \right] (dx^t)^2, \quad \alpha \neq 0$$

where

$$d\omega_1^2 = \sum_{i=1}^{k} X_i \left[\prod_{\substack{j\neq i \\ j\leq k}} (\sigma_{ij} + \sigma_{ji}) \right] (dx^i)^2, \tag{3.66}$$

$$d\omega_2^2 = \sum_{l=k+1}^{p} X_l \left[\prod_{\substack{m\neq l \\ k+1\leq m\leq p}} (\sigma_{lm} + \sigma_{ml}) \right] (dx^l)^2 . \tag{3.67}$$

The conditions that this metric correspond to S_n require that we have the same functions X_a as in the previous case and now

$$\sigma_t = \frac{x^t - g_1}{g_{t-p+1} - g_1}, \qquad \sigma_t + \alpha = \frac{x^t - g_2}{g_{t-p+2} - g_2} \tag{3.68}$$

Here we have adopted the convention

$$g_{n-p+1+l} = g_l \quad \text{for } k+1 \le l \le p. \tag{3.69}$$

Consequently the infinitesimal distance has the form

$$ds^2 = \left[\frac{\Pi^n_{t=p+1}(x^t - g_1)}{\Pi_{u\neq t}(g_u - g_1)}\right] d\omega_1^2 + \left[\frac{\Pi^n_{t=p+1}(x^t - g_2)}{\Pi_{u\neq t}(g_u - g_2)}\right] d\omega_2^2 \tag{3.70}$$
$$- \frac{1}{4}\sum_{t=p+1}^{n} \left[\frac{\Pi_{\substack{u\neq t \\ p+1\le u\le n}} (x^t - x^u)}{\Pi^{n+1}_{u=p+1}(x^t - g_u)}\right] (dx^t)^2 .$$

A standard choice of coordinates on S_n for this infinitesimal distance can be taken as

$$(s_1, \ldots, s_{n+1}) = (u_1 v_1, \ldots, u_1 v_{k+1}, u_2 w_1, \ldots, u_2 w_{p+1}, u_3, \ldots, u_{n-p-k-1}) \tag{3.71}$$

with u_i, v_j and w_k coordinates as in (3.61). This procedure can be iterated without difficulty to find all separable coordinate systems on S_n. If we do this we obtain an infinitesimal distance of the form

$$ds^2 = \sum_{I=1}^{p} \{ \sum_{i\in N_I} (H_i^I)^2 (dx^i)^2 \} [\prod_{l\in N_{p+1}} (\sigma_l + \alpha_I)]$$
$$+ \sum_{j\in N_{p+1}} (H_j^{p+1})^2 (dx^j)^2, \qquad \alpha_I \neq \alpha_J \text{ if } I \neq J. \tag{3.72}$$

Here $\{N_1, \ldots, N_{p+1}\}$ is a partition of the integers $1, \ldots, n$ into non empty mutually exclusive sets N_I, i.e., $N_I \cap N_J = \emptyset$. It follows from Eisenhart's types (3.38) (i)-(iv) that $(\partial_j) H_i^I = 0$ if $j \notin N_I$. The curvature conditions

can now be written down. The conditions $R_{ijji} = -H_i^2 H_j^2$ $(i \neq j)$ are equivalent to the equations

$$R^{(p+1)}_{ijji} = -(H_i^{p+1})^2 (H_u^{p+1})^2, \qquad i, j \in N_{p+1}, \tag{3.73}$$

$$(H_i^I)^{-2}(H_j^I)^{-2} R^{(I)}_{ijji} + [\prod_{k\in N_{p+1}} (\sigma_k + \alpha_I)][\frac{1}{4}\sum_{l\in N_{p+1}} (H_l^{(p+1)})^{-2}$$
$$\times \frac{\sigma_l'^2}{(\sigma_l + \alpha_I)^2} + 1] = 0, \qquad i, j \in N_I, \tag{3.74}$$

$$2\frac{\sigma_l''}{(\sigma_l + \alpha_I)} - (\frac{\sigma_l'}{\sigma_l + \alpha_I})^2 - (\frac{\sigma_l'}{(\sigma_l + \alpha_I)})$$
$$\times [\frac{\partial}{\partial x^l} \log(H_l^{(p+1)})^2 + (H_l^{(p+1)})^2 \sum_{\substack{m\neq l \\ m\in N_{p+1}}} \frac{1}{(H_m^{(p+1)})^2 (x^l - x^m)}]$$
$$= -4(H_l^{(p+1)})^2, \qquad l \in N_{p+1}, \tag{3.75}$$

$$\frac{1}{4}\sum_{l\in N_{p+1}} \frac{1}{(H_l^{(p+1)})^2} \frac{\sigma_l'^2}{(\sigma_l + \alpha_I)(\sigma_l + \alpha_J)} = -1. \tag{3.76}$$

Here we have used the notation $R^{(I)}_{hijk}$ to refer to the curvature tensor of the Riemannian manifold with infinitesimal distance

$$d\omega_I^2 = \sum_{I\in N_I} (H_i^{(I)})^2 (dx^i)^2 . \tag{3.77}$$

These equations have the solutions

$$[\prod_{l\in N_{p+1}} (\sigma_l + \alpha_I)] = [\prod_{l\in N_{p+1}} (x^l - e_I)] / [\prod_{m\in N_{p+1}} (e_m - e_I)], \tag{3.78}$$

$$(H_l^{p+1})^2 = -\frac{1}{4}\left[\frac{\Pi_{\substack{m\in N_{p+1} \\ (m\neq l)}} (x^m - x^l)}{\Pi^{n_{p+1}+1}_{n=1} (x^l - e_n)}\right], \qquad l \in N_{p+1} \tag{3.79}$$

$$R^{(I)}_{ijji} = -(H^{(I)}_i)^2 (H^{(I)}_j)^2 \qquad I = 1, \ldots, p+1;\ i, j \in N_I \tag{3.80}$$

where $n_{p+1} = \dim N_{p+1}$. The infinitesimal distance can always be written in the form

$$ds^2 = \sum_{I=1}^{p} d\omega_I^2 \left[\frac{\Pi_{l=1}^{n_1}(x_l - e_I)}{\Pi_{m \neq I}(e_m - e_I)} \right] - \frac{1}{4} \sum_{i=1}^{n_1} \left[\frac{\Pi_{j \neq i}(x^i - x^j)}{\Pi_{j=1}^{n_1+1}(x^i - e_j)} \right] (dx^i)^2 \tag{3.81}$$

where each $d\omega_I^2$ is the infinitesimal distance of a S_{p_a}. The coordinates on each S_{p_a} are again separable. Clearly we must have the constraint $\Sigma_{I=1}^{p} p_I + n_1 = n$.

Using this infinitesimal distance we can construct all separable coordinate systems inductively. The basic building blocks of separable coordinate systems are the elliptic coordinates on spheres of various dimensions. We will prescribe a graphical procedure for obtaining admissible coordinate systems, essentially giving the embeddings of spheres inside spheres which are admissible so as to correspond to separable coordinates.

3. THE CONSTRUCTION OF SEPARABLE COORDINATE SYSTEMS ON S_n

As we have seen in Section 2, the basic building blocks of separable coordinate systems on S_n are the p-sphere elliptic coordinates

$$_p s_j^2 = \frac{\Pi_{i=1}^{p}(x^i - e_j)}{\Pi_{j \neq i}(e_i - e_j)}, \qquad \begin{matrix} p = 1, \ldots, n \\ j = 1, \ldots, n+1 \end{matrix} \tag{3.82}$$

$$\sum_{j=1}^{p+1} {_p s_j^2} = 1.$$

Two important examples of these coordinates are

$$\text{(i)}\quad p = 1:\quad {_1 s_1^2} = \frac{(x^1 - e_1)}{(e_2 - e_1)}, \quad {_1 s_2^2} = \frac{(x^1 - e_2)}{(e_1 - e_2)} \tag{3.83}$$

where ${_1 s_1^2} + {_1 s_2^2} = 1$, $e_1 < x^1 < e_2$.

$$\text{(ii)}\quad p = 2:\quad {_2 s_1^2} = \frac{(x^1 - e_1)(x^2 - e_1)}{(e_2 - e_1)(e_3 - e_1)}, \quad {_2 s_2^2} = \frac{(x^1 - e_2)(x^2 - e_2)}{(e_1 - e_2)(e_3 - e_2)},$$
$${_2 s_3^2} = \frac{(x^1 - e_3)(x^2 - e_3)}{(e_3 - e_1)(e_3 - e_2)} \tag{3.84}$$

where ${_2 s_1^2} + {_2 s_2^2} + {_2 s_3^2} = 1$, $e_1 < x^1 < e_2 < x^2 < e_3$.

We will develop a graphical calculus for calculating admissible coordinate systems. We represent elliptical coordinates on S_n by the 'irreducible' block.

$$\boxed{e_1 \mid e_2 \mid \cdots\cdots \mid e_{n+1}} \tag{3.85}$$

Each separable coordinate system will be associated with a directed tree graph. Consider for example the sphere S_2. There are two possibilities: (i) the irreducible block $\boxed{e_1 \mid e_2 \mid e_3}$. Most treatments of elliptic coordinates on S_2 correspond to the choice $e_1 = 0$, $e_2 = 1$, $e_3 = a > 1$. This is just a reflection of the fact that for Jacobi elliptic coordinates the variables x^i and e_i can always be subjected to the transformation

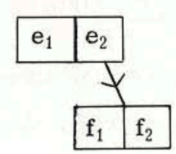
$$x^{i'} = ax^i + b, \quad e_j' = ae_j + b; \qquad i = 1, \ldots, n, \quad j = 1, \ldots, n+1. \tag{3.86}$$

Thus we can always choose $e_1 = 0$ and $e_2 = 1$. (Note in particular that $\boxed{e_1 \mid e_2}$ can always be replaced by $\boxed{0 \mid 1}$. Putting $x^1 = \cos^2\phi$ we recover ${_1 s_1} = \cos\phi$, ${_1 s_2} = \sin\phi$ $(0 \leq \phi \leq 2\pi)$.)

(ii) The second system is the usual choice of spherical coordinates,

$$s_1 = \sin\theta\cos\phi, \quad s_2 = \sin\theta\sin\phi, \quad s_3 = \cos\theta. \tag{3.87}$$

This system can be considered as the result of attaching a circle to a circle and is the prototype for the construction of more complicated systems. The graph



(3.88)

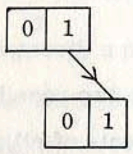
is taken to correspond to the choice of coordinates

$$s_1^2 = {}_1u_1^2 = \frac{(x^1 - e_1)}{(e_2 - e_1)}, \quad s_2^2 = ({}_1u_2^2)({}_1v_1^2) = \frac{(x^1 - e_2)}{(e_1 - e_2)}\frac{(x^2 - f_1)}{(f_2 - f_1)}$$

$$s_3^2 = ({}_1u_2^2)({}_1v_2^2) = \frac{(x^1 - e_2)}{(e_1 - e_2)}\frac{(x^2 - f_2)}{(f_1 - f_2)} \tag{3.89}$$

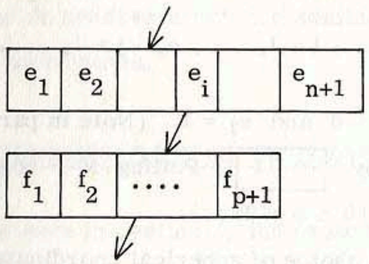
$$e_1 < x^1 < e_2, \quad f_1 < x^2 < f_2 .$$

Clearly, choosing angle variables on the s_i's, the choice of spherical coordinates corresponds to the graph



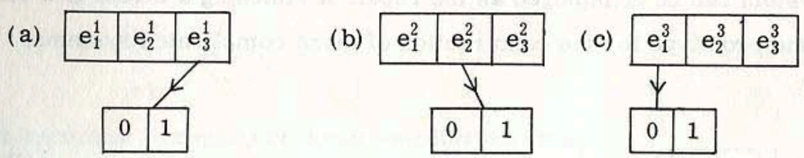
(3.90)

Only the square of origin of the arrow is of importance for a given arrow connecting two irreducible blocks, not the target square. The general branching law for an arrow connecting two irreducible blocks is readily given:



(3.91)

We should also note here that, because of the availability of transformations of the type (3.86), some graphs that look different do in fact correspond to the same coordinate system. Indeed, consider graphs of type



(3.92)

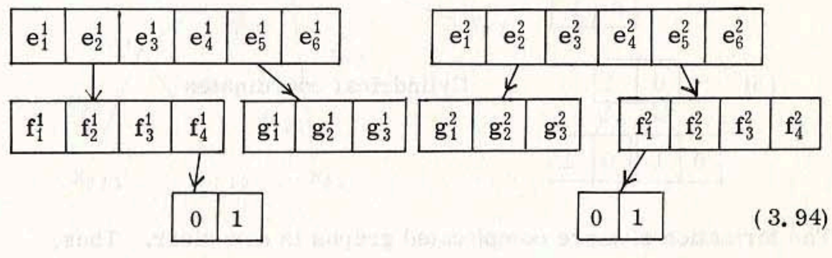
These graphs correspond to Lamé rotational coordinates on the sphere S_3. There are, however, only two distinct such coordinate systems. In fact, if

the coordinates x^i and e_j^1 $(i = 1, 2, \quad j = 1, 2, 3)$ are subjected to the transformation

$$x^i = -x^i = y^i ; \quad e_1^1 \to -e_1^1 = e_3^3, \quad e_2^1 \to -e_2^1 = e_2^3,$$

$$e_3^1 \to -e_3^1 = e_1^3 , \tag{3.93}$$

we see that the graphs (3.92) (a) and (c) correspond to the same type of coordinates. Graphs that are related in this way can be recognized by the feature that, if the branch below a given irreducible block $\boxed{e_1 \; \cdots \; e_p}$ is obtained from that of another graph by reflection about a vertical at the centre of the corresponding $\boxed{e_1^1 \cdots e_p^1}$ block, then the two graphs are equivalent. (We are of course assuming that all other features of the graphs are identical.) Graphs that are essentially the same can be related by several transformations of the type (3.86) and the situation gets more complicated, e.g.,



(3.94)

If the two irreducible blocks of S_n and S_p occur as indicated in (3.91) as part of some larger graph, this means that the elliptic coordinates ${}_nu_1, \ldots, {}_nu_{n+1}$ and ${}_pv_1, \ldots, {}_pv_{p+1}$ of these blocks must occur in the combinations

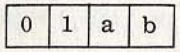
$$w_1 = {}_nu_1, \ldots, \quad w_i = ({}_nu_i)({}_pv_1), \ldots$$

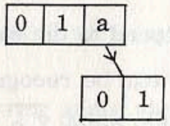
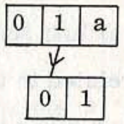
$$w_{i+p+1} = ({}_nu_i)({}_pv_{p+1}), \quad w_{i+p+2} = {}_nu_{i+1}, \ldots$$

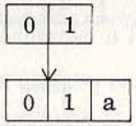
$$w_{p+n+2} = {}_nu_{n+1} .$$

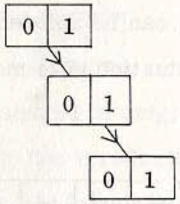
Arrows may emanate from different squares $(e_i$'s) of the same block but cannot be directed at the same block. With these rules we may construct graphs corresponding to all separable coordinate systems on S_n.

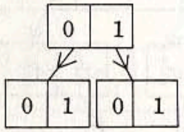
For $n = 3$ we have the following possibilities [20]:

(1)  Jacobi elliptic coordinates (3.96)

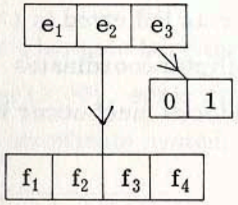
(2) (a)  (b)  Lamé rotational coordinates (3.97)

(3)  Lamé subgroup reduction (3.98)

(4)  Spherical coordinates (3.99)

(5)  Cylindrical coordinates (3.100)

The formation of more complicated graphs is now clear. Thus,



(3.101)

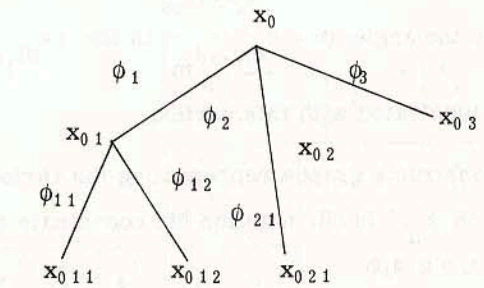
is a coordinate system on S_6 with coordinates

$$s_1^2 = ({}_2u_1)^2, \quad s_2^2 = ({}_2u_2)^2({}_3v_1)^2, \quad s_3^2 = ({}_2u_2)^2({}_3v_2)^2 \tag{3.102}$$
$$s_4^2 = ({}_2u_2)^2({}_3v_3)^2, \quad s_5^2 = ({}_2u_2)^2({}_3v_4)^2, \quad s_6^2 = ({}_2u_3)^2({}_1w_1)^2$$
$$s_7^2 = ({}_2u_3)^2({}_1w_2)^2 .$$

Vilenkin [21] has studied polyspherical coordinates on S_n and developed a graphical technique for constructing them. For example, he considers the coordinates on S_6:

$$\begin{aligned}
x_0 &= \cos\phi_3 \cos\phi_2 \cos\phi_1 \\
x_{03} &= \sin\phi_3 \\
x_{02} &= \cos\phi_3 \sin\phi_2 \cos\phi_{21} \\
x_{01} &= \cos\phi_3 \cos\phi_2 \sin\phi_1 \cos\phi_{12} \cos\phi_{11} \\
x_{021} &= \cos\phi_3 \sin\phi_2 \sin\phi_{21} \\
x_{012} &= \cos\phi_3 \cos\phi_2 \sin\phi_1 \sin\phi_{12} \\
x_{011} &= \cos\phi_3 \cos\phi_2 \sin\phi_1 \cos\phi_{12} \sin\phi_{11}
\end{aligned} \tag{3.103}$$

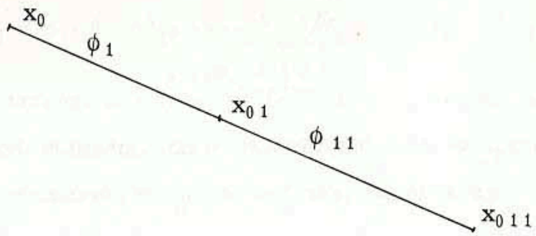
and represents these coordinates by the graph



For him, spherical coordinates on S_2

$$\begin{aligned}
x_0 &= \cos\phi_1 \\
x_{01} &= \sin\phi_1 \cos\phi_{11} \\
x_{011} &= \sin\phi_1 \sin\phi_{11}
\end{aligned} \tag{3.104}$$

correspond to the graph

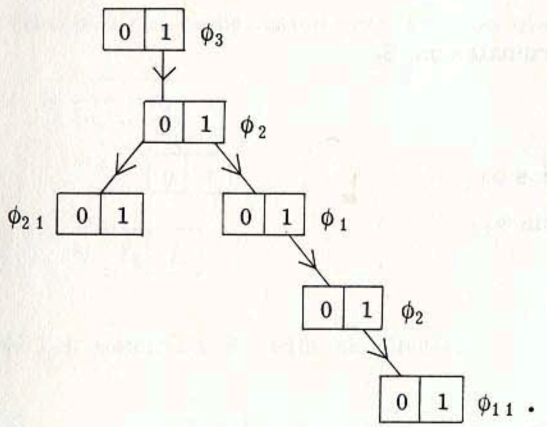


Vilenkin denotes coordinates of <u>rank r</u> by $x_{0i_1 \ldots i_r}$ and in the example of (3.103) arranges coordinates in the order

$$x_{011}, \quad x_{012}, \quad x_{021}, \quad x_{01}, \quad x_{02}, \quad x_{03}, \quad x_0, \qquad (3.105)$$

i.e., coordinates of higher rank precede those of lower rank while coordinates of equal rank are ordered lexicographically. Coordinates of the form $x_{0i_1 \ldots i_s \ldots i_{s+1} \ldots i_m}$ are called <u>subordinate</u> to the coordinate $x_{0i_1 \ldots i_s}$. Further, the coordinate $x_{0j_1 \ldots j_m}$ <u>essentially precedes</u> the coordinate $x_{0i_1 \ldots i_s}$ if $m \geq s$, and $j_k = i_k$ for $1 \leq k \leq s-1$ and $j_s < i_s$. The coordinate $x_{0i_1 \ldots i_s}$ <u>essentially follows</u> $x_{0j_1 \ldots j_m}$. To extract coordinates on S_n from this notation let $x_{0i_1 \ldots i_m}$ be a vertex of nonzero rank. A rotation $g(\phi)$ by the angle $\phi = \phi_{i_1 \ldots i_m}$ in the $(x_{0i_1 \ldots i_{m-1}}, x_{0i_1 \ldots i_m})$ plane is then associated with this vertex.

In this way Vilenkin constructs graphs representing the various possible polyspherical coordinates on S_n. In our notation his coordinate system (3.103) is represented by the graph



From these considerations we see that Vilenkin's polyspherical coordinates are the special case of separable coordinates on S_n consisting of those

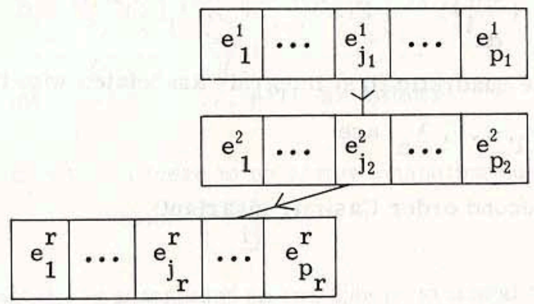
graphs which contain only the irreducible blocks of type $\boxed{0 \mid 1}$.

4. <u>PROPERTIES OF SEPARABLE SYSTEMS IN S_n</u>

Here we make more precise our graphical techniques by means of a prescription for writing down the standard coordinates s_i, $i = 1, \ldots, n+1$, on S_n in terms of the separable coordinates. A given standard coordinate coming from a given graph consists of a product of r factors which we denote

$$x^{j_1}_{p_1} \cdots {}^{j_r}_{p_r} = ({}_{p_1}u_{j_1}) \ldots ({}_{p_r}u_{j_r}) .$$

This is obtained by tracing the complete length of a branch of a given tree graph, i.e.



We can then set up an ordering $<$ for the products $x^{j_1 \ldots j_r}_{P_1 \ldots P_r}$. We say that

$$x^{j_1 \ldots j_r}_{P_1 \ldots P_r} < x^{i_1 \ldots i_s}_{Q_1 \ldots Q_s}$$

if

$$P_1 = Q_1, \; j_1 = i_1, \; \ldots, \; P_t = Q_t, \; j_t < i_t, \; P_{t+1} \neq Q_{t+1}, \; \ldots, \; j_s \neq i_s .$$

Then if we arrange the products in increasing order, say $x_1, \ldots, x_{n+1}$, we can identify this ordered n-tuple with $s_1, \ldots, s_{n+1}$. For the example (3.102) given above, the choice of coordinates corresponds to this ordering.

Having settled on a prescription for writing down the coordinates corresponding to a given coordinate system on S_n, we can now discuss the separation equations for both the Hamilton-Jacobi and Helmholtz equations. Let us first consider the coordinates corresponding to the irreducible block $\boxed{e_1 \mid e_2 \mid \cdots \mid e_{n+1}}$. The Hamilton-Jacobi equation in these coordinates is

$$H = \sum_{i=1}^{n} \frac{1}{[\Pi_{j\neq i}(x^i - x^j)]} P_i^2 = E \tag{3.106}$$

where

$$P_i = \sqrt{[\prod_{j=1}^{n+1} (x^i - e_j)]}\ \frac{\partial W}{\partial x^i} .$$

The separation equations are

$$[\prod_{j=1}^{n+1} (x^i - e_j)]\left(\frac{dW_i}{dx^i}\right)^2 + [E(x^i)^{n-1} + \sum_{j=2}^{n} \lambda_j (x^i)^{n-j}] = 0 \tag{3.107}$$

If we set $E = \lambda_1$ then the quadratic first integrals associated with the separation parameters $\lambda_1, \ldots, \lambda_n$ are

$$I_1^n = \sum_{i>j} I_{ij}^2 \quad \text{(second order Casimir invariant)} \tag{3.108}$$

$$I_2^n = \sum_{i>j} S_1^{ij} I_{ij}^2$$

$$I_n^n = \sum_{i>j} S_{n-1}^{ij} I_{ij}^2$$

where $S_l^{ij} = \frac{1}{l!} \Sigma'_{i_1,\ldots,i_l \neq} e_{i_1} \cdots e_{i_l}$ and the summation extends over $i_1, \ldots, i_l \neq i, j$ and $i_l \neq i_m$ for $l \neq m$. For the associated Helmholtz equation the eigenvalues of Δ_n have the form $\sigma(\sigma + n - 1)$ and the Helmholtz equation becomes

$$\sum_{i=1}^{n} \frac{1}{[\Pi_{j\neq i}(x^i - x^j)]} \left\{ \sqrt{(\mathcal{P}_i)} \frac{\partial}{\partial x^i}\left(\sqrt{(\mathcal{P}_i)} \frac{\partial \Psi}{\partial x^i}\right) \right\}$$

$$= -\sigma(\sigma + n - 1)\Psi \tag{3.109}$$

where $\mathcal{P}_i = \Pi_{j=1}^{n}(x^i - e_j)$. The separation equations are

$$\sqrt{(\mathcal{P}_i)} \frac{d}{dx^i}\left(\sqrt{(\mathcal{P}_i)} \frac{d\Psi_i}{dx^i}\right) + [\sigma(\sigma + n - 1)(x^i)^{n-1} + \sum_{j=2}^{n} \tilde{\lambda}_j (x^i)^{n-j}]\Psi_i = 0 . \tag{3.110}$$

The identification $\tilde{\lambda}_1 = \sigma(\sigma + n - 1)$ enables us to further identify the symmetry operators whose eigenvalues are $\tilde{\lambda}_j$ with the expressions (3.108) where I_{ij} is replaced by the corresponding symmetry operator $\hat{I}_{ij}$.

For an irreducible block appearing in an admissible graph the generalizations of these equations can readily be computed. Consider the block shown as part of a given graph:

$$\boxed{e_1 \mid e_2 \mid \cdots \mid e_j \mid \cdots \mid e_{p+1}} \ .$$

Then define d_i $(i = 1, \ldots, p+1)$ as follows:

$d_i = 0$ if there is no arrow emanating downward from the block $\boxed{e_i}$;

otherwise d_i is a constant on the sphere attached to e_i.

From the form of the metric we see the variables $x^1, \ldots, x^p$ coming from this block satisfy an equation of the form

$$\sum_{i=1}^{p} \frac{1}{[\Pi_{j\neq i}(x^i - x^j)]} \hat{P}_i^2 + \sum_{i=1}^{p} \left[\frac{\Pi_{j\neq i}(e_i - e_j)}{\Pi_{j=1}^{p}(x^j - e_i)}\right] d_i = E_p . \tag{3.111}$$

Using the relation

$$\frac{1}{\Pi_{j=1}^{p}(x^j - e_k)} = \frac{1}{\Pi_{i>j}(x^i - x^j)} \left[\sum_{l=1}^{p} \frac{T_l}{(x^l - e_k)}\right] \tag{3.112}$$

where

$$\hat{P}_i = \sqrt{[\prod_{j=1}^{p+1} (x^i - e_j)]}\ \frac{\partial W}{\partial x^i}$$

$$T_l = (-1)^{l+1} \prod_{i>j} (x^i - x^j) \quad \text{with } i, j \neq l,$$

we see that the separation equations have the form

$$\Big[\prod_{j=1}^{p+1} (x^i - e_j)\Big]\Big(\frac{dW_i}{dx^i}\Big)^2 + \sum_{k=1}^{p+1} \frac{\prod_{j\neq k}(e_k - e_j)\, d_k}{(x^i - e_k)} \tag{3.113}$$

$$+ \Big[E_p (x^i)^{p-1} + \sum_{l=2}^{p} \lambda_l (x^i)^{p-l}\Big] = 0.$$

For the corresponding Helmholtz equation the situation is somewhat more complicated. With each ${}_p u_j$ $(j = 1, \ldots, p+1)$ we associate an index k_j which is calculated as follows: if the irreducible block occurs as the r^{th} step down from the trunk of the graph and if we write out the S_i in terms of our coordinates, then k_j is the number of coordinates for which $x^{j_1 \ldots j \ldots j_q}_{p_1 \ldots p \ldots p_q}$ (r^{th} column) occurs. The Helmholtz equation assumes the form

$$\sum_{i=1}^{p} \frac{1}{\big[\prod_{j\neq i} x^i - x^j\big]} \left\{ \sqrt{(\hat{\mathcal{P}}_i/Q_i)}\, \frac{\partial}{\partial x^i} \Big(\sqrt{(\hat{\mathcal{P}}_i Q_i)}\, \frac{\partial \Psi}{\partial x^i}\Big) \right\} \tag{3.114}$$

$$+ \sum_{i=1}^{p} \left[\frac{\prod_{j\neq i}(e_i - e_j)}{\prod_{j=1}^{p} (x^j - e_i)} \right] t_i \Psi = -\sigma(\sigma + p - 1)\Psi$$

where

$$\hat{\mathcal{P}}_i = \sum_{j=1}^{p+1} (x^i - e_j), \quad Q_i = \sum_{j=1}^{p+1} (x^i - e_j)^{k_j - 1},$$

$t_i = 0$ if $k_i = 1$ and $t_i = j_i(j_i + k_i - 1)$ if $k_i \neq 1$. The separation equations become

$$\sqrt{(\hat{\mathcal{P}}_i/Q_i)}\, \frac{d}{dx^i}\Big(\sqrt{(\hat{\mathcal{P}}_i Q_i)}\, \frac{d\Psi_i}{dx^i}\Big) \tag{3.115}$$

$$+ \left\{ \sum_{k=1}^{p} \frac{\prod_{j\neq k}(e_k - e_j)}{(x^i - e_k)}\, t_k + \Big[\sigma(\sigma + p - 1)(x^i)^{p-1} + \sum_{l=2}^{p} \tilde{\lambda}_l (x^i)^{p-l}\Big] \right\} \Psi_1 = 0$$

If we take the coordinates (3.102) and choose

$${}_2u_j^2 = \frac{\prod_{i=1}^{2} (x^i - e_j)}{\prod_{j\neq i} (e_i - e_j)}, \quad j = 1, 2, 3, \quad i = 1, 2, \tag{3.116}$$

$${}_3v_l^2 = \frac{\prod_{i=3}^{5} (x^i - f_l)}{\prod_{m\neq l} (f_m - f_l)}, \quad l = 1, 2, 3, 4, \quad i = 3, 4, 5,$$

$${}_1w_s^2 = \frac{(x^6 - g_s)}{(g_t - g_s)}, \quad t, s = 1, 2, \quad t \neq s,$$

then the separation equations for the Hamilton-Jacobi equation are

$$\text{(i)} \quad \Big[\prod_{j=1}^{3} (x^i - e_j)\Big]\Big(\frac{dW_i}{dx^i}\Big)^2 + \frac{(e_2 - e_3)(e_2 - e_1)}{(x^i - e_2)}\, d_2 \tag{3.117}$$

$$+ \frac{(e_3 - e_2)(e_3 - e_1)}{(x^i - e_3)}\, d_3 + E x^i + \lambda_1 = 0, \qquad i = 1, 2,$$

$$\text{(ii)} \quad \Big[\prod_{m=1}^{4} (x^l - f_m)\Big]\Big(\frac{dW_l}{dx^l}\Big)^2 + d_2 (x^l)^2 + \lambda_2 x^l + \lambda_3 = 0, \qquad l = 3, 4, 5,$$

$$\text{(iii)} \quad \Big[\prod_{s=1}^{2} (x^6 - g_s)\Big]\Big(\frac{dW_6}{dx^6}\Big)^2 + d_3 = 0,$$

and for the Helmholtz equation the corresponding separation equations are

$$\text{(i)} \quad \sqrt{\left[\frac{\prod_{j=1}^{3}(x^i - e_j)}{(x^i - e_2)^3 (x^i - e_3)}\right]}\, \frac{d}{dx^i}\Big(\sqrt{(\prod_{j=1}^{3}(x^i - e_j)(x^i - e_2)^3 (x^i - e_3))}\, \frac{d\Psi_i}{dx^i}\Big)$$

$$+ \left[\frac{(e_2 - e_3)(e_2 - e_1)}{(x^i - e_2)}\, j_1(j_1 + 2) + \frac{(e_3 - e_2)(e_3 - e_1)}{(x^i - e_3)}\, j_2^2 \right.$$

$$\left. + j(j+5)x^i + \lambda_1 \right] \Psi_i = 0, \qquad i = 1, 2, \tag{3.118}$$

$$\text{(ii)} \quad \sqrt{(\prod_{m=1}^{4}(x^l - f_m))}\, \frac{d}{dx^i}\Big(\sqrt{(\prod_{m=1}^{4}(x^l - f_m)}\, \frac{d\Psi_l}{dx^i}\Big)\Big)$$

$$+ \big[j_1(j_1+2)(x^l)^2 + \lambda_2 x^l + \lambda_3\big]\Psi_l = 0, \quad l = 3, 4, 5,$$

(iii) $\sqrt{(\Pi_{s=1}^{2}(x^6 - g_s))}\,\frac{d}{dx^6}\left(\sqrt{(\Pi_{s=1}^{2}(x^6 - g_s)}\,\frac{d\Psi_6}{dx^6})\right) + j_2^2\psi_6 = 0 .$

Having computed the separation equations for the Hamilton-Jacobi equation in (3.117), we can now compute the five quadratic first integrals for the separation constants. We do this as follows: in (3.113) we put $\lambda_1 = E_p$. Given ${}_p u_i$, two coordinates s_i, s_k are said to be connected if they both contain ${}_p u_j$. The corresponding quadratic first integrals are then calculated from the formulas (3.108) with I_{ij}^2 replaced by $\sum_{r>s} I_{rs}^2$ where the sum extends over all indices r connected to i and s connected to j. The quadratic first integrals correspond to I_1^m type operators of the next irreducible block of dimension m connected farther up the branch in question. For example, consider the coordinates (3.102). The corresponding quadratic first integrals are

$$L_1 = \sum_{i>j} I_{ij}^2 , \tag{3.119}$$

$$L_2 = e_1\left(\sum_{i=2}^{5} I_{6i}^2 + I_{7i}^2\right) + e_2\left(I_{16}^2 + I_{17}^2\right) + e_3\left(\sum_{j=2}^{5} I_{1j}^2\right)$$

$$L_3 = \sum_{k>l}{}' I_{kl}^2 , \qquad k,\ l = 2,\ 3,\ 4,\ 5,$$

$$L_4 = (f_1 + f_2) I_{45}^2 + (f_1 + f_3) I_{35}^2 + (f_1 + f_4) I_{34}^2$$
$$+ (f_2 + f_3) I_{25}^2 + (f_2 + f_4) I_{24}^2 + (f_3 + f_4) I_{23}^2 ,$$

$$L_5 = f_1 f_2 I_{45}^2 + f_1 f_3 I_{35}^2 + f_1 f_4 I_{34}^2 + f_2 f_3 I_{25}^2$$
$$+ f_2 f_4 I_{24}^2 + f_3 f_4 I_{23}^2 ,$$

$$L_6 = I_{67}^2 .$$

For the Hamilton-Jacobi equation these quadratic first integrals have the constant values

$$L_1 \sim E_1 , \quad L_2 \sim \lambda_1 , \quad L_3 \sim d_2 , \quad L_4 \sim \lambda_2 , \quad L_5 \sim \lambda_3 , \quad L_6 \sim d_3$$

and for the Helmholtz equation with $I_{ij} \to \hat{I}_{ij}$ the resulting operators L_i $(i = 1, \ldots, 6)$ have the eigenvalues $\tilde{L}_1 \sim j(j+5)$, $\tilde{L}_2 \sim \lambda_1$, $\tilde{L}_3 \sim j_1(j_1+2)$, $\tilde{L}_4 \sim \lambda_2$, $\tilde{L}_5 \sim \lambda_3$, $\tilde{L}_6 \sim j_2^2$.

4 Separation of variables in Euclidean n–space E_n

1. MATHEMATICAL PRELIMINARIES

The same methods as used in Chapter 3 enable us to find all separable coordinate systems for E_n. The linear first integrals (Lie symmetries) of E_n form a $\frac{1}{2}n(n+1)$-dimensional Lie algebra $E(n)$ with basis

$$I_{ij} = z_i p_j - z_j p_i , \qquad i > j , \qquad i, j = 1, \dots, n. \tag{4.1}$$

$$P_i = p_i .$$

This basis satisfies the commutation relations

$$[I_{ab}, I_{cd}] = \delta_{bc} I_{ad} + \delta_{ad} I_{bc} + \delta_{bd} I_{ca} + \delta_{ac} I_{db} , \tag{4.2}$$

$$[I_{ab}, P_c] = -\delta_{ac} P_b + \delta_{bc} P_a ,$$

$$[P_a, P_b] = 0 .$$

Just as was observed for S_n, two coordinate systems $\{x^i\}$ and $\{y^j\}$ that are related by a group motion are regarded as being essentially the same. In this case, if we are given the cartesian coordinates $z_i(x^j)$ in terms of a separable set of coordinates $\{x^i\}$ then

$$\underset{\sim}{z}' = 0\underset{\sim}{z} + \underset{\sim}{a} \tag{4.3}$$

defines a new vector $\underset{\sim}{z}'$ related to $\underset{\sim}{z}$ by means of a Euclidean group motion specified by the orthogonal matrix 0 and constant vector $\underset{\sim}{a}$. Then, clearly,

$$ds^2 = d\underset{\sim}{z}' \cdot d\underset{\sim}{z}' = d\underset{\sim}{z} \cdot d\underset{\sim}{z} = g_{ij} dx^i dx^j \tag{4.4}$$

Chosen coordinates z and z' related in this way are then regarded as being 'equivalent'.

2. SEPARATION OF VARIABLES ON E_n

As was the case for S_n, all separable coordinates systems in E_n can be chosen to be orthogonal.

Theorem 4.1: Let $\{x^i\}$ be a coordinate system on E_n for which the Hamilton-Jacobi equation admits separation of variables and let q be the number of ignorable variables. Then it is always possible to choose an equivalent coordinate system $\{x^i\}$ such that $g^{ij} = \delta^{ij} H_i^{-2}$, i.e., the coordinates are orthogonal. Furthermore the ignorable variables can always be taken such that

$$p_{\alpha_1} = I_{12}, \dots, p_{\alpha_p} = I_{2p-1,2p},$$

$$p_{\alpha_{p+1}} = P_{2p+1}, \dots, p_{\alpha_q} = P_{p+q} .$$

Proof: We use methods similar to those used in Chapter 3. Any element of the algebra $E(n)$ is conjugate to one of the two forms [19]

$$L = I_{12} + b_2 I_{34} + \dots + b_\nu I_{2\nu-1,2\nu} + \beta P_{2\nu+1} \tag{4.5a}$$

where $\beta = 0$ if $n = 2\nu$

$$L' = P_n . \tag{4.5b}$$

Let $\{x^i\}$ be a separable system with q = 1. It follows from the block diagonal form (3.1) that this system must be orthogonal. Furthermore, without loss of generality we can assume that $p_{\alpha_1} = L$ or $p_{\alpha_1} = L'$. For the first case we can always choose the ignorable variable α_1 such that it is related to the cartesian coordinates $(z_1, \dots, z_n)$ by

$$(z_1, \ldots, z_n) = (\rho_1 \cos(x^{\alpha_1}+w_1),\ \rho_1 \sin(x^{\alpha_1}+w_1), \ldots \tag{4.6}$$

$$\rho_\nu \cos(b_\nu x^{\alpha_1}+w_\nu),\ \rho_\nu \sin(b_\nu x^{\alpha_1}+w_\nu),\ \rho_{\nu+1} + \beta x^{\alpha_1},$$

$$y_{2\nu+2}, \ldots, y_n).$$

The infinitesimal metric then has the form

$$ds^2 = d\rho_1^2 + \ldots + d\rho_\nu^2 + \rho_1^2(dx^{\alpha_1}+dw_1)^2 \tag{4.7}$$

$$+ \ldots + \rho_\nu^2(b_\nu dx^{\alpha_1}+dw_\nu)^2 + (d\rho_{\nu+1} + \beta dx^{\alpha_1})^2$$

$$+ dy^2_{2\nu+2} + \ldots + dy_n^2 .$$

If there is only one ignorable variable the coordinate system must be orthogonal and consequently

$$\rho_1^2 dw_1 + \sum_{j=2}^{\nu} b_j \rho_j^2 dw_j + \beta d\rho_{\nu+1} = 0. \tag{4.8}$$

This is possible only if $b_2 = \ldots = b_\nu = \beta = 0$ and $dw_1 = 0$. (By redefining α_1 we can then take $w_1 = 0$.) Therefore, if we have only one ignorable variable, $p_\alpha = I_{12}$ or P_n.

Now suppose we have q Lie symmetries p_{α_i}, $i = 1, \ldots, q$. Then they must be of the form

$$L_1 = I_{12} + \sum_{l>p}^{s} b_l^1 I_{2l-1,2l} + \sum_{m=2s+1}^{n} \gamma_m^1 P_m \tag{4.9}$$

$$L_2 = I_{34} + \sum_{l>p}^{s} b_l^2 I_{2l-1,2l} + \sum_{m=2s+1}^{n} \gamma_m^2 P_m$$

$$\vdots$$

$$L_p = I_{2p-1,2p} + \sum_{l>p}^{s} b_l^p I_{2l-1,2l} + \sum_{m=2s+1}^{n} \gamma_m^p P_m$$

$$L_{p+1} = \sum_{m=2s+1}^{n} \gamma_m^{p+1} P_m$$

$$\vdots$$

$$L_q = \sum_{m=2s+1}^{n} \gamma_m^q P_m .$$

The condition $[L_i, L_l] = 0$ implies

$$b_k^i \gamma_{2k-1}^l = b_k^i \gamma_{2k}^l = 0 \tag{4.10}$$

for $i = 1, \ldots, p$; $l = 1, \ldots, q$; $k = p+1, \ldots, s$. We are assuming that there is always one b_k^i non-zero for each k and some i. Then $\gamma_{2k-1}^l = \gamma_{2k}^l = 0$ for $k = p+1, \ldots, s$ and $l = 1, \ldots, q$. The cartesian coordinates are

$$(y_1, \ldots, y_n) = (\rho_1 \cos(x^{\alpha_1}+w_1),\ \rho_1 \sin(x^{\alpha_1}+w_1), \ldots, \rho_p \cos(x^{\alpha_p}+w_p),$$

$$\rho_p \sin(x^{\alpha_p}+w_p),\ \rho_{p+1}\cos(\sum_{l=1}^{p} b_{p+1}^l x^{\alpha_l}+w_{p+1}), \ldots,$$

$$\rho_s \sin(\sum_{l=1}^{p} b_s^l x^{\alpha_l}+w_s),\ \sum_{l=1}^{q} \gamma_{2s+1}^l x^{\alpha_l} + w_{2s+1}, \ldots,$$

$$\sum_{l=1}^{q} \gamma_n^l x^{\alpha_l} + w_n). \tag{4.11}$$

This set of candidate ignorable variables can take the necessary block diagonal form only if $dw_i = 0$, $b_k^i = 0$ for $i = 1, \ldots, p$ and $k = p+1, \ldots, s$. Also $dw_l = 0$ for $l = 2s+1, \ldots, n$. We can thus assume that $w_1 = \ldots = w_p = 0$, $w_{2s+1} = \ldots = w_n = 0$. This implies $\gamma_m^i = 0$ for $i = 1, \ldots, q$, $m = 2s+q - p+1, \ldots, n$ and we can also assume $\gamma_m^i = 0$ for $i = 1, \ldots, p$ and $m = 2s+1, \ldots, 2s+q - p+1$. Consequently we can take

$$L_1 = I_{12}, \ldots, L_p = I_{2p-1,2p}, \qquad L_{p+1} = P_{2s+1}, \ldots, L_q = P_{2s+q-p} \tag{4.12}$$

and there are no non-zero elements $g^{\alpha_i \alpha_j}$; $1 \le i < j \le q$ in the metric. By a suitable relabelling of coordinates we can always choose $s = p$. All separable coordinates in E_n are orthogonal.

To find all possible separable coordinate systems on E_n we proceed by analogy with what we have done in Chapter 3. If we choose orthogonal coordinates in which none of the σ_{ij} is a constant function then

$$H_i^2 = X_i\left[\prod_{j\neq i}(x^i - x^j)\right], \qquad i = 1, \ldots, n, \tag{4.13}$$

where, as usual,

$$ds^2 = \sum_{i=1}^{n} H_i^2 (dx^i)^2 .$$

The conditions $R_{ijji} = 0$ are equivalent to (3.33) in which the right-hand side is zero. These conditions have the solution

$$\left(\frac{1}{X_i}\right)^{(n+1)} = 0, \qquad i = 1, \ldots, n, \tag{4.14}$$

and $1/X_i = \Sigma_{l=0}^{n} a_l (x^i)^{n-l} = g(x^i)$. Again we look for choices of $g(x)$ that are compatible with a positive definite metric. There are only two possibilities:

(i) $g(x) = \prod_{i=1}^{n} (x - e_i)$ Elliptic coordinates (4.15)

$$e_1 < x^1 < e_2 < \ldots < x^{n-1} < e_n < x^n$$

(ii) $g(x) = \prod_{i=1}^{n-1} (x - e_i)$ Parabolic coordinates (4.16)

$$x^1 < e_1 < x^2 < e_2 < \ldots < x^{n-1} < e_{n-1} < x^n .$$

These metrics give coordinates in n dimensions that are the analogue of elliptic and parabolic coordinates, familiar in Euclidean spaces of dimension $n=2, 3$, [1], [22]. To these systems we may associate cartesian coordinates by

(i) $$y_j^2 = c^2 \frac{\prod_{i=1}^{n}(x^i - e_j)}{\prod_{i\neq j}(e_i - e_j)}, \qquad j = 1, \ldots, n, \quad x \in \mathbb{R}, \tag{4.17}$$

(ii) $$y_1 = \frac{c}{2}(x^1 + \ldots + x^n + e_1 + \ldots + e_{n-1})$$

$$y_j^2 = -c^2 \frac{\prod_{i=1}^{n-1}(x^i - e_{j-1})}{\prod_{i\neq j}(e_i - e_{j-1})} \qquad j = 2, \ldots, n. \tag{4.18}$$

These two systems are fundamental for generating all separable systems on E_n. As an example of the relevance of these systems we consider the case when some of the σ_{ij} functions are constants. We first treat, as we did for S_n, the case in which the metric coefficients have the form (3.39). Then, as was shown in Chapter 3, these coefficients reduce to

$$H_i^2 = [X_i \prod_{j\neq i}(x^i - x^j)]\left(\prod_{h=k+1}^{n} \sigma_h\right), \quad H_l^2 = X_l\left[\prod_{m\neq l}(x^l - x^m)\right]. \tag{4.19}$$

The conditions $R_{kllk} = 0$ imply that the quadratic form $ds'^2 = \Sigma_l H_l^2 (dx^l)^2$ is that of a flat space. The remaining non-zero conditions are

$$\tilde{H}_i^{-2}\tilde{H}_j^{-2}\tilde{R}_{ijji} + \left(\prod_{l=k+1}^{n}\sigma_l\right)\left[\prod_{m=k+1}^{n}\frac{1}{4H_m^2}\left(\frac{\sigma'_m}{\sigma_m}\right)^2\right] = 0, \tag{4.20}$$

$$2\frac{\sigma''_l}{\sigma_l} - \left(\frac{\sigma'_l}{\sigma_l}\right)^2 - \left(\frac{\sigma'_l}{\sigma_l}\right)\left[\frac{\partial}{\partial x^l}\log H_l^2 + H_l^2 \sum_{m\neq l}\frac{1}{H_m^2(x^l - x^m)}\right] = 0, \tag{4.21}$$

with $\tilde{R}_{ijji}$ as in (3.41). These equations are satisfied provided that $\tilde{R}_{ijji} = -\tilde{H}_i^2\tilde{H}_j^2$ and the function X_l and $\Sigma = (\Pi_{l=k+1}^{n}\sigma_l)$ are given by:

$$\frac{1}{X_l} = \prod_{m=k+1}^{N}(x^m - e_l), \qquad l = k+1, \ldots, n, \tag{4.22}$$

$$\Sigma = \frac{\Pi_{l=k+1}^{N}(x^l - e_m)}{\Sigma_{l\neq m}(e_l - e_m)} \quad \text{for some } m \text{ fixed}, \tag{4.23}$$

where $N = n, n-1$. The functions $1/X_i$ are given by

$$\frac{1}{X_i} = -4\prod_{j=1}^{k+1}(x^i - e'_j). \tag{4.24}$$

The systems are related to cartesian coordinates on E_n according to

$$(y_1, \ldots, y_n) = (w_1 s_1, \ldots, w_1 s_{k+1}, w_2, \ldots, w_{n-k}) \tag{4.25}$$

where $\Sigma_{i=1}^{k+1} s_i^2 = 1$ and $s_j^2 = \dfrac{\Pi_{i=1}^{k}(x^i - e'_j)}{\Pi_{j\neq i}(e'_i - e'_j)}$,

$$\text{(i)}\quad w_l^2 = \frac{\prod_{m=1}^{n-k}(x^m - e_l)}{\prod_{m\neq l}(e_m - e_l)}\,, \quad l = 1, \ldots, n-k, \tag{4.26}$$

$$\text{(ii)}\quad w_l^2 = \frac{\prod_{m=1}^{n-k}(x^m - e_l)}{\prod_{m\neq l}(e_m - e_l)}\,, \quad l = 1, \ldots, n-k-1,$$

$$w_{n-k} = \tfrac{1}{2}\Big(\sum_{m=1}^{n-k} x^m + e_1 + \ldots + e_{n-k}\Big)\,.$$

There exists an additional possibility that could be discounted for S_n viz $\sigma_l = a_l$; $l = k+1, \ldots, n$. This corresponds to the case in which the infinitesimal distance can be written

$$ds^2 = ds_1^2 + ds_2^2 \tag{4.27}$$

where ds_1^2 is the infinitesimal distance for elliptic or parabolic coordinates in E_k and ds_2^2 is a similar infinitesimal distance on E_{n-k}. We can mimic the procedure adopted for S_n. The only essential difference is that the infinitesimal distance can in general be expressed as a sum of distances that can be identified with Euclidean subspaces. This reflects the fact that if $\{z_i\}$, $i = 1, \ldots, n_1$, and $\{w_j\}$, $j = 1, \ldots, n_2$, are separable coordinate systems in Euclidean spaces E_{n_1} and E_{n_2} with respective infinitesimal distances ds_1^2, ds_2^2, then the coordinates z_i, w_j, $i = 1, \ldots, n_1$, $j = 1, \ldots, n_2$, can be regarded as a separable coordinate system on $E_{n_1+n_2}$ with corresponding infinitesimal distance $ds^2 = ds_1^2 + ds_2^2$. This is, of course, not possible for S_n. This property of Euclidean space coordinates naturally extends to separable coordinate systems $\{x_p^i\}$ $i = 1, \ldots, n_p$, $p = 1, \ldots, Q$ on E_p in such a way that

$$ds^2 = ds_1^2 + \ldots + ds_Q^2\,.$$

In general the infinitesimal distance can be written as a sum of basic forms

$$ds^2 = \sum_{I=1}^{Q} ds_I^2 \tag{4.28}$$

where

$$ds_I^2 = \sum_{i\,1}^{n_I} \left[\frac{\prod_{l=1}^{N_I}(x^l = e_i^I)}{\prod_{j\neq i}(e_j^I - e_i^I)} \right] d\omega_i^2 + d\sigma_I^2\,. \tag{4.29}$$

Here the $d\sigma_I^2$ is the infinitesimal distance corresponding to elliptic or parabolic coordinates for a flat space of dimension N_I. Also $n_I \leq N_I$ for elliptic coordinates with a strict inequality for parabolic coordinates.

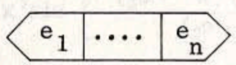
The $d\omega_i^2$ is the infinitesimal distance of some separable coordinate system on the sphere S_{p_I} and $n = \Sigma_{I=1}^{Q}(N_I + p_I)$. To establish a graphical procedure for construction of separable coordinates we need only analyse one of the basic forms ds_I^2. We should also mention here that if $N_I = 1$ then the basic form is written

$$ds_I^2 = w^2 d\omega^2 + dw^2\,. \tag{4.30}$$

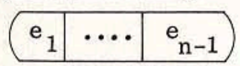
A basic form could in fact correspond to elliptic or parabolic coordinates on E_{N_I} and no $d\omega_i^2$ term. We associate this with $n_I = 0$ in (4.30).

3. THE CONSTRUCTION OF SEPARABLE COORDINATES ON E_n

For our construction we need only invent graphical representations for elliptic and parabolic coordinates in E_n, the analogues of the irreducible blocks on S_n. We adopt the following notation:

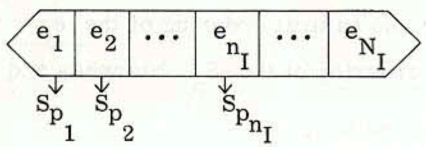
(1) Elliptic coordinates 

, $n \geq 1$,

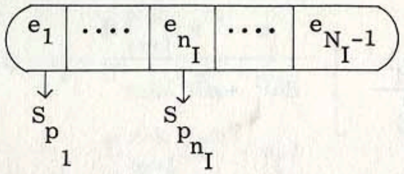
(2) Parabolic coordinates 

, $n \geq 2$.

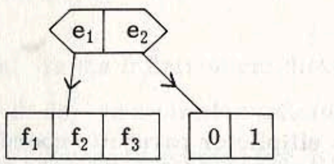
It is clear that only elliptic coordinates exist in one dimension. The graphical representation of a basic form corresponding to the infinitesimal distance ds_I^2 in (4.28) is

(i) 

(4.31)

(ii) 

Attached to each leg descending from the top block is the appropriate graph of the coordinate system on the S_{p_i} giving rise to the form $d\omega_i^2$. The general graph corresponding to a separable system can then be constructed as a sum of disconnected graphs of type (4.31) (i) or (ii). We first illustrate this technique for the separable systems of E_3 (Table 4.1). As an additional non-standard example consider the graph:



which defines a coordinate system in E_5. The coordinates can be chosen as

$$y_i^2 = c^2 \left[\frac{(x^1 - e_1)(x^2 - e_1)}{(e_2 - e_1)} \right] ({}_2u_i)^2 , \qquad i = 1, 2, 3,$$

$$y_4^2 = c^2 \left[\frac{(x^1 - e_2)(x^2 - e_2)}{(e_1 - e_2)} \right] \cos^2 x^5 , \tag{4.32}$$

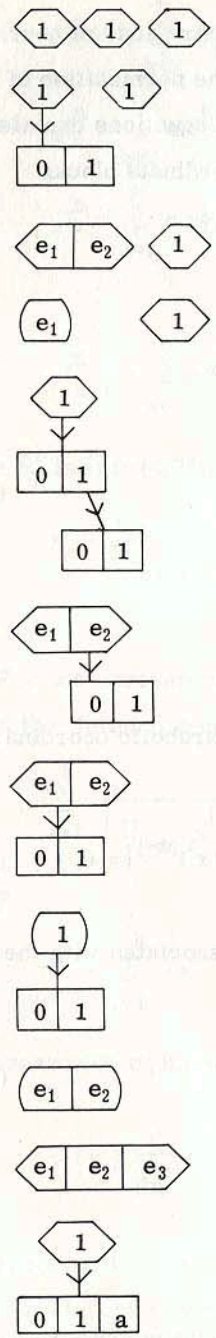
$$y_5^2 = c^2 \left[\frac{(x^1 - e_2)(x^2 - e_2)}{(e_1 - e_2)} \right] \sin^2 x^5 ,$$

where

$$({}_2u_i)^2 = \frac{(x^3 - f_k)(x^4 - f_i)}{(f_j - f_i)(f_k - f_i)} , \quad i, j, k \text{ distinct.}$$

We can set up a natural ordering for separable systems in E_n. For a given basic form we can impose the natural ordering of the e_i's in the leading irreducible block on the ordering of the S_{p_i} branches and then write down coordinates in a standard way.

Table 4.1 Separable coordinates of E_3



(1)	Cartesian
(2)	Cylindrical
(3)	Elliptic cylindrical
(4)	Parabolic cylindrical
(5)	Spherical
(6)	Prolate spheroidal
(7)	Oblate spheroidal
(8)	Parabolic
(9)	Paraboloidal
(10)	Ellipsoidal
(11)	Conical

The ordering of the disconnected parts of the graph is presumed already given. There are equivalences relating graphs of various coordinate systems that we have already discussed for the n-sphere and, of course, there is an additional equivalence corresponding to the permutation of disconnected parts of a given graph. The separation equations can also be readily computed. For the elliptic and parabolic coordinate blocks

(1) $\langle e_1 \mid \dots \mid e_n \rangle$

(2) $(e_1 \mid \dots \mid e_{n-1})$,

the Hamilton-Jacobi equation has the form

$$H = \sum_{i=1}^{n} \frac{1}{[\Pi_{j\neq i}(x^i - x^j)]} P_i^2 = E , \tag{4.33}$$

where

$$P_i = \sqrt{[\prod_{j=1}^{N_k} (x^i - e_j)]} \frac{\partial W}{\partial x^i}$$

with $N_1 = n$ (elliptic coordinates) and $N_2 = n-1$ (parabolic coordinates). The separation equations are

$$[\prod_{j=1}^{N_k} (x^i - e_j)] \left[\frac{dW_i}{dx^i}\right]^2 + [E(x^i)^{n-1} + \sum_{j=2}^{n} \lambda_j (x^i)^{n-j}] = 0. \tag{4.34}$$

If we identify $E = \lambda_1$, the quadratic first integrals associated with the separation parameters $\lambda_1, \dots, \lambda_n$ are

$$(1) \quad {}_1I_1^n = P_1^2 + \dots + P_n^2 , \tag{4.35}$$

$${}_1I_2^n = \sum_{i>j} I_{ij}^2 + c^2 \sum_{i=1}^{n} S_1^i P_i^2 ,$$

$${}_1I_n^n = \sum_{i>j} S_n^{ij} I_{ij}^2 + c^2 \sum_{i=1}^{n} S_{n-1}^i P_i^2 ,$$

where $S_l^i = \frac{1}{l\,!} \Sigma_{i_1, \dots, i_l \neq} e_{i_1} \dots e_{i_l}$ and the sum is over $i_1, \dots, i_l \neq i$.

$$(2) \quad {}_2I_1^n = P_1^2 + \dots + P_n^2 , \tag{4.36}$$

$${}_2I_2^n = c \sum_{k=2}^{n} \{I_{1k}, P_k\}_+ + c^2 S_1 P_1^2 + \sum_{j=2}^{n} c^2 S_1^j P_j^2 ,$$

$${}_2I_3^n = \sum_{k=2}^{n} c S_1^k \{I_{1k}, P_k\}_+ + \sum_{i>j\geq 2} I_{ij}^2 + c^2 S_2 P_1^2 + c^2 \sum_{j=2}^{n} S_2^j P_j^2 ,$$

$${}_2I_4^n = \sum_{k=2}^{n} c S_2^j \{I_{1k}, P_j\}_+ + \sum_{i>j\geq 2} S_1^{ij} I_{ij}^2 + c^2 S_3 P_1^2 + c^2 \sum_{j=2}^{n} S_3^j P_j^2 ,$$

$$\vdots$$

$${}_2I_{n-1}^n = \sum_{k=2}^{n} c S_{n-2}^k \{I_{1k}, P_k\}_+ + \sum_{i>j\geq 2} S_{n-3}^{ij} I_{ij}^2 + c^2 S_{n-1} P_1^2 ,$$

where S_n^j is as in (4.35), $\{ \ , \ \}_+$ is the anti commutator bracket and

$$S_l = \frac{1}{l\,!} \sum_{i_1, \dots, i_l \neq} e_{i_1} \dots e_{i_l} .$$

For the corresponding Helmholtz equation the eigenvalues of Δ_n are λ and the Helmholtz equation reads

$$\sum_{i=1}^{n} \frac{1}{[\Pi_{j\neq i}(x^i - x^j)]} \left\{ \sqrt{(\Theta_i)} \frac{\partial}{\partial x^i} (\sqrt{(\Theta_i)} \frac{\partial \Psi}{\partial x^i}) \right\} = \lambda \Psi, \tag{4.37}$$

where

$$\Theta_i = \sum_{j=1}^{N_k} (x^i - e_j) .$$

The separation equations are

$$\sqrt{(\Theta_i)} \frac{d}{dx^i} (\sqrt{(\Theta_i)} \frac{d\Psi_i}{dx^i}) + [(-\lambda)(x^i)^n + \sum_{j=2}^{n-1} \lambda_j (x^i)^{n-j}] \Psi_i = 0 . \tag{4.38}$$

For a basic form such as ds_I^2 the separation equations for the Hamilton-Jacobi equation have the form

$$\Big[\prod_{j=1}^{N_{I_k}} (x^i - e_j)\Big]\Big(\frac{dW_i}{dx^i}\Big)^2 + \sum_{l=1}^{n_I} \frac{\Pi_{j\neq k}(e_l - e_j)}{(x^i - e_l)} k_l \qquad (4.39)$$

$$+ \Big[E_I(x^i)^{N_{I_k}-1} + \sum_{l=2}^{N_{I_k}} \lambda_l (x^i)^{N_{I_k}-l}\Big] = 0,$$

where k_l is the constant value of the Hamiltonian on the sphere whose infinitesimal distance is $d\omega_l^2$. For the Helmholtz equation the corresponding contribution of this basic form is the equation

$$\sum_{i=1}^{N_{I_k}} \frac{1}{[\Pi_{j\neq i}(x^i - x^j)]} \Big[\sqrt{\Big(\frac{\Theta_i}{Q_i}\Big)} \frac{\partial}{\partial x^i}\Big(\sqrt{(\Theta_i Q_i)}\, \frac{\partial \Psi}{\partial x^i}\Big) \qquad (4.40)$$

$$+ \sum_{i=1}^{n_I} \left[\frac{\Pi_{j\neq i}(e_i - e_j)}{\prod_{j=1}^{n_{I_k}} (x^j - e_i)}\right] j_i(j_i + p_i - 1)\, \Psi = -k_I^2 \Psi ,$$

where $\Theta_i = \Pi_{k=1}^{N_{I_k}} (x^i - e_k)$, $Q_i = \Pi_{k=1}^{N_{I_k}} (x^i - e_k)^{d_k - 1}$,

$$d_k = \begin{cases} p_k + 1 & \text{if } k = 1, \ldots, n_I \\ 1 & \text{if } k = n_I + 1, \ldots, N_{I_k} . \end{cases}$$

The separation equations are

$$\sqrt{\Big(\frac{P_i}{Q_i}\Big)} \frac{d}{dx^i}\Big(\sqrt{(P_i Q_i)}\, \frac{d\Psi_i}{dx^i}\Big) + \left[\sum_{k=1}^{n_I} \left[\frac{\Pi_{j\neq k}(e_k - e_j)}{(x^i - e_k)}\right] j_i(j_i + p_i - 1)\right.$$

$$\left. + k_I^2 (x^i)^{N_{I_k}-1} + \sum_{l=2}^{N_{I_k}} \lambda_l (x^i)^{N_{I_k}-l}\right] \Psi_i = 0 . \qquad (4.41)$$

In the example on E_5 the separation equations for the Hamilton-Jacobi equation are

$$\Big[\prod_{j=1}^{2} (x^i - e_j)\Big]\Big(\frac{dW_i}{dx^i}\Big)^2 + \frac{(e_2 - e_1)}{(x^i - e_2)} k_1 + \frac{(e_1 - e_2)}{(x^i - e_1)} k_2$$

$$-\lambda x^i + \lambda_1 = 0, \qquad i = 1, 2 . \qquad (4.42)$$

and for the Helmholtz equation they are

$$\sqrt{\left[\frac{\Pi_{j=1}^2 (x^i - e_j)}{(x^i - e_1)^2 (x^i - e_2)}\right]} \frac{d}{dx^i}\left[\prod_{j=1}^{2} (x^i - e_j)(x^i - e_1)^2 (x^i - e_2)\, \frac{d\Psi_i}{dx^i}\right]$$

$$+ \left[\frac{(e_2 - e_1)}{(x^i - e_2)}\, j_1^2 + \frac{(e_1 - e_2)}{(x^i - e_1)}\, j_2(j_2 + 1) - \lambda x^i + \tilde{\lambda}_1\right] \Psi_i = 0.$$

For the elliptic case the only new prescription required is that P_i^2 be replaced by $\Sigma_r P_r^2$ where the sum extends over all indices r connected to i. Similar comments apply to expressions of the form $\{I_{kl}, P_l\}$. For our example with coordinates (4.32), the quadratic first integrals that describe separation are

$$L_1 = I_{12}^2 + I_{23}^2 + I_{13}^2 , \qquad (4.43)$$

$$L_2 = f_1 I_{23}^2 + f_2 I_{13}^2 + f_3 I_{12}^2 ,$$

$$L_3 = I_{45}^2 ,$$

$$L_4 = \sum_{i=1}^{5} P_i^2 ,$$

$$L_6 = \sum_{i=1}^{3} (I_{i4}^2 + I_{i5}^2) + c^2 [e_2(P_1^2 + P_2^2 + P_3^2) + e_1(P_4^2 + P_5^2)] .$$

The operator L_6 corresponds to the separation constant λ_1.

5 Separation of variables on H_n

1. MATHEMATICAL PRELIMINARIES

Using the same methods as in Chapters 3 and 4, we can find all separable coordinate systems for H_n. The linear first integrals (Lie symmetries) of H_n form a $\frac{1}{2}n(n+1)$-dimensional Lie algebra $SO(1, n)$ with basis

$$I_{ij} = v_i p_j - v_j p_i, \quad i > j, \quad i, j = 1, \ldots, n, \tag{5.1}$$
$$I_{0j} = v_0 p_j + v_j p_0, \quad j = 1, \ldots, n.$$

This basis satisfies the commutation relations

$$[I_{ab}, I_{cd}] = \delta_{bc} I_{ad} + \delta_{ad} I_{bc} + \delta_{bd} I_{ca} + \delta_{ac} I_{bb}, \tag{5.2}$$
$$[I_{ab}, I_{0c}] = \delta_{bc} I_{a0} - \delta_{ac} I_{b0},$$
$$[I_{0a}, I_{0b}] = I_{ab} \qquad a, b, c, d = 1, \ldots, n.$$

As we have discussed previously, two coordinate systems related by a group motion are regarded as being essentially the same. In the case of H_n the group $SO(1, n)$ [22] consists of matrices L such that

$$ds^2 = d\underset{\sim}{v}' \cdot J d\underset{\sim}{v}' = d\underset{\sim}{v} \cdot J d\underset{\sim}{v} \tag{5.3}$$

where $J = \mathrm{diag}(1, -1, \ldots, -1)$ and

$$\underset{\sim}{v}' = L\underset{\sim}{v}. \tag{5.4}$$

For $n = 3$ there are just the well known Lorentz transformations that occur frequently in relativistic physics [2]. If we are given the components of the vector $\underset{\sim}{v}$ in terms of a separable coordinate system $\{x^i\}$ then a new

vector $\underset{\sim}{v}' = L\underset{\sim}{v}$ expressed in terms of these same coordinates is regarded as specifying an equivalent coordinate system.

For H_n, however, there are more cases to consider and the problem is more complicated.

2. SEPARATION OF VARIABLES ON H_n

The classification of separable coordinate systems on H_n is greatly facilitated by the fact that all such coordinate systems can be taken as orthogonal. Indeed we have:

Theorem 5.1: Let $\{x^i\}$ be a coordinate system on H_n for which the Hamilton-Jacobi equation (I) admits a separation of variables. Then, by passing to an equivalent system of coordinates if necessary, we have $g^{ij} = \delta^{ij} H_i^{-2}$, i.e., we have only orthogonal separation. In terms of the standard coordinates on the hyperboloid $v_0, v_1, \ldots, v_n$ the ignorable variables x^α can be chosen such that the corresponding $n - n_1$ Lie symmetries p_{α_i} $(i = 1, \ldots, q)$ are of one of the following three types [23]:

$$\text{(i)} \quad p_{\alpha_1} = I_{01}, \; p_{\alpha_2} = I_{23}, \ldots, p_{\alpha_q} = I_{2q-2, 2q-1}, \tag{5.5}$$

$$\text{(ii)} \quad p_{\alpha_1} = I_{12}, \; p_{\alpha_2} = I_{34}, \ldots, p_{\alpha_q} = I_{2q-1, 2q},$$

$$\text{(iii)} \quad p_{\alpha_1} = I_{02} - I_{12}, \ldots, p_{\alpha_s} = I_{0s+1} - I_{1s+1},$$
$$p_{\alpha_{s+1}} = I_{s+2, s+3}, \ldots, p_{\alpha_q} = I_{q+1, q+2}.$$

Proof: This is based on the general block diagonal form (3.1) of the contravariant metric tensor for a separable coordinate system. Any element of the symmetry algebra $SO(1, n)$ is conjugate to an element of one of the types

$$\text{(i)} \quad L = I_{01} + b_2 I_{23} + \ldots + b_\nu I_{2\nu, 2\nu+1}, \tag{5.6}$$

$$\text{(ii)} \quad L = I_{12} + b_2 I_{34} + \ldots + b_\nu I_{2\nu-1, 2\nu},$$

(iii) $L = (I_{02} - I_{12}) + b_2 I_{34} + \dots + b_\nu I_{2\nu-1, 2\nu}$.

If one of these elements corresponds to the ignorable variable x^{α_1}, i.e., $L = p_{\alpha_1}$, then by local Lie theory the standard coordinates on H_n can be taken in each case as

$$\text{(i)}\quad (v_0, v_1, \dots, v_n) = (\rho_1 \cosh(x^{\alpha_1}+\omega_1),\ \rho_1 \sinh(x^{\alpha_1}+\omega_1),\qquad (5.7)$$
$$\rho_2 \cos(b_2 x^{\alpha_1}+\omega_2),\ \rho_2 \sin(b_2 x^{\alpha_1}+\omega_2), \dots$$
$$\rho_\nu \cos(b_\nu x^{\alpha_1}+\omega_\nu),\ \rho_\nu \sin(b_\nu x^{\alpha_1}+\omega_\nu),$$
$$v_{2\nu+2}, \dots, v_n),$$
$$\rho_1^2 - \dots - \rho_\nu^2 - v_{2\nu+2}^2 - \dots - v_n^2 = 1,$$

$$\text{(ii)}\quad (v_0, v_1, \dots, v_n) = (v_0,\ \rho_1 \cos(x^{\alpha_1}+\omega_1),\ \rho_1 \sin(x^{\alpha_1}+\omega_1), \dots$$
$$\rho_\nu \cos(b_\nu x^{\alpha_1}+\omega_\nu),\ \rho_\nu \sin(b_\nu x^{\alpha_1}+\omega_\nu),$$
$$v_{2\nu+1}, \dots, v_n),$$
$$v_0^2 - \rho_1^2 - \dots - \rho_\nu^2 - v_{2\nu+1}^2 - \dots - v_n^2 = 1,$$

$$\text{(iii)}\quad (v_0, v_1, \dots, v_n) = (\tfrac{1}{2}[\rho_1 \{(x^{\alpha_1}+\omega_1)^2 + 1\} + v],$$
$$\tfrac{1}{2}[\rho_1 \{(x^{\alpha_1}+\omega_1)^2 - 1\} + v],$$
$$\rho_1 (x^{\alpha_1}+\omega_1),\ \rho_2 \cos(b_2 x^{\alpha_1}+\omega_2),$$
$$\rho_2 \sin(b_2 x^{\alpha_1}+\omega_2), \dots,$$
$$\rho_\nu \cos(b_\nu x^{\alpha_1}+\omega_\nu),\ \rho_\nu \sin(b_\nu x^{\alpha_1}+\omega_\nu),$$
$$v_{2\nu+2}, \dots, v_n),$$
$$\rho_1 v - \rho_2^2 - \dots - \rho_\nu^2 - v_{2\nu+2}^2 - \dots - v_n^2 = 1.$$

The infinitesimal distances corresponding to these three possibilities are

$$\text{(i)}\quad ds^2 = d\rho_1^2 - d\rho_2^2 - \dots - d\rho_\nu^2 + \rho_1^2 (dx^{\alpha_1} + d\omega_1)^2 \qquad (5.8)$$
$$- \rho_2^2 (b_2 dx^{\alpha_1} + d\omega_2)^2 - \dots \rho_\nu^2 (b_\nu dx^{\alpha_1} + d\omega_\nu)^2$$
$$- dv_{2\nu+2}^2 - \dots - dv_n^2,$$

$$\text{(ii)}\quad ds^2 = dv_0^2 - d\rho_1^2 - \dots - d\rho_\nu^2 - \rho_1^2 (dx^{\alpha_1} + d\omega_1)^2$$
$$- \dots - \rho_\nu^2 (b_\nu dx^{\alpha_1} + d\omega_\nu)^2 - dv_{2\nu+1}^2 \dots - dv_n^2$$

$$\text{(iii)}\quad ds^2 = d\rho_1 dv - \rho_1^2 (dx^{\alpha_1} + d\omega_1)^2 - d\rho_2^2 - \dots - d\rho_\nu^2$$
$$- \rho_2^2 (b_2 dx^{\alpha_1} + d\omega_2)^2 - \dots - \rho_\nu^2 (b_\nu dx^{\alpha_1} + d\omega_\nu)^2$$
$$- dv_{2\nu+2}^2 - \dots - dv_n^2.$$

If there is only one ignorable variable, x^{α_1}, then the coordinate system must be orthogonal, which is only possible if $b_1 = b_2 = \dots = b_\nu = 0$, i.e., we have the three cases

$$\text{(i)}\quad p_{\alpha_1} = I_{01} \qquad (5.9)$$
$$\text{(ii)}\quad p_{\alpha_1} = I_{12}$$
$$\text{(iii)}\quad p_{\alpha_1} = I_{01} - I_{12}.$$

Indeed, the requirements that the contravariant metric have the form (3.1) (orthogonal in this case) imply

$$\text{(i)}\quad d\omega_1 = \sum_{j=2}^{\nu} \left(\frac{\rho_j^2}{\rho_1^2}\right) b_j d\omega_j, \qquad (5.10)$$
$$\text{(ii), (iii)}\quad d\omega_1 = -\sum_{j=2}^{\nu} \left(\frac{\rho_j^2}{\rho_1^2}\right) b_j d\omega_j.$$

Since the differentials $d\rho_j$, $d\omega_j$, $(j \geq 2)$ are independent and the only conditions on ρ_1^2 are the requirements given in (5.7) (i)-(iii), $d^2\omega_1 = 0$ implies that $b_j = 0$, $j = 2, \dots, v$, and $d\omega_1 = 0$. By suitably redefining α_1, we can put $\omega_1 = 0$. Hence the result we want to prove follows for one ignorable variable.

Suppose now that we have $n - n_1$ Lie symmetries p_{α_i}, $i = 1, \dots, q$, in involution. Then they must have one of the forms:

$$\text{(i)}\quad p_{\alpha_1} = I_{01} + \sum_{l=q+1}^{N} b_l^1 I_{2l,2l+1}, \qquad (5.11)$$

$$p_{\alpha_2} = I_{23} + \sum_{l=q+1}^{N} b_l^2 I_{2l,2l+2},$$

$$\vdots$$

$$p_{\alpha_q} = I_{2q-2,2q-1} + \sum_{l=q+1}^{N} b_l^q I_{2l,2l+1},$$

$$\text{(ii)}\quad p_{\alpha_1} = I_{12} + \sum_{l=q+1}^{N} b_l^1 I_{2l-1,2l},$$

$$p_{\alpha_2} = I_{34} + \sum_{l=q+1}^{N} b_l^2 I_{2l-1,2l},$$

$$\vdots$$

$$p_{\alpha_q} = I_{2q-1,2q} + \sum_{l=q+1}^{N} b_l^q I_{2l-1,2l},$$

$$\text{(iii)}\quad p_{\alpha_1} = (I_{02} - I_{12}) + \sum_{l=q}^{N} b_l^1 I_{2l-s+2,2l-s+3},$$

$$p_{\alpha_2} = (I_{03} - I_{13}) + \sum_{l=q}^{N} b_l^2 I_{2l-s+2,2l-s+3},$$

$$\vdots$$

$$p_{\alpha_s} = (I_{0s+1} - I_{1s+1}) + \sum_{l=q}^{N} b_l^s I_{2l-s+2,2l-s+3},$$

$$p_{\alpha_{s+1}} = I_{s+2,s+3} + \sum_{l=q}^{N} b_l^{s+1} I_{2l-s+2,2l-s+3},$$

$$\vdots$$

$$p_{\alpha_q} = I_{2q-s,2q-s+1} + \sum_{l=q}^{N} b_l^q I_{2l-s+2,2l-s+3}.$$

Of these three possibilities we see that case (ii) involves only elements of of the SO(n) subalgebra and so is equivalent to the problem of the corresponding case already treated on the n-sphere S_n. The two other cases give rise to the coordinates

$$\text{(i)}\quad (v_0, v_1, \ldots, v_n) = (\rho_1 \cosh(x^{\alpha_1}+\omega_1),\ \rho_1 \sinh(x^{\alpha_1}+\omega_1), \qquad (5.12)$$
$$\rho_2 \cos(x^{\alpha_2}+\omega_2),\ \rho_2 \sin(x^{\alpha_2}+\omega_2), \ldots,$$
$$\rho_v \sin(x^{\alpha_q}+\omega_q),$$
$$\rho_{q+1} \cos(\sum_{i=1}^{q} b_{q+1}^i x^{\alpha_i} + \omega_{q+1}), \ldots,$$
$$\rho_N \sin(\sum_{i=1}^{q} b_N^i x^{\alpha_i} + \omega_N),$$
$$v_{2N+2}, \ldots, v_n),$$

$$\text{(ii)}\quad (v_0, v_1, \ldots, v_n) = (\tfrac{1}{2}[\rho_1 \{ \sum_{i=1}^{s} (x^{\alpha_i}+\omega_i)^2 + 1\} + v], \qquad (5.13)$$
$$\tfrac{1}{2}[\rho_1 \{ \sum_{i=1}^{s} (x^{\alpha_i}+\omega_i)^2 - 1\} + v],$$
$$\rho_1 (x^{\alpha_1}+\omega_1), \ldots$$
$$\ldots, \rho_s (x^{\alpha_s}+\omega_s),\ \rho_{s+1}\cos(x^{\alpha_{s+1}}+\omega_{s+1}),$$
$$\ldots, \rho_q \sin(x^{\alpha_q}+\omega_q),\ \rho_{q+1}\cos(\sum_{i=1}^{q} b_q^i x^{\alpha_i}+\omega_{q+1}),$$
$$\ldots, \rho_N \sin(\sum_{i=1}^{q} b_N^i x^{\alpha_i}+\omega_N),\ v_{2N-s+4}, \ldots, v_n).$$

If the ignorable variables x^{α_i} correspond to part of a separable coordinate system, the associated covariant metric should be in block diagonal form (3.1). Just as in the case $q = 1$, this is only possible if $b_l^j = 0$ for $l = q+1, \ldots, N$, $j = 1, \ldots, q$. We can therefore assume that the ignorable variables can be chosen such that

$$\text{(i)}\quad p_{\alpha_1} = I_{01},\ p_{\alpha_2} = I_{23},\ \ldots,\ p_{\alpha_q} = I_{2q-2,2q-1}, \qquad (5.14)$$

$$\text{(ii)}\quad p_{\alpha_1} = I_{12},\ \ldots,\ p_{\alpha_q} = I_{2q-1,2q},$$

$$\text{(iii)}\quad p_{\alpha_1} = I_{02} - I_{12},\ \ldots,\ p_{\alpha_s} = I_{0s+1} - I_{1s+1},\ p_{\alpha_{s+1}} = I_{s+2,s+3},$$
$$p_{\alpha_q} = I_{2q-s,2q-s+1}.$$

In each case the functions $\omega_1, \ldots, \omega_q$ can be chosen to be zero by suitable redefinitions of the α_i's. Thus each such coordinate system is orthogonal.

As with the n-sphere and Euclidean n-space, we can directly apply Eisenhart's [14] methods to enumerate the various possible separable coordinate systems on H_n. We proceed in analogy with the treatment in Chapter 3.

We first consider the separable coordinate system $\{x^i\}$ corresponding to the infinitesimal distance

$$ds^2 = \sum_{i=1}^{n} X_i \Big[\prod_{j \neq i} (x^i - x^j) \Big] (dx^i)^2 . \tag{5.15}$$

The curvature conditions for H_n are

$$R_{ijji} = H_i^2 H_j^2 , \quad i \neq j . \tag{5.16}$$

These are equivalent to the equations

$$\Big[\prod_{l \neq j} (x^j - x^l) \Big]^{-1} \left\{ \frac{-2}{(x^i - x^j)^2} \left(\frac{1}{X_j}\right) + \frac{-1}{(x^i - x^j)} \left(\frac{1}{X_j}\right)' \right\} \tag{5.17}$$

$$+ \Big[\prod_{l \neq i} (x^i - x^l) \Big]^{-1} \left\{ \frac{-2}{(x^i - x^j)^2} \left(\frac{1}{X_i}\right) + \frac{-1}{(x^j - x^i)} \left(\frac{1}{X_i}\right)' \right\}$$

$$+ \sum_{l \neq i, j} \frac{1}{X_l (x^l - x^i)(x^l - x^j) \prod_{k \neq l} (x^l - x^k)} = +4 .$$

These equations have the solution

$$\left(\frac{1}{X_i}\right)^{(n+1)} = 4(n+1)! , \tag{5.18}$$

i.e.,

$$\frac{1}{X_i} = 4(x^i)^{n+1} = \sum_{l=0}^{n} a_l (x^i)^{n-l} = f(x^i) .$$

We normally write

$$f(x) = 4 \prod_{i=1}^{n+1} (x - e_i) . \tag{5.19}$$

To determine which metrics of this type occur on H_n we must require that the corresponding infinitesimal distance be that of a positive definite

Riemannian space. There are four classes of solution to this requirement:

Class (A). $e_i \neq e_j$ for $i \neq j$, $i, j = 1, 2, \ldots, n+1$.

If $n = 2p+1$ there are $p+1$ distinct possibilities given by the inequalities

$$(1)\quad e_1 < e_2 < x^1 < e_3 < x^2 < \ldots < e_{2p+2} < x^{2p+1} , \tag{5.20}$$

$$(2)\quad x^1 < e_1 < e_2 < e_3 < x^2 < \ldots < e_{2p+2} < x^{2p+1} ,$$

$\vdots$

$$(i)\quad x^1 < e_1 < x^2 < \ldots < x^{i-1} < e_{i-1} < e_i < e_{i+1} < x^i < \ldots < e_{i+2} < \ldots < e_{2p+2} < x^{2p+1} ,$$

$\vdots$

$$(p+1)\quad x^1 < e_1 < x^2 < \ldots < x^p < e_p < e_{p+1} < e_{p+2} < x^{p+1} < \ldots < e_{2p+2} < x^{2p+1} .$$

In fact, with convention e_j, $x^j = 0$ for j a non-positive integer, these inequalities can be summarized in the form

$$(i)\quad \ldots x^{i-2} < e_{i-2} < x^{i-1} < e_{i-1} < e_i < e_{i+1} < x^i < e_{i+2} < \ldots < e_{2p+2} < x^{2p+1} , \qquad i = 0, \ldots, p . \tag{5.21}$$

To express the coordinates for these systems in a compact way we introduce the symbols $E_i^{(j)}$ defined by

$$E_i^{(j)} = e_{j+i+1} , \quad i = 1, \ldots, 2p+2, \quad j = 1, \ldots, p+1, \tag{5.22}$$

and use the definition

$$e_r = e_s \text{ if } r = s (\mathrm{mod}\ n+1) , \tag{5.23}$$

r, s positive integers. Then the standard coordinates on H_n can always be chosen to be

$$v_0^2 = \frac{\Pi_{i=1}^{n}(x^i - E_1^{(j)})}{\Pi_{k\neq 1}(E_k^{(j)} - E_1^{(j)})}, \qquad v_\ell^2 = -\frac{\Pi_{i=1}^{n}(x^i - E_{\ell+1}^{(j)})}{\Pi_{k\neq \ell+1}(E_k^{(j)} - E_{\ell+1}^{(j)})}. \tag{5.24}$$

These ideas work equally well when $n = 2p + 2$, i.e., n is even. In this case the x^i, e_j satisfy the inequalities

$$\ldots < x^{i-2} < e_{i-2} < x^{i-1} < e_{i-1} < e_i < e_{i+1} < x^i < \ldots < e_{2p+1} < x^{2p} \tag{5.25}$$

where $i = 0, \ldots, p$. If we use the E_i's as defined in (5.22) then the standard coordinates for H_n are given by equations (5.24).

Class (B). $e_1 = \alpha + i\beta$, $e_2 = \alpha - i\beta$, $\alpha, \beta \in \mathbb{R}$, $e_3 = f_1, \ldots, e_{n+1} = f_{n-1}$.

There is only one possible choice for the coordinates x^i, i.e.,

$$x^1 < f_1 < x^2 < f_2 < \ldots < x^{n-1} < f_{n-1} < x^n.$$

A suitable choice of standard coordinates on H_n is

$$(v_0 + iv_1)^2 = \left(\frac{i}{\beta}\right) \frac{\Pi_{k=1}^{n}(x^k - \alpha - i\beta)}{\Pi_{k=1}^{n-1}(f_k - \alpha - i\beta)}, \tag{5.26}$$

$$v_j^2 = -\frac{\Pi_{k=1}^{n}(x^k - f_{j-1})}{[(\alpha - f_{j-1})^2 + \beta^2]\,\Pi_{k\neq j-1}(f_k - f_{j-1})}, \quad j = 2, \ldots, n.$$

Class (C). $e_1 = e_2 = a$, $e_j = g_{j-2}$, $j = 3, \ldots, n+1$ with $g_k \neq g_\ell$ if $k \neq \ell$ and $g_k \neq a$ for any k. This case divides into two families of coordinate systems. With the same conventions as for type (A) coordinates, the ranges of variation of the coordinates x^i can be given as

$$\text{(a)} \quad \ldots x^{i-1} < g_{i-1} < x^i < g_i < a < x^{i+1} < g_{i+1} < \ldots < g_{n-1} < x^n, \tag{5.27}$$

$$\text{(b)} \quad \ldots x^{i-1} < g_{i-1} < x^i < g_i < x^{i+1} < a < g_{i+1} < \ldots < g_{n-1} < x^n.$$

If $n = 2p + 1$ (i.e., n is odd), then $i = 0, \ldots, p$ and if $n = 2p$ (i.e., n is even) then $i = 0, \ldots, p$, so that in either case there are $p + 1$

distinguishable cases to consider. In fact if we define symbols

$$G_j^{(i)} = g_{j+1}, \quad j = 1, \ldots, n-1, \quad i = 0, \ldots, p,$$

where $g_r = g_s$ if $r = s \pmod{n-1}$, r, s positive integers, then we can write down a suitable set of standard coordinates for H_n as

$$(v_0 - v_1)^2 = \varepsilon \frac{\Pi_{i=1}^{n}(x^i - a)}{\Pi_{j=1}^{n-1}(G_j^{(i)} - a)}, \tag{5.28}$$

$$(v_0^2 - v_1^2) = \frac{\partial}{\partial a}\left[\frac{\Pi_{i=1}^{n}(x^i - a)}{\Pi_{j=1}^{n-1}(G_j^{(i)} - a)}\right],$$

$$v_j^2 = -\frac{\Pi_{i=1}^{n}(x^i - G_{j-1}^{(i)})}{(a - G_{j-1}^{(i)})^2\,\Pi_{\ell\neq j-1}(G_\ell^{(i)} - G_{j-1}^{(i)})},$$

$j = 2, \ldots, n$. Here $\varepsilon = +1$ if we have case (a) and $\varepsilon = -1$ if we have case (b).

Class (D). $e_1 = e_2 = e_3 = b$, $e_j = h_{j-3}$, $j = 4, \ldots, n+1$ with $h_k \neq h_\ell$ if $k \neq \ell$ and $h_k \neq a$ for any k. This case divides into a family of solutions with coordinates varying in the ranges

$$\ldots x^{i-1} < h_{i-1} < x^i < h_i < x^{i+1} < b < x^{i+2} < h_{i+1} \ldots < x^{n-1} < h_{n-2} < x^n \tag{5.29}$$

where, as usual, $h_i = 0$ if $i \leq 0$. Just as in the previous case there are $p + 1$ distinct cases to consider where $p = \frac{1}{2}(n-1)$ if n is odd and $p = \frac{1}{2}n$ if n is even. Defining symbols

$$H_j^{(i)} = h_{j+1}, \quad j = 1, \ldots, n-2, \quad i = 0, \ldots, p,$$

with $h_r = h_s$ if $r = s \pmod{n-2}$, r, s positive integers, we find that a suitable set of standard coordinates for H_n is

$$(v_0 - v_1)^2 = -\frac{\Pi_{i=1}^{n}(x^i - b)}{\Pi_{j=1}^{n-2}(H_j^{(i)} - b)}, \tag{5.30}$$

$$2v_2(v_0 - v_1) = -\frac{\partial}{\partial b}\left[\frac{\Pi_{i=1}^{n}(x^i - b)}{\Pi_{j=1}^{n-2}(H_j^{(i)} - b)}\right],$$

$$v_0^2 - v_1^2 - v_2^2 = -\tfrac{1}{2}\frac{\partial^2}{\partial b^2}\left[\frac{\Pi_{i=1}^{n}(x^i - b)}{\Pi_{j=1}^{n-2}(H_j^{(i)} - b)}\right],$$

$$v_j^2 = -\frac{\Pi_{i=1}^{n}(x^i - H_{j-2}^{(i)})}{\Pi_{l=j-2}^{n}(H_l^{(i)} - H_{j-2}^{(i)})(b - H_{j-2}^{(i)})^3}, \quad j = 3, \ldots, n.$$

Clearly the variety of separable systems for H_n is much richer than that for S_n and E_n. There are four classes of separable coordinate systems and in classes A, C, D the number of distinct types of coordinates increases with the dimension.

We now proceed to examine the possible metrics in which some of the σ_{ij} are constant functions. The simplest such case occurs when the components of the contravariant metric have the form

$$H_i^2 = [X_i \prod_{i\neq j}(x^i - x^j)](\prod_{l=k+1}^{n}\sigma_l), \quad i, j = 1, \ldots, k, \tag{5.31}$$

$$H_l^2 = [X_l \prod_{m\neq l}(x^l - x^m)], \quad l, m = k+1, \ldots, n.$$

The conditions $R_{kllk} = H_k^2 H_l^2$ are the same as for (5.15), i.e. (5.17) but with $n \to n - k = n'$. With $\tilde{H}_i^2 = [X_i \prod_{j\neq i}(x^i - x^j)]$, the conditions $R_{ijji} = H_i^2 H_j^2$ and $R_{illi} = H_i^2 H_l^2$ are equivalent to

$$\tilde{H}_i^{-2}\tilde{H}_j^{-2}\tilde{R}_{ijji} + (\prod_{l=k+1}^{n}\sigma_l)\left[\sum_{l'=k+1}^{n}\frac{1}{4H_{l'}^2}\left(\frac{\sigma'_{l'}}{\sigma_{l'}}\right)^2 - 1\right] = 0, \tag{5.32}$$

$$2\left(\frac{\sigma''_l}{\sigma_l}\right) - \left(\frac{\sigma'_l}{\sigma_l}\right)^2 - \left(\frac{\sigma'_l}{\sigma_l}\right)\left[\frac{\partial}{\partial x^l}\log H_l^2 + H_l^2 \sum_{m\neq l}\frac{1}{H_m^2(x^l - x^m)}\right] = 4H_l^2, \tag{5.33}$$

where $\tilde{R}_{ijji}$ is the Riemann curvature tensor for the Riemannian manifold with infinitesimal distance $ds_2^2 = \sum_{i=1}^{k}\tilde{H}_i^2(dx^i)^2$. For the functions X_l we have

$$\left(\frac{1}{X_l}\right)^{(n-k+1)} = 4(n - k + 1)! \tag{5.34}$$

but for the metric coefficients $\tilde{H}_i^2$ there are several possibilities as follows:

Case (A).

$$\frac{1}{X_l} = 4\prod_{m=1}^{n-k+1}(x^l - E_m^{(j)}), \quad l = k+1, \ldots, n, \tag{5.35}$$

with $E_m \neq E_n$ if $m \neq n$ and all E_p real. Then

$$(\prod_{l=k+1}^{n}\sigma_l) = \frac{\Pi_{l=k+1}^{n}(x^l - E_J)}{\Pi_{m\neq J}(E_m - E_J)}, \tag{5.36}$$

where $\tilde{R}_{ijji} = \tilde{H}_i^2\tilde{H}_j^2$ if $J = 1$
$\qquad = -\tilde{H}_i^2\tilde{H}_j^2$ if $J \neq 1$,

i.e.,

$$\frac{1}{X_i} = 4\prod_{j=1}^{k+1}(x^i - E'_j), \quad i = 1, \ldots, k, \quad J = 1, \tag{5.37}$$

$$\frac{1}{X_i} = -4\prod_{j=1}^{k+1}(x^i - e_j), \quad i = 1, \ldots, k, \quad J \neq 1. \tag{5.38}$$

The coordinates in these two cases can be taken as:

(a) $(v_0, v_1, \ldots, v_n) = (u_0w_0, u_0w_1, \ldots, u_0w_k, u_1, \ldots, u_{n-k})$,

where

$$u_0^2 - \sum_{i=1}^{n-k}u_i^2 = 1, \quad w_0^2 - \sum_{j=1}^{k}w_j^2 = 1 \tag{5.39}$$

and

$$w_i^2 = \varepsilon_i \frac{\Pi_{j=1}^{k}(x^j - E'_{i+1})}{\Pi_{j\neq i+1}(E'_j - E'_{i+1})}, \quad i = 0, 1, \ldots, k, \tag{5.40}$$

$$u_l^2 = \varepsilon_l \frac{\Pi_{m=k+1}^{n}(x^m - E_{l+1})}{\Pi_{m\neq l}(E_m - E_{l+1})}, \quad l = 0, 1, \ldots, n-k, \tag{5.41}$$

where $\varepsilon_l = +1$ if $l = 0$ and -1 otherwise. This case corresponds to $J = 1$.

(b) $(v_0, v_1, \ldots, v_n) = (u_0, u_1, \ldots, u_{J-1} w_1, \ldots, u_{J-1} w_{k+1}, u_J, \ldots, u_{n-k})$, (5.42)

where

$$u_0^2 - \sum_{i=1}^{n-k} u_i^2 = 1, \qquad \sum_{l=1}^{k+1} w_l^2 = 1.$$

The u_l coordinates are given as in (5.41) and

$$w_i^2 = \frac{\Pi_{j=1}^{k}(x^j - e_i)}{\Pi_{j\neq i}(e_i - e_j)}, \quad i, j = 1, \ldots, k+1. \tag{5.43}$$

In this latter case the w_i^2 are given in terms of elliptic coordinates on the k-sphere S_k and consequently

$$e_1 < x^1 < e_2 < \ldots < x^k < e_{k+1}. \tag{5.44}$$

Case (B).

$$\frac{1}{X_l} = 4[(x^l - \alpha)^2 + \beta^2] \prod_{m=1}^{n-k-1} (x^l - f_m) \tag{5.45}$$

with f_m all real and different, α, β real. Then

$$\left(\prod_{l=k+1}^{n} \sigma_l\right) = \frac{\Pi_{l=k+1}^{n}(x^l - f_J)}{\Pi_{m\neq J}(f_m - f_J)}. \tag{5.46}$$

The standard coordinates on H_n are

$$(v_0, v_1, \ldots, v_n) = (u_0, u_1, \ldots, u_{J-1} w_1, \ldots, u_{J-1} w_{k+1}, u_J, \ldots, u_{n-k}) \tag{5.47}$$

where

$$u_0^2 - \sum_{i=1}^{n-k+1} u_i^2 = 1, \qquad \sum_{l=1}^{k+1} w_l^2 = 1.$$

The w_i^2 are given as in (5.43). The u_j coordinates are

$$(u_0 + iu_1)^2 = \left(\frac{i}{\beta}\right) \frac{\Pi_{l=k+1}^{n}(x^l - \alpha - i\beta)}{\Pi_{l=1}^{n-k-1}(f_l - \alpha - i\beta)}, \tag{5.48}$$

$$u_m^2 = \frac{\Pi_{l=k+1}^{n}(x^l - f_{m-1})}{[(\alpha - f_{m-1})^2 + \beta^2]\,\Pi_{l\neq m-1}(f_l - f_{m-1})}, \qquad m = 2, \ldots, n-k.$$

Case (C).

$$\frac{1}{X_l} = 4(x^l - a)^2 \prod_{m=1}^{n-k-1} (x^l - G_m). \tag{5.49}$$

There are two possibilities:

(a) $$\left(\prod_{l=k+1}^{n} \sigma_l\right) = \frac{\Pi_{l=k+1}^{n}(x^l - G_J)}{\Pi_{m\neq J}(G_m - G_J)(a - G_J)^2} \tag{5.50}$$

and $R_{ijji} = -H_i^2 H_j^2$, $i, j = 1, \ldots, k$; $i \neq j$.

(b) $$\left(\prod_{l=k+1}^{n} \sigma_l\right) = \varepsilon \frac{\Pi_{l=k+1}^{n}(x^l - a)}{\Pi_{m=1}^{n-k-1}(G_j - a)} \tag{5.51}$$

and $R_{ijji} = 0$, $i, j = 1, \ldots, k$; $i \neq j$.

The coordinates in each of these cases may be taken as

(a) $(v_0, v_1, \ldots, v_n) = (u_0, u_1, \ldots, u_{J+1} w_1, \ldots, u_{J+1} w_{k+1}, \ldots, u_{n-k})$ (5.52)

with u_l, w_i satisfying the same conditions as those coordinates in (5.46). The w_i^2 are given as in (5.43). The u_l coordinates are

$$(u_0 - u_1)^2 = \varepsilon \frac{\Pi_{l=k+1}^{n}(x^l - a)}{\Pi_{m=1}^{n-k-1}(G_m - a)} \tag{5.53}$$

$$u_0^2 - u_1^2 = \frac{\partial}{\partial a}\left(\frac{\Pi_{l=k+1}^{n}(x^l - a)}{\Pi_{m=i}^{n-k-1}(G_m - a)}\right)$$

$$u_j^2 = -\frac{\Pi_{l=k+1}^{n}(x^l - G_{j-1})}{(a-G_{j-1})^2 \Pi_{l\neq j-1}(G_l - G_{j-1})}, \quad j = 2, \ldots, n-k.$$

$$\text{(b)} \quad (v_0, v_1, \ldots, v_n) = (\tfrac{1}{2}[(u_0-u_1)(w_1^2+\ldots+w_k^2+1) + (u_0+u_1) \tag{5.54}$$

$$\tfrac{1}{2}[(u_0-u_1)(w_1^2+\ldots+w_k^2-1) + (u_0+u_1)],\ w_1(u_0-u_1),$$

$$\ldots,\ w_k(u_0-u_1),\ u_2, \ldots, u_{n-k}).$$

The u_j are given as in (a) and the w_i are one of the two possible elliptic coordinate systems on E_k [1], i.e.,

$$\text{(i) Elliptic:} \quad w_j^2 = c^2 \frac{\Pi_{i=1}^{k}(x^i - e_j)}{\Pi_{i\neq j}(e_i - e_j)}, \quad j = 1, \ldots, k. \tag{5.55}$$

$$\text{(ii) Parabolic:} \quad w_1 = \tfrac{1}{2}c(x^1 + \ldots + x^n + e_1 + \ldots + e_{k-1}), \tag{5.56}$$

$$w_j^2 = -c^2 \frac{\Pi_{i=1}^{k}(x^i - e_{j-1})}{\Pi_{i\neq j}(e^i - e^j)}, \quad j = 2, \ldots, k.$$

$$\text{Case (D).} \quad \left(\frac{1}{X_l}\right) = 4(x^l - b)^3 \prod_{m=1}^{n-k-2}(x^l - G_m). \tag{5.57}$$

There are two possibilities:

$$\text{(a)} \quad \left(\prod_{l=k+1}^{n} \sigma_l\right) = \frac{\Pi_{l=k+1}^{n}(x^l - H_{J-2})}{\Pi_{l\neq J}(H_l - H_{J-2})(b - H_{J-2})^3} \tag{5.58}$$

and $\tilde{R}_{ijji} = -\tilde{H}_i^2 \tilde{H}_j^2$, $i, j = 1, \ldots, k$, $i \neq j$;

$$\text{(b)} \quad \left(\prod_{l=k+1}^{n} \sigma_l\right) = -\frac{\Pi_{i=1}^{n}(x^l - b)}{\Pi_{j=1}^{n-2}(H_j - b)} \tag{5.59}$$

The coordinates in each of these cases may be taken as

$$\text{(a)} \quad (v_0, v_1, \ldots, v_n) = (u_0, u_1, \ldots, u_{J-2}w_1, \ldots, u_{J-2}w_{k+1}, \ldots u_{n-k}) \tag{5.60}$$

with u_l, w_i satisfying the same conditions as in (5.42). The w_i^2 are given as in (5.43). The u_l coordinates are

$$(u_0 - u_1)^2 = -\frac{\Pi_{l=k+1}^{n}(x^l - b)}{\Pi_{l=1}^{n-k-2}(H_l - b)} \tag{5.61}$$

$$2u_2(u_0 - u_1) = -\frac{\partial}{\partial b}\left(\frac{\Pi_{l=k+1}^{n}(x^l - b)}{\Pi_{l=1}^{n-k-2}(H_l - b)}\right)$$

$$u_0^2 - u_1^2 - u_2^2 = -\tfrac{1}{2}\frac{\partial^2}{\partial b^2}\left(\frac{\Pi_{l=k+1}^{n}(x^l - b)}{\Pi_{l=1}^{n-k-2}(H_l - b)}\right)$$

$$u_j^2 = -\frac{\Pi_{l=k+1}^{n}(x^l - H_{j-2})}{\Pi_{l\neq j-2}(H_l - H_{j-2})(b - H_{j-2})^3} \qquad j = 3, \ldots, n-k.$$

$$\text{(b)} \quad (v_0, v_1, \ldots, v_n) = (\tfrac{1}{2}[(u_0-u_1)(w_1^2+\ldots+w_k^2+1) + (u_0+u_1)], \tag{5.62}$$

$$\tfrac{1}{2}[(u_0-u_1)(w_1^2+\ldots+w_k^2-1) + (u_0+u_1)],\ w_1(u_0-u_1), \ldots$$

$$\ldots\ w_k(u_0-u_1),\ u_2, \ldots, u_{n-k}).$$

The u_j are given as in (5.60) and the w_i correspond to the two possible nondegenerate systems on E_k: elliptic or parabolic coordinates. This completes the treatment of the case (5.31).

We recall that in general the infinitesimal distance can be written in the form

$$ds^2 = \sum_{I=1}^{p} \{ \sum_{i\in N_I} (H_i^{(I)})^2 (dx^i)^2 \} [\prod_{l\in N_{p+1}} (\sigma_l + \alpha_I)] \tag{5.63}$$

$$+ \sum_{j\in N_{p+1}} (H_j^{(p+1)})^2 (dx^j)^2$$

where $\{N_1, \dots, N_{p+1}\}$ is a partition of the integers $1, \dots, n$ into mutually exclusive sets N_I; $N_I \cap N_J = \emptyset$ $(I \neq J)$. In addition, $\partial_j H_i^{(I)} = 0$ if $j \notin N_I$. The curvative conditions $R_{ijji} = H_i^2 H_j^2$ $(i \neq j)$ are equivalent to the relations

$$R_{ijji}^{(p+1)} = (H_i^{p+1})^2 (H_j^{p+1})^2, \quad i, j \in N_{p+1}, \tag{5.64}$$

$$(H_i^{(I)})^{-2} (H_j^{(I)})^{-2} R_{ijji}^{(I)} + \prod_{k\in N_{p+1}} [(\sigma_k + \alpha_I)][\frac{1}{4} \sum_{l\in N_{p+1}} (H_l^{p+1})^{-2}$$

$$\times \frac{\sigma_l'^2}{(\sigma_l + \alpha_I)^2} - 1] = 0, \quad i, j \in N_I, \tag{5.65}$$

$$2\frac{\sigma_l''}{(\sigma_l + \alpha_I)} - \left(\frac{\sigma_l'}{\sigma_l + \alpha_I}\right)^2 - \frac{\sigma_l'}{(\sigma_l + \alpha_I)} \left[\frac{\partial}{\partial x^l} \log H_l^2 \right. \tag{5.66}$$

$$\left. + H_l^2 \sum_{\substack{m\neq l \\ m\in N_{p+1}}} \frac{1}{H_m^2 (x^l - x^m)} \right] = 4H_l^2,$$

$$\frac{1}{4} \sum_{l\in N_{p+1}} \frac{1}{H_l^2} \frac{\sigma_l'^2}{(\sigma_l + \alpha_I)(\sigma_l + \alpha_J)} = 1, \tag{5.67}$$

where, as in Chapter 3, we denote $R_{ijkl}^{(I)}$ as the Riemannian curvature tensor of the Riemannian manifold with infinitesimal distance

$$ds_I^2 = \sum_{i\in N_I} (H_i^{(I)})^2 (dx^i)^2 . \tag{5.68}$$

We recall that the metric coefficients $(H_i^{(p+1)})^2$ have the form

$$(H_i^{(p+1)})^2 = X_i [\prod_{j\neq i} (x^i - x^j)], \quad i \in N_{p+1}, \tag{5.69}$$

where $(1/X_i)^{n_{p+1}} = 4(n_{p+1} + 1)!$ There are then various possibilities which we list in abbreviated form:

$$[\mathrm{I}] \quad (\frac{1}{X_i}) = 4 \prod_{j=1}^{n+1} (x^i - E_j), \quad E_j \neq E_k, \; j \neq k; \; i \in N_{p+1}. \tag{5.70}$$

Then

$$[\prod_{l\in N_{p+1}} (\sigma_l + \alpha_I)] = \varepsilon \frac{\prod_{i\in N_{p+1}} (x^i - E_I)}{\prod_{J\neq I} (E_J - E_I)} \tag{5.71}$$

where $\varepsilon_I = +1$ if $I = 1$, and $\varepsilon_I = -1$ otherwise. In addition,

$$R_{ijji}^{(I)} = \varepsilon_I (H_i^{(I)})^2 (H_j^{(I)})^2, \tag{5.72}$$

$$[\mathrm{II}] \quad (\frac{1}{X_i}) = 4[(x^i - \alpha)^2 + \beta^2] \prod_{j=1}^{n-2} (x^i - f_j), \quad f_i \neq f_j \text{ if } i \neq j, \; i \in N_{p+1}. \tag{5.73}$$

Then

$$[\prod_{l\in N_{p+1}} (\sigma_l + \alpha_I)] = -\frac{\prod_{i\in N_{p+1}} (x^i - f_I)}{[(\alpha - f_I)^2 + \beta^2] \prod_{J\neq I} (f_J - f_I)} \tag{5.74}$$

and

$$R_{ijji}^{(I)} = -(H_i^{(I)})^2 (H_j^{(I)})^2. \tag{5.75}$$

$$[\mathrm{III}] \quad \frac{1}{X_i} = 4(x^i - a)^2 \prod_{j=1}^{n-1} (x^i - G_j), \quad G_j \neq G_k, \; j \neq k, \; i \in N_{p+1}. \tag{5.76}$$

There are then two possibilities:

$$\prod_{l\in N_{p+1}} (\sigma_l + \alpha_1) = \varepsilon \frac{\prod_{i\in N_{p+1}} (x^i - a)}{\prod_{j=1}^{n-1} (G_j - a)}, \tag{5.77}$$

$$\prod_{l\in N_{p+1}} (\sigma_l + \alpha_I) = -\frac{\prod_{i\in N_{p+1}} (x^i - G_I)}{(a - G_I)^2 \prod_{J\neq I} (G_J - G_I)} \tag{5.78}$$

and

$$R^{(I)}_{ijji} = 0, \quad R^{(I)}_{ijji} = -(H^{(I)}_i)^2 (H^{(I)}_j)^2 \quad \text{for } I \neq 1. \tag{5.79}$$

$$[IV] \quad (\frac{1}{X_i}) = 4(x^i - b)^3 \prod_{j=1}^{n-3} (x^i - H_j), \quad H_j \neq H_k, \quad j \neq k, \quad i \in N_{p+1}. \tag{5.80}$$

There are again two possibilities:

$$[\prod_{l \in N_{p+1}} (\sigma_l + \alpha_1)] = - \frac{\prod_{i \in N_{p+1}} (x^i - b)}{\prod_{j=1}^{n-2} (H_j - b)}, \tag{5.81}$$

$$[\prod_{l \in N_{p+1}} (\sigma_l + \alpha_I)] = - \frac{\prod_{i \in N_{p+1}} (x^i - H_I)}{\prod_{J \neq I} (H_J - H_I)(b - H_I)^3} \tag{5.82}$$

and $R^{(I)}_{ijji} = 0$, $R^{(I)}_{ijji} = -(H^{(I)}_i)^2 (H^{(I)}_j)^2$ for $I \neq 1$.

These results give us the branching laws for the construction of graphs representing the various possible coordinate systems on H_n. In each case the infinitesimal distance has the general form

$$ds^2 = \sum_{I=1}^{p} ds^2_I \left[\frac{\prod_{l \in N_{p+1}} (x^l - e_I)}{U_I} \right] + \frac{1}{4} \sum_{i \in N_{p+1}} \frac{\prod_{j \neq i} (x^i - x^j)}{\prod_{j=1}^{n+1} (x^i - e_j)} (dx^i)^2, \tag{5.83}$$

where ds^2_I is the infinitesimal distance of a Riemannian manifold of constant (possibly zero) curvature and U_I is a fixed number related to the choice of coordinates x^i, $i \in N_{p+1}$. If ε_I is a double or triple root of $\prod_{j=1}^{n+1} (x - \varepsilon_j)$ then ds^2_I is the infinitesimal distance of a flat space. This can of course only occur once.

3. THE CONSTRUCTION OF SEPARABLE COORDINATE SYSTEMS ON H_n

We have shown that the basic building blocks for separable coordinate systems on H_n are the four types of general coordinates (types A, B, C, D) and the systems we have already developed for the n-sphere S_n and real Euclidean n-space E_n. For $n = 1$ the coordinates are given by

A. $$v_0^2 = \frac{(x^1 - E_1)}{(E_2 - E_1)}, \quad v_1^2 = - \frac{(x^1 - E_2)}{(E_1 - E_2)} \tag{5.84}$$

where $E_1 < E_2 < x^1$.

B. $$(v_0 + iv_1)^2 = (\frac{i}{\beta})(x^1 - \alpha - i\beta), \quad \alpha, \beta \text{ real}, \tag{5.85}$$

or

$$v_0^2 - v_1^2 = 1, \quad 2v_0 v_1 = \frac{1}{\beta}(x^1 - \alpha)$$

and x^1 any real number.

C. $$(v_0 - v_1)^2 = \varepsilon (x^1 - a), \quad v_0^2 - v_1^2 = 1 \tag{5.86}$$

and we distinguish two cases

(i) $\varepsilon = +1$, $\quad x^1 > a$,

(ii) $\varepsilon = -1$, $\quad x^1 < a$.

In fact we need not make this distinction, since we can always define the standard coordinates in this system as

$$(v_0 - v_1)^2 = x^1, \quad v_0^2 - v_1^2 = 1, \quad x^1 > 0. \tag{5.87}$$

It may seem somewhat superfluous to be discussing various coordinate systems on H_1 when there is only one variable. However, as we shall see subsequently, the coordinates (5.84) and (5.86) are intimately related to the polyspherical and horospherical coordinates for which Vilenkin [21] has developed graphical techniques. This relationship will be made more explicit later in this section. There is for $n = 1$ no system of type D.

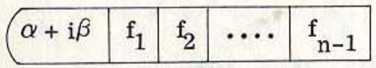
The main problem remaining in developing graphical methods for describing all such coordinate systems is to devise a suitable notation for the 'irreducible' coordinate types A, B, C. D. We do this as follows:

(I) For type A we use the notation

$$E_1^{(j)} \ \boxed{E_3^{(1)}} \boxed{\ \cdots\ } \boxed{E_{n+1}^{(1)}}$$

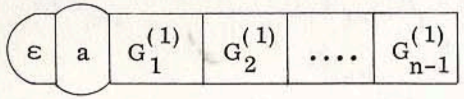
to denote the coordinate system (5.24).

(II) For type B we use the notation



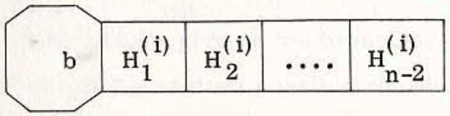
to denote the coordinate system (5.26).

(III) For type C we use the notation



to denote the coordinate system (5.28).

(iv) For type D we use the notation



to denote the coordinate system (5.30). For the case when the metric coefficients have the form (5.31) the various possible graphical representations of coordinates are shown in Table 5.1. In the case n = 1, for type A we can make the transformation

$$x^{1\,\prime} = ax^{i} + b, \quad E_i' = aE_i + b, \quad i = 1, 2, \tag{5.88}$$

and take $E_1 = 0$, $E_2 = 1$. Putting $x^1 = \cosh^2 t$, we get

$$v_0 = \cosh t, \quad v_1 = \sinh t,$$

the natural hyperbolic coordinates on the hyperbola $v_0^2 - v_1^2 = 1$. This is the analogue of Vilenkin's [21] polyspherical coordinates which occur when graphs are composed of the irreducible blocks $\boxed{0 \mid 1}$, (0 | 1). Indeed, consider the graph

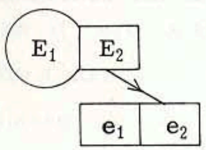
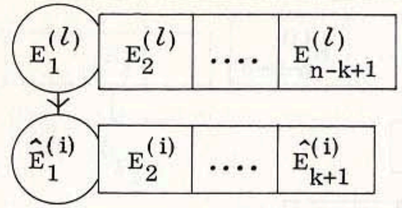
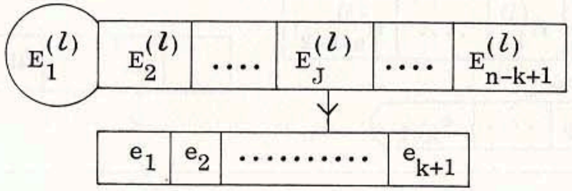


Table 5.1

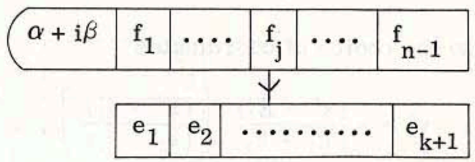
(A) (i)



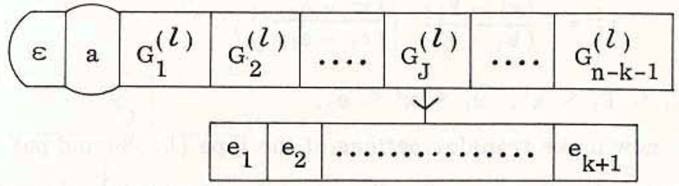
(ii)



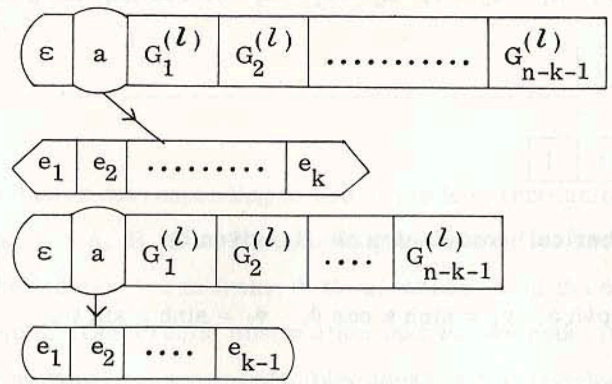
(B)



(C) (i)



(ii)



(D) (i)

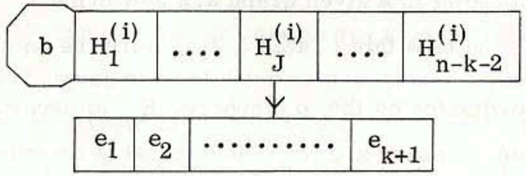
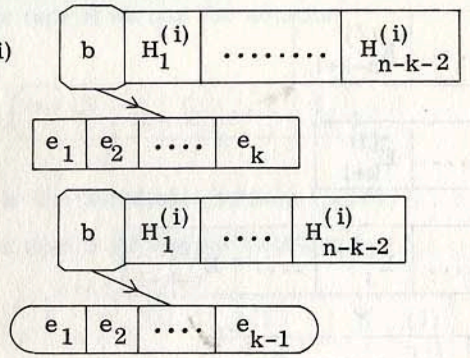


Table 5.1 (continued)

(ii)



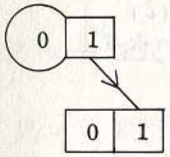
This graph corresponds to the choice of coordinates

$$v_0^2 = \frac{(x^1 - E_1)}{(E_2 - E_1)}\ ,\quad v_1^2 = -\frac{(x^1 - E_2)}{(E_1 - E_2)}\left[\frac{(x^2 - e_1)}{(e_2 - e_1)}\right],\tag{5.89}$$

$$v_2^2 = -\frac{(x^1 - E_2)}{(E_1 - E_2)}\left[\frac{(x^2 - e_2)}{(e_1 - e_2)}\right],$$

where $E_1 < E_2 < x^1$, $e_1 < x^2 < e_2$.

If we now make transformations of the type (5.88) and put $x^1 = \cosh^2 a$, and $x^2 = \cos^2\phi$, $E_1 = e_1 = 0$, $E_2 = e_2 = 1$, we see that the graph



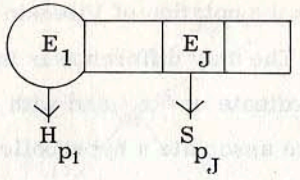
represents 'spherical' coordinates on H_2 given by

$$v_0 = \cosh a,\quad v_1 = \sinh a\cos\phi,\quad v_2 = \sinh a\sin\phi\ .\tag{5.90}$$

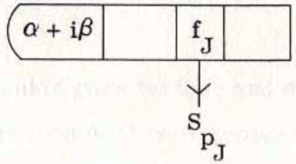
The rules for drawing arrows in a given graph are now evident. We adopt an obvious shorthand to indicate this (Table 5.2). Here the notation S_{p_J} refers to separable coordinates on the p_J sphere, E_p to separable coordinates in Euclidean p space and H_{p_1} to separable coordinates on the p_1 dimensional hyperboloid. We also here introduce the notation ${}_n^L v_i$ to

Table 5.2

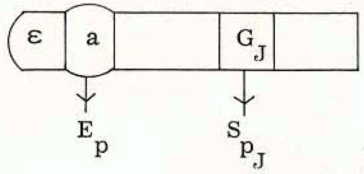
(A)



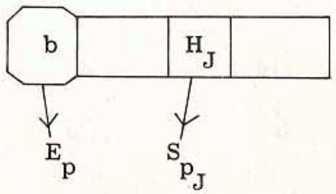
(B)



(C)



(D)

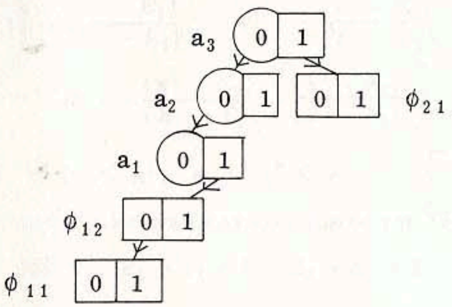


denote coordinates corresponding to one of the four irreducible blocks. In this notation, L = A, B, C or D according to whether we have type A, B, C or D. The indices i run from 0 to n, where n is the dimension of the hyperboloid. One crucial observation that we can make from the rules for drawing arrows between irreducible blocks is that irreducible blocks of type B, C, D can only occur once in any given graph. Vilenkin's graphical techniques are valid for polyspherical analogues on H_n, i.e., those graphs composed of (0 | 1), $\boxed{0\,|\,1}$ irreducible blocks. He also gives a treatment of horospherical coordinates. These coordinates are composed of the irreducible blocks (0 | 1), $\boxed{0\,|\,1}$ and (0). (For n = 1 graphs, $\varepsilon\,(0)$ are not distinguished by ε and we merely use the symbol (0).) For polyspherical coordinates the notation of Vilenkin is defined in the same way as on the n-sphere. The only difference is that with the root of the tree we associate the coordinate $x \equiv x_n$ and with each of the vertices of first rank $x_{01}, \ldots, x_{0k}$ we associate a hyperbolic rotation in the (x_{0m}, x_0) plane by the 'angle' ϕ_m:

$$x'_0 = x_0 \cosh\phi_m + x_{0m} \sinh\phi_m \,, \tag{5.91}$$

$$x'_{0m} = x_0 \sinh\phi_m + x_{0m} \cosh\phi_m \,.$$

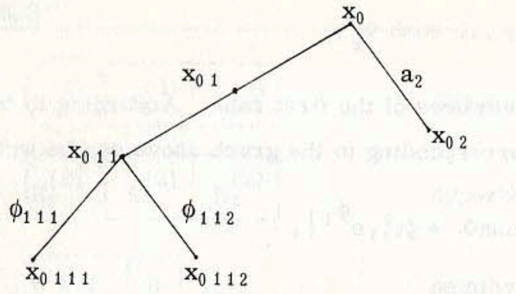
Apart from this modification, the scheme adhered to is the same as for the sphere. The graph



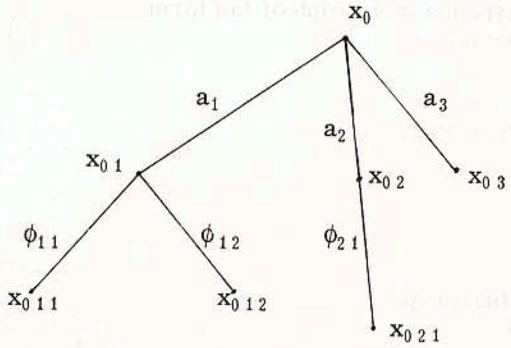
corresponds to the choice of coordinates

$$
\begin{aligned}
x_0 &= \cosh a_3 \cosh a_2 \cosh a_1 \,, && (5.92)\\
x_{03} &= \sinh a_3 \,,\\
x_{02} &= \cosh a_3 \sinh a_2 \cos\phi_{21} \,,\\
x_{01} &= \cosh a_3 \cosh a_2 \sinh a_1 \cos\phi_{12} \cos\phi_{11} \,,\\
x_{021} &= \cosh a_3 \sinh a_2 \sin\phi_{21} \,,\\
x_{012} &= \cosh a_3 \cosh a_2 \sinh a_1 \sin\phi_{12} \,,\\
x_{011} &= \cosh a_3 \cosh a_2 \sinh a_1 \cos\phi_{12} \sin\phi_{11} \,.
\end{aligned}
$$

In our notation this coordinate system would be represented by the graph



Vilenkin goes further and discusses a graphical method for dealing with horospherical coordinates as follows: if we consider a tree containing a vertex x_{011} but not x_{012} as shown below



then with all vertices except the vertex x_{011} we associate the same transformation as for polyspherical graphs. With the vertex x_{011} we associate a birational transformation $h(t)$ which Vilenkin calls a horospherical rotation in the subspace (x_0, x_{01}, x_{011}) by an angle t. Specifically, the action of $h(t)$ is given by the formulas

$$x'_0 = x_0 + \frac{2tx_{011} + t^2}{2(x_0 + x_{01})} \,, \tag{5.93}$$

$$x'_{01} = x_{01} - \frac{2tx_{011} + t^2}{2(x_0 + x_{01})} \,,$$

$$x'_{011} = x_{011} + t \,.$$

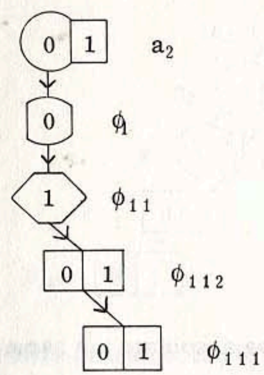
He then introduces a parameter ϕ_{11} by putting

$$t = \phi_{11} e^{\phi_1} \cosh \phi_2 \ldots \cosh \phi_k ,$$

where k is the number of vertices of the first rank. According to this scheme, the coordinates corresponding to the graph above can be written

$$\begin{aligned}
x_0 &= \cosh a_2 \left[\cosh\phi_1 + \tfrac{1}{2}\phi_{11}^2 e^{\phi_1} \right] , \\
x_{02} &= \sinh a_2 , \\
x_{01} &= \cosh a_2 \left[\sinh \phi_1 - \tfrac{1}{2}\phi_{11}^2 e^{\phi_1} \right] , \\
x_{011} &= e^{\phi_1} \phi_{11} \cos \phi_{111} \cos \phi_{112} \cosh a_2 , \\
x_{0112} &= e^{\phi_1} \phi_{11} \sin \phi_{112} \cosh a_2 , \\
x_{0111} &= e^{\phi_1} \phi_{11} \sin \phi_{111} \cos \phi_{112} \cosh a_2 .
\end{aligned} \qquad (5.94)$$

In our notation this corresponds to a graph of the form

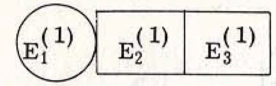
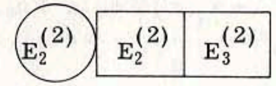
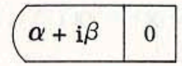
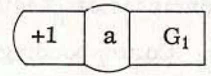
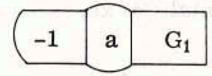
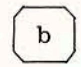
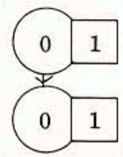
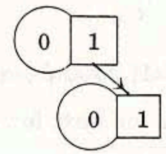
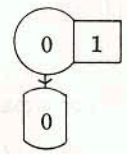


For $n = 2$ there are nine possible graphs, shown in Table 5.3. These graphs simply represent all the coordinate systems which permit variable separation on H_2. The notation for these coordinates originates in [24], see also [25].

4. PROPERTIES OF SEPARABLE SYSTEMS ON H_n

We can now discuss the separation equations for both the Hamilton-Jacobi equation and the Helmholtz equation. We first consider the irreducible blocks A, B, C, D. In each case the Hamilton-Jacobi equation has the form

Table 5.3

(1)		elliptic
(2)		hyperbolic
(3)		semihyperbolic
(4)		hyperbolic parabolic
(5)		elliptic parabolic
(6)		semicircular parabolic
(7)		equidistant
(8)		spherical
(9)		horicyclic

$$H = \sum_{i=1}^{n} \frac{1}{[\Pi_{j\neq i}(x^i - x^j)]} P_i^2 = E , \tag{5.95}$$

where $P_i = \sqrt{(\Theta_i)}(\partial W/\partial x^i)$ and $\Theta_i = 4(x^i)^{n+1} + a_n(x^i)^n + \dots + a_0$. The separation equations are

$$\Theta_i(x^i)\left(\frac{dW_i}{dx^i}\right)^2 + \left[E(x^i)^{n+1} + \sum_{j=2}^{n} \lambda_j (x^i)^{n-j}\right] = 0.$$

With the identification $E = \lambda_1$, the constants of motion associated with the separation parameters $\lambda_1, \dots, \lambda_n$ are given below. Corresponding to each of block type we give the corresponding polynomial $\Theta(x)$.

I. $$\Theta(x) = \prod_{i=1}^{n+1} (x - E_i^{(j)}), \quad E_i^{(j)} \neq E_k^{(j)} \quad \text{for } i \neq k. \tag{5.96}$$

$$I_p^n = \sum_{i>j} \varepsilon_j S_{p-1}^{ij} I_{ij}^2 , \quad p = 1, \dots, n, \tag{5.97}$$

where we define the symbol

$$S_p^{ij\dots n}(E_1, \dots, E_{n+1}) = \frac{1}{p!} \sum_{i_1,\dots,i_l} E_{i_1} \dots E_{i_l} , \tag{5.98}$$

$i_1, \dots, e_l = i, j, \dots, n$ and $i_k \neq i_m$ $(k \neq m)$, $k, m = 1, \dots, l$, and it is understood that $S_0^{ij} = 1$ and $S_{-p}^{ij} = 0$ (we need this for the next few types) and $\varepsilon_j = 2\delta_{j0} - 1$.

II. $$\Theta(x) = [(x-\alpha)^2 + \beta^2] \prod_{i=1}^{n-1} (x - f_i) . \tag{5.99}$$

$$\begin{aligned} I_p^n = {} & -S_{p-1} I_{01}^2 + \sum_{i>j>1} [(\alpha^2+\beta^2) S_{p-3}^{ij} + 2\alpha S_{p-2}^{ij} \\ & + S_{p-1}^{ij}] I_{ij}^2 + \sum_{j>1}^{n+1} [S_{p-2}^{j}(\alpha[I_{1j}^2 - I_{0j}^2] \\ & - \beta \{I_{1j}, I_{0j}\}_+) + S_{p-1}^{j}(I_{1j}^2 - I_{0j}^2)] , \end{aligned} \tag{5.100}$$

where $S_p^{i,j\dots k} = S_p^{i,j,\dots,k}(f_1, \dots, f_{n-1})$.

III. $$\Theta(x) = (x-a)^2 \prod_{j=1}^{n-1} (x - G_j) . \tag{5.101}$$

$$\begin{aligned} I_p^n = {} & -S_{p-1} I_{01}^2 + \sum_{i>j>1} (a^2 S_{p-3}^{ij} + 2a S_{p-2}^{ij} + S_{p-1}^{ij}) I_{ij}^2 \\ & + \sum_{j=2}^{n} [(S_{p-1}^{j} + a S_{p-2}^{j})(I_{1j}^2 - I_{0j}^2) - S_{p-2}^{j}(I_{1j} + I_{0j})^2] , \end{aligned} \tag{5.102}$$

where $S_p^{i,j,\dots,} = S_p^{i,j,\dots k}(G_1, \dots, G_{n-1})$.

IV. $$\Theta(x) = (x-b)^3 \prod_{j=1}^{n-2} (x - H_j). \tag{5.103}$$

$$\begin{aligned} I_p^n = {} & (S_{p-1} + b S_{p-2})(I_{12}^2 - I_{01}^2 - I_{02}^2) \\ & + 2S_{p-2} \{I_{01}, I_{12} - I_{02}\}_+ \\ & + \sum_{i>j>2} (S_{p-1}^{ij} + 3b S_{p-2}^{ij} + 3b^2 S_{p-3}^{ij} + b^3 S_{p-4}^{ij}) I_{ij}^2 \\ & + \sum_{j>2} (S_{p-1}^{j} + 2b S_{p-2}^{j} + b^2 S_{p-3}^{j})(-I_{j0}^2 + I_{1j}^2 + I_{2j}^2) \\ & + \sum_{j>2} (S_{p-2}^{j} + b S_{p-3}^{j}) \{I_{0j} - I_{1j}, I_{j2}\}_+ \\ & + \sum_{j\geq 2} S_{p-3}^{j} (I_{0j} - I_{1j})^2 , \end{aligned} \tag{5.104}$$

$S_p^{i,j,\dots k} = S_p^{i,j,\dots k}(H_1, \dots, H_{n-2})$.

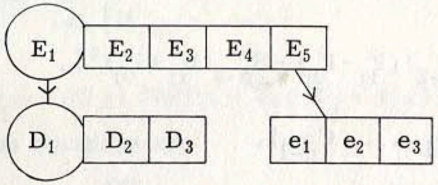
The associated Helmholtz equation in these various coordinates becomes

$$\sum_{i=1}^{n} \frac{1}{[\Pi_{j\neq i}(x^i - x^j)]} \left\{ \sqrt{(\Theta_i)} \frac{\partial}{\partial x^i} \left(\sqrt{(\Theta_i)} \frac{\partial \psi}{\partial x^i} \right) \right\} = \sigma(\sigma+n-1)\psi. \tag{5.105}$$

The separation equations are

$$\sqrt{(\Theta_i)} \frac{d}{dx^i}\left(\sqrt{(\Theta_i)} \frac{d\psi_i}{dx^i}\right) + \left[\sigma(\sigma+n-1)(x^i)^n + \sum_{j=2}^{n-1} \tilde{\lambda}_j (x^i)^{n-j}\right]\psi_i = 0. \tag{5.106}$$

The identification $\tilde{\lambda}_1 = \sigma(\sigma + n - 1)$ enables us to further identify the symmetry operators $\hat{I}_p^n$ whose eigenvalues are $\hat{\lambda}_j$ with the expressions I_p^n, where $I_{ij} \to \hat{I}_{ij}$ and $[\hat{I}_j^n, \hat{I}_k^n] = 0$. For graphs consisting entirely of irreducible blocks of type A the same generalizations apply directly as were developed for S_n. In particular, we can set up a natural ordering of coordinates as we did for S_n, e.g.,



A standard choice of coordinates is

$$(v_0, v_1, \dots, v_9) = \big(({}_4\overset{A}{v}_0)({}_2\overset{A}{v}_0), ({}_4\overset{A}{v}_0)({}_2\overset{A}{v}_1), ({}_4\overset{A}{v}_0)({}_2\overset{A}{v}_2), \qquad (5.107)$$

$${}_4\overset{A}{v}_1,\ {}_4\overset{A}{v}_2,\ {}_4\overset{A}{v}_3,\ ({}_4\overset{A}{v}_4)({}_3u_1),\ ({}_4\overset{A}{v}_4)({}_3u_2),\ ({}_4\overset{A}{v}_4)({}_3u_3)\big),$$

$$({}_4\overset{A}{v}_j)^2 = \varepsilon_j \left[\frac{\Pi_{i=1}^{4}(x^i - E_{j+1})}{\Pi_{k\neq j+1}(E_k - E_{j+1})}\right], \quad \varepsilon_j = 2\delta_{j0} - 1, \qquad (5.108)$$

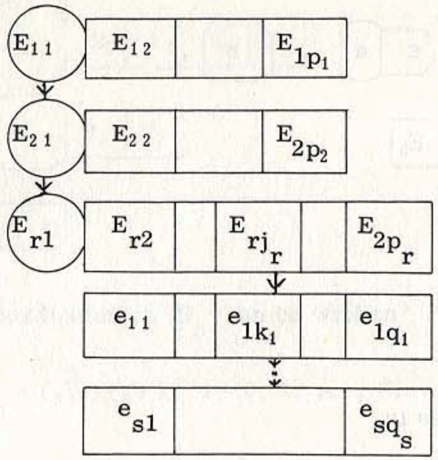
$$({}_2\overset{A}{v}_j)^2 = \varepsilon_j \left[\frac{\Pi_{i=5}^{6}(x^i - E'_{j+1})}{\Pi_{k=j+1}(E'_k - E'_{j+1})}\right],$$

$$({}_3u_j)^2 = \frac{\Pi_{i=7}^{8}(x^i - e_j)}{\Pi_{k\neq j}(e_k - e_j)} .$$

We see that for graphs involving only irreducible blocks of type A for H_q and the single type for S_p, there is a natural ordering induced by the ordering already adopted for the n-sphere. Recall that this is done as follows: a standard coordinate coming from a given graph consists of a product of r factors

$$x^{j_1\dots j_r k_1\dots k_s}_{p_1\dots p_r q_1\dots q_s} = \Big(\prod_{i=1}^{r} {}_{p_i}\overset{A}{v}_{j_i}\Big)\Big(\prod_{i=1}^{s} {}_{q_i}u_{k_i}\Big) .$$

These factors come from a typical branch



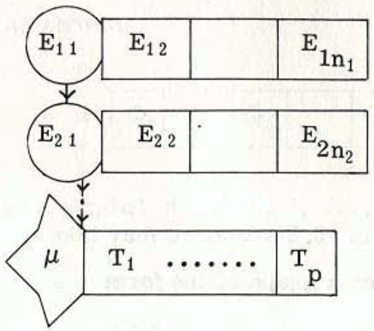
We can set up an ordering $<$ for products $x^{j_1\dots j_r k_1\dots k_s}_{p_1\dots p_r Q_1\dots Q_s}$. We say that

$$x^{j_1\dots j_r k_1\dots k_s}_{p_1\dots p_r q_1\dots q_s} < x^{j'_1\dots j'_r k'_1\dots k'_s}_{p'_1\dots p'_r q'_1\dots q'_s} \qquad (5.109)$$

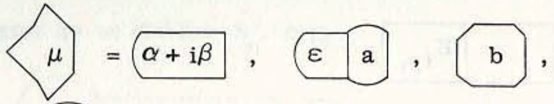
if $\quad p_1 = P'_1,\ \ p_2 = p'_2,\ \ j_2 = j'_2,$

$$p_t = p'_t,\ \ j_t < j'_t,\ \ p_{t+1} \neq p'_{t+1},\ \ j_{t+1} \neq j'_{t+1},\ \ Q_s \neq Q'_s,\ \ k_s \neq k'_s.$$

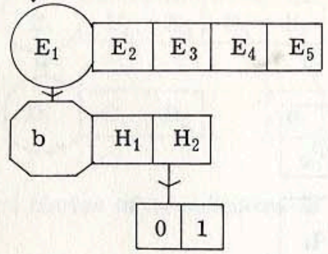
This has an obvious extension when $t > r$. The coordinates given in the example (5.107) have this ordering. This basic ordering can readily be extended to the case in which a graph contains one irreducible block of type B, C, D. We note from the rules for drawing graphs for H_n that in any given graph there can be at most one irreducible block of type B, C, or D. In fact, the branch of an admissible graph containing one of these types must look like



and



e.g.,



A standard choice of coordinates is

$$(v_0,\dots,v_9) = ({}_4\overset{A}{v}_0)({}_4\overset{D}{v}_0),\ ({}_4\overset{A}{v}_0)({}_4\overset{D}{v}_1),\ ({}_4\overset{A}{v}_0)({}_4\overset{D}{v}_2),\ ({}_4\overset{A}{v}_0)({}_4\overset{D}{v}_3),\ ({}_4\overset{A}{v}_0)({}_4\overset{D}{v}_4)({}_2u_1),\ ({}_4\overset{A}{v}_0)({}_4\overset{D}{v}_0)({}_2u_2),\ {}_4\overset{A}{v}_1,\ ({}_4\overset{A}{v}_2),\ ({}_4\overset{A}{v}_3),\ ({}_4\overset{A}{v}_4)). \tag{5.110}$$

Here the coordinates ${}_4\overset{D}{v}_j$ are given according to the standard formulas

$$({}_4\overset{D}{v}_0 - {}_4\overset{D}{v}_1)^2 = -\frac{\Pi_{i=5}^{9}(x^i - b)}{\Pi_{j=1}^{2}(H_j - b)} \tag{5.111}$$

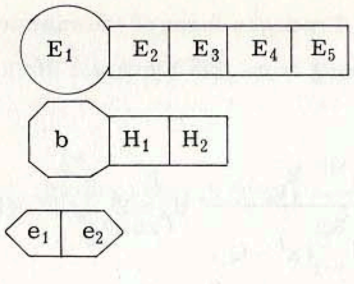
$$2\,{}_4\overset{D}{v}_2({}_4\overset{D}{v}_0 - {}_4\overset{D}{v}_1) = -\frac{\partial}{\partial b}\left[\frac{\Pi_{i=5}^{9}(x^i - b)}{\Pi_{j=1}^{2}(H_j - b)}\right]$$

$$({}_4\overset{D}{v}_0)^2 - ({}_4\overset{D}{v}_1)^2 - ({}_4\overset{D}{v}_2)^2 = -\tfrac{1}{2}\frac{\partial^2}{\partial b^2}\left[\frac{\Pi_{i=5}^{9}(x^i - b)}{\Pi_{j=1}^{2}(H_j - b)}\right]$$

$$({}_4\overset{D}{v}_3)^2 = -\frac{\Pi_{i=5}^{9}(x^i - H_1)}{(H_2 - H_1)(b - H_1)^3}$$

$$({}_4\overset{D}{v}_4)^3 = -\frac{\Pi_{i=5}^{9}(x^i - H_2)}{(H_1 - H_2)(b - H_2)^3}\,.$$

The coordinates ${}_4\overset{A}{v}_j$ are the same as in (5.24) and we may take ${}_2u_2 = \cos\phi$, ${}_2u_1 = \sin\phi$. If we consider a graph of the form



then the coordinates on H_n can be written

$$v_j = ({}_4\overset{A}{v}_0)({}_6v_j),\quad i = 0, 1, \dots, 6,\quad v_k = {}_4v_{k-6},\quad k = 7, \dots, 10, \tag{5.112}$$

where

$${}_6v_0 = \tfrac{1}{2}[({}_4\overset{D}{v}_0 - {}_4\overset{D}{v}_1)(w_1^2 + w_2^2 + 1) + ({}_4\overset{D}{v}_0 + {}_4\overset{D}{v}_1)], \tag{5.113}$$

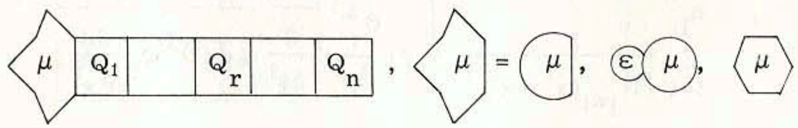
$${}_6v_1 = \tfrac{1}{2}[({}_4\overset{D}{v}_0 - {}_4\overset{D}{v}_1)(w_1^2 + w_2^2 - 1) + ({}_4\overset{D}{v}_0 + {}_4\overset{D}{v}_1)],$$

$${}_6v_2 = w_1({}_4\overset{D}{v}_0 - {}_4\overset{D}{v}_1),$$

$${}_6v_3 = w_2({}_4\overset{D}{v}_0 - {}_4\overset{D}{v}_1),$$

$${}_6v_j = {}_4\overset{D}{v}_{j-2},\quad j = 4, 5, 6.$$

These observations enable us to set up a natural ordering for any coordinate system on H_n. In dealing with graphs that contain blocks of type B, C, D we need consider only a standardized choice of coordinates for the irreducible block components given by (5.26), (5.28) and (5.30). This enables us to write down any coordinate system, given its graph. We can now discuss the form of the separation equations for a given graph. Consider a block as shown:



We define a symbol d_i $(i = 0, 1, \dots, n)$ as follows:

$d_i = 0$ if there is no arrow emanating down and from the i^{th} block or μ symbol, otherwise d_i is a non-zero parameter. From the form of the metric we see that the variables $x^1, \ldots, x^{n_L}$ coming from this block satisfy an equation of the form

$$\sum_{i=1}^{n_L} \frac{1}{[\Pi_{j\neq i}(x^i - x^j)]} P_i^2 + \sum_{i=0}^{n} \frac{k_i}{\Pi_{j=1}^{n_L}(x^j - Q_i)} d_i = E_{n_L} \qquad (5.114)$$

where $n_L = n, \quad L = A,$

$\qquad = n+1, L = C,$

$\qquad = n+2, L = D,$

and $k_0 = \Pi_{i=1}^{n}(Q_i - \mu), \quad k_i = \Pi_{j\neq i}(Q_j - Q_i)(\mu - Q_i)^{\varepsilon_L}$,

and the symbol $\varepsilon_L = 1, \quad L = A,$

$\qquad = 2, \quad L = C,$

$\qquad = 3, \quad L = D,$

and $P_i = \sqrt{(\Theta_i^L)}(dW/dx^i)$. The corresponding separation equations have the form

$$\Theta_i^L\left(\frac{dW_i}{dx^i}\right)^2 + \sum_{i=0}^{n} \frac{k_i d_i}{(x^i - Q_i)} + E_{n_L}(x^i)^{n_L - 1} + \sum_{l=2}^{n_L} \lambda_l (x^i)^{n_L - l} = 0, \qquad (5.115)$$

$$i = 1, \ldots, n_L .$$

For the associated Helmholtz equation the situation is more complicated. With each block we associate an index k_j which is calculated as follows: $k_0 = 1$, k_i $(i = 1, \ldots,)$ is the number of different coordinates v_i $(i = 0, \ldots, n)$ in which the coordinate ${}_{n_L}\overset{L}{v}_i$ occurs in our choice of standard coordinates. The Helmholtz equation assumes the form

$$\sum_{i=1}^{n_L} \frac{1}{[\Pi_{j\neq i}(x^i - x^j)]} \left\{ \sqrt{\left(\frac{\Theta_i^L}{Q_i}\right)} \frac{\partial}{\partial x^i} \left(\sqrt{(\Theta_i^L Q_i)} \frac{\partial \psi}{\partial x^i} \right) + \sum_{i=0}^{n} \frac{k_i}{\Pi_{j=1}^{n_L}(x^j - Q_i)} \right\} \psi = -\sigma(\sigma + n_L - 1)\psi ,$$

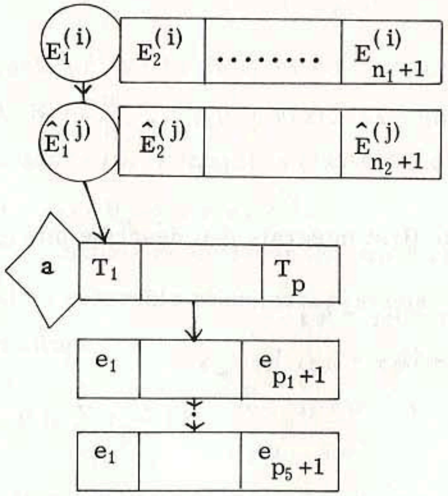
where $Q = \Pi_{j=1}^{n}(x^i - Q_j)^{k_j - 1}$, $t_i = 0$ if $k_i = L$ and $t_0 = \lambda^2$, $t_i = j_i(j_i + k_i - 1)$ if $k_i \neq 1$. The separation equations become

$$\sqrt{\left(\frac{\Theta_i^L}{Q_i}\right)} \frac{d}{dx^i} \left(\sqrt{(\Theta_i^L Q_i)} \frac{d\psi_i}{dx^i} \right) \qquad (5.116)$$

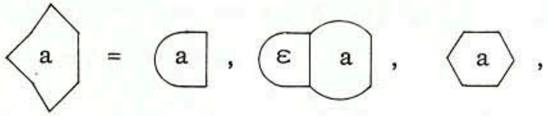
$$+ \left\{ \sum_{k=0}^{n} \frac{k_i}{(x^i - Q_i)} t_k + [\sigma(\sigma + n_L - 1)^{n_L - 1} + \sum_{l=2}^{n_L} \tilde{\lambda}_l (x^i)^{n_L - l}] \right\} \psi_i = 0.$$

Having given the coordinates and computed the associated separation equations for (I) and (II), we can also compute the quadratic first integrals corresponding to the separation constants. Given ${}_p\overset{L}{v}_j$, two coordinates v_i and v_k on H_n are said to be connected if they both contain the factor ${}_p\overset{L}{v}_j$. The corresponding quadratic first integrals can be calculated from the corresponding expressions for the quadratic first integrals of given irreducible blocks as follows:

(i) If the factor ${}_p\overset{L}{v}_j$ occurs in a branch of one of the forms

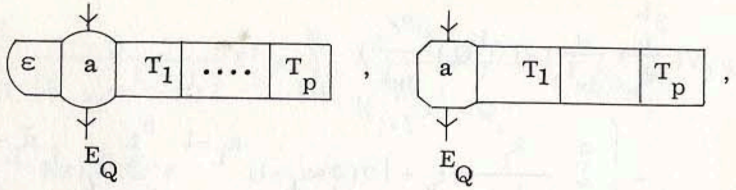


where

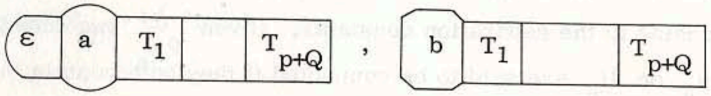


then the quadratic first integrals can be obtained by replacing the expressions for the quadratic first integrals of the lowest irreducible block by the expressions obtained by summing over connected indices.

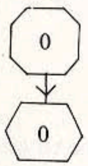
(ii) If the corresponding branch contains a components of either of the forms



the situation is somewhat different. The operators corresponding to these branches can be obtained from those of the irreducible blocks



by placing $T_{p+1} = \ldots = T_{p+Q} = a$, b, respectively, and supplementing the operators thus obtained by the corresponding operators of E_Q. Indeed if these considerations are pursued it can be shown that all Killing tensors can be obtained in this way. We conclude with an example. The graph



is defined on H_3. The quadratic first integrals that describe this graph are

$$I_1^3 = I_{12}^2 + I_{23}^2 + I_{13}^2 - I_{01}^2 - I_{02}^2 - I_{03}^2 ,$$
$$I_2^3 = 2\,\{I_{01}, I_{12} - I_{02}\}_+ + \{I_{03} - I_{13}, I_{32}\}_+ ,$$
$$I_3^3 = (I_{02} - I_{12})^2 .$$

6 Separation of variables on conformally Euclidean spaces

1. MATHEMATICAL PRELIMINARIES

Thus far we have addressed the problem of the solution by means of the additive separation of variables ansatz (1.2) of the Hamilton-Jacobi equation (1.3). A natural extension of this technique can be used to find solutions of the 'null' Hamilton-Jacobi equation

$$H(p_1, \ldots, p_n; x^1, \ldots, x^n) = 0, \quad p_i = \frac{\partial W}{\partial x^i}, \quad i = 1, \ldots, n \tag{6.1}$$

of the form

$$W = \sum_{i=1}^{n} W_i(x^i; c_1, \ldots, c_{n-1}) \tag{6.2}$$

where rank $[(\partial^2 W/\partial x^i \partial c_j)] = n - 1$. We will again be interested in the case in which $H = \Sigma^n_{i,j=1} g^{ij} p_i p_j + V(x)$. One immediate observation is that if W is a solution of (6.1) then it is also a solution of $QH = H' = 0$ where $Q = Q(p_1, \ldots, p_n; x^1, \ldots, x^n)$.

In this chapter we consider the problem of classification of all inequivalent separable coordinate systems $\{x^i\}$ for the null Hamilton-Jacobi equation

$$H = \sum_{i=1}^{n} \left(\frac{\partial W}{\partial z_i}\right)^2 = \sum_{i,j=1}^{n} g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j} = 0 \tag{6.3}$$

that can be solved by the separation of variables ansatz (6.2).

In addition, we are interested in product separable solutions of the corresponding Laplace equation

$$\Delta_n \psi = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{(g)}\, g^{ij} \frac{\partial \psi}{\partial x^j}\right) = 0 \qquad (6.4)$$

of the form

$$\psi = R \prod_{i=1}^{n} \psi(x^i; c_1, \ldots, c_{n-1}), \qquad (6.5)$$

with R a known function. This is a variant of pure separation which is known as R-separation. It has long been known that this type of solution occurs for Laplace's equation in three dimensions [1], [26].

The separation constants appearing in the solution of Laplace's equation (6.4) are a generalization of the first and second order symmetry operators defined in Chapter 3. The appropriate concept is that of a conformal Lie symmetry operator. A first order partial differential operator $\tilde{L} = \Sigma_i a^i(\partial/\partial x^i) + a$ is a conformal Lie symmetry operator for (6.4) if and only if $\{L, \Delta_n\} = Q\Delta_n$ for some function $Q = Q(x^i)$. Similarly, a second order partial differential operator

$$\tilde{m} = \sum_{i,j} a^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_k b^k \frac{\partial}{\partial x^k} + c$$

is a second order conformal symmetry for (6.4) if and only if $\{\tilde{m}, \Delta_n\} = S\Delta_n$ for some first order operator $S = \Sigma_i r^i(\partial/\partial x^i) + r$.

For the additive separation of variables of (6.2) a theorem analogous to that of Benenti (i.e. Theorem 3.1) can be formulated [27].

<u>Theorem 6.1:</u> Let M be a positive definite Riemannian manifold of dimension n for which the Hamilton-Jacobi equation

$$H = \sum_{i,j=1}^{n} g^{ij} \frac{\partial W}{\partial y^i} \frac{\partial W}{\partial y^j} = 0$$

admits an additive separation of variables in a system of coordinates $\{y^i\}$. Then there exists a system of coordinates $\{x^i\}$ 'equivalent' to $\{y^i\}$ such that the contravariant metric tensor has the form

$$(g^{ij}) = Q \left[\begin{array}{c|c} \delta^{ab} H_a^{-2} & \bigcirc \\ \hline \bigcirc & \tilde{g}^{\alpha\beta} \end{array}\right] \qquad (6.6)$$

where the functions H_a^{-2} and $\tilde{g}^{\alpha\beta}$ have the same form as in Benenti's theorem, i.e.

$$H_a^{-2} = \frac{S^{a1}}{S}, \quad \tilde{g}^{\alpha\beta} = \sum_b A_b^{\alpha\beta}(x^b) \frac{S^{b1}}{S}.$$

The comments (i)-(iii) that followed Benenti's theorem remain unchanged, with the exception that $\lambda_1 = 0$. The variables x^α are no longer simply ignorable variables satisfying $[p_\alpha, H] = 0$ as $\partial_\alpha Q$ need not be zero. However, we do have that

$$[p_\alpha, H] = (\partial_\alpha \ln Q) H. \qquad (6.7)$$

Linear forms in p_i's that satisfy a relation of this type will be referred to as conformal Lie symmetries. For Laplace's equation (6.4) the space of first order conformal Lie symmetries has dimension $\frac{1}{2}(n+1)(n+2)$ and a basis

$$P_i = p_{y^i}; \quad i = 1, \ldots, n, \qquad (6.8)$$

$$M_{jk} = -M_{kj} = y^j p_{y^k} - y^k p_{y^j}; \quad i \le j < k \le n,$$

$$D = -\sum_j y^j p_{y^j},$$

$$K_j = ((y^j)^2 - \sum_{l \ne j} (y^l)^2) p_{y^j} + 2y^j \sum_{l \ne j} y^l p_{y^l}, \qquad j = 1, \ldots, n.$$

This algebra can be identified with the Lie algebra SO(n+1, 1) via the equations

$$P_j = I_{1,j+1} + I_{0,j+1} \qquad j = 1, \ldots, n \qquad (6.9)$$

$$M_{jk} = I_{j+1,k+1} \qquad 1 \le j < k \le n-1$$

$$M_{n\ell} = I_{\ell+1,n+1} \qquad \ell = 1, \ldots, n-1$$

$$D = I_{10}$$

$$K_j = I_{1,j+1} - I_{0,j+1} \qquad j = 1, \ldots, n$$

where the I_{jk}, I_{0j} satisfy the $SO(n+1, 1)$ commutation relations

$$\{I_{jk}, I_{rs}\} = \delta_{rk} I_{js} - \delta_{sk} I_{rj} - \delta_{jr} I_{ks} + \delta_{js} I_{kr} \qquad (6.10)$$

$$\{I_{jk}, I_{0\ell}\} = \delta_{k\ell} I_{0j} - \delta_{j\ell} I_{0k}$$

$$\{I_{0j}, I_{0k}\} = I_{jk}.$$

In fact this correspondence can be made explicit by passing to pentaspherical coordinates [26]

$$Y_0 = \left(\sum_{j=1}^{n} (\eta^j)^2 + \eta^2\right), \quad Y_1 = \left(\sum_{j=1}^{n} (\eta^j)^2 - \eta^2\right) \qquad (6.11)$$

$$Y_k = 2\eta^{k-1}\eta \qquad k = 2, \ldots, n+1$$

where the Cartesian coordinates y^i are given by

$$y^i = \eta^i/\eta = Y_{i+1}/(Y_0 - Y_1) \qquad (6.12)$$

$$i = 1, \ldots, n.$$

These coordinates satisfy

$$-Y_0^2 + \sum_{i=1}^{n+1} Y_i^2 = 0 \qquad (6.13)$$

and in terms of the Y_i $(i = 0, \ldots, n+1)$ variables the conformal Lie symmetries I_{ij}, I_{0k} have the more familiar form

$$I_{ij} = Y_i p_{Y_j} - Y_j p_{Y_i}\ ; \quad j = 1, \ldots, n+1;\ i > j \qquad (6.14)$$

$$I_{0k} = Y_0 p_{Y_k} + Y_k p_{Y_0} \qquad k = 1, \ldots, n+1.$$

The conformal Lie symmetry operators for (6.4) form a $\frac{1}{2}(n+1)(n+2)$-dimensional Lie algebra which is isomorphic to $SO(n+1, 1)$. A convenient

basis is

$$\Theta_i = \frac{\partial}{\partial y^i}\ ; \quad i = 1, \ldots, n \qquad (6.15)$$

$$\mathfrak{M}_{jk} = y^j \frac{\partial}{\partial y^k} - y^k \frac{\partial}{\partial y^j}\ ; \quad 1 \leq j < k \leq n$$

$$\mathfrak{D} = -\frac{(n-2)}{2} - \sum_{j=1}^{n} y^j \frac{\partial}{\partial y^j}$$

$$\mathfrak{K}_j = \left((y^j)^2 - \sum_{\ell \neq j} (y^\ell)^2\right) \frac{\partial}{\partial y^j} + 2y^j \sum_{\ell \neq j} y^\ell \frac{\partial}{\partial y^\ell} + (n-2)\, y^j$$

This basis can be related to the standard $SO(n+1, 1)$ basis by taking

$$\Theta_j = \mathcal{I}_{1,j+1} + \mathcal{I}_{0,j+1}, \quad j = 1, \ldots, n \qquad (6.16)$$

$$\mathfrak{M}_{jk} = \mathcal{I}_{j+1,k+1}, \quad 1 \leq j < k \leq n-1$$

$$\mathfrak{M}_{n\ell} = \mathcal{I}_{\ell+1,n+1} \quad \ell = 1, \ldots, n-1$$

$$\mathfrak{D} = \mathcal{I}_{01}$$

$$\mathfrak{K}_j = \mathcal{I}_{1,j+1} - \mathcal{I}_{0,j+1}, \quad j = 1, \ldots, n.$$

In pentaspherical coordinates we have

$$\mathcal{I}_{ij} = Y_i \frac{\partial}{\partial Y_j} - Y_j \frac{\partial}{\partial Y_i}\ ; \quad i, j = 1, \ldots, n+1;\ i > j \qquad (6.17)$$

$$\mathcal{I}_{0k} = Y_0 \frac{\partial}{\partial Y_k} + Y_k \frac{\partial}{\partial Y_0}, \quad k = 1, \ldots, n+1.$$

The infinitesimal distance corresponding to the contravariant metric tensor given as in (6.6) is

$$ds^2 = Q\Big[\sum_a H_a^2 (dx^a)^2 + \sum_{\alpha,\beta} g_{\alpha\beta}\, dx^\alpha dx^\beta\Big] \qquad (6.18)$$

The Riemannian space with infinitesimal distance

$$d\tilde{s}^2 = \tilde{g}_{ij}\, dx^i dx^j = \sum_a H_a^2 (dx^a)^2 + \sum_{\alpha,\beta} g_{\alpha\beta}\, dx^\alpha dx^\beta \qquad (6.19)$$

is such that the metric coefficients $\tilde{g}^{ij}$ have the form (6.6) with $Q = 1$. This Riemannian space is also conformally Euclidean. A necessary and sufficient condition for this is that the conformal curvature tensor $C_{\ell ijk}$ vanish identically, i.e., [3]

$$C_{\ell ijk} = R_{\ell ijk} + \frac{1}{(n-2)}(\tilde{g}_{j\ell} R_{ik} - \tilde{g}_{\ell k} R_{ij} + \tilde{g}_{ik} R_{\ell k}) + \frac{R}{(n-1)(n-2)}(\tilde{g}_{\ell k}\tilde{g}_{ij} - \tilde{g}_{\ell j}\tilde{g}_{ik}) = 0. \tag{6.20}$$

Here $R_{\ell ijk}$, R_{ij} and R are the Riemann tensor, Ricci tensor and scalar curvature, respectively, of the Riemannian space with infinitesimal distance $d\tilde{s}^2$. The function $Q = e^{2\lambda}$ satisfies the equations

$$\lambda_{ij} = \frac{1}{(n-2)}\left(\frac{1}{2(n-1)}\tilde{g}_{ij}R - R_{ij}\right) - \tfrac{1}{2}\tilde{g}_{ij}[\tilde{g}^{k\ell}\lambda,_k \lambda,_\ell] \tag{6.21}$$

where $\lambda_{ij} = \lambda,_{ij} - \lambda,_j \lambda,_j$, $\lambda,_i = \partial\lambda/\partial x^i$ and $\lambda,_{ij}$ is the second covariant derivative of λ with respect to $\tilde{g}_{k\ell}$.

2. SEPARABLE COORDINATE SYSTEMS ON CONFORMALLY EUCLIDEAN SPACES

As with the real n-sphere, S_n, real Euclidean n-space E_n and the hyperboloid H_n, a crucial step in finding all separable coordinate systems for conformally Euclidean spaces is to show that all such coordinate systems can be taken as orthogonal.

Theorem 6.2: Let $\{x^j\}$ be a coordinate system on E_n for which the Hamilton-Jacobi equation (6.3) admits a separation of variables. Then, by passing to an equivalent system of coordinates if necessary, we have $g^{ij} = \delta^{ij}H_i^2$, i.e., there is only orthogonal separation. In terms of the choice of "standard" pentaspherical coordinates in (6.9) the variables x^{α_ℓ}, $\ell = 1, \ldots, q$, can be chosen such that the associated first order Lie symmetries take one of the forms

$$\text{(i)}\quad p_{\alpha_1} = I_{01},\ p_{\alpha_2} = I_{23}, \ldots, p_{\alpha_q} = I_{2q-2,2q-1}, \tag{6.22}$$

$$\text{(ii)}\quad p_{\alpha_1} = I_{12},\ p_{\alpha_2} = I_{34}, \ldots, p_{\alpha_q} = I_{2q-1,2q},$$

$$\text{(iii)}\quad p_{\alpha_1} = I_{02} - I_{12}, \ldots, p_{\alpha_s} = I_{0s+1} - I_{1s+1},$$

$$p_{\alpha_{s+1}} = I_{s+2,s+3}, \ldots, p_{\alpha_q} = I_{q+1,q+2}.$$

Proof: This is based on the general block diagonal form (6.6) of the contravariant metric tensor in a separable coordinate system and is closely related to the corresponding theorem for H_{n+1}. This is because in pentaspherical coordinates Y^j the corresponding choices of coordinates are virtually identical. What we have to show is that the conformal Lie symmetry vectors p_{x^α} can always be chosen such that the resulting separable coordinate system is orthogonal. Any element L of the symmetry algebra of SO(n+1, 1) can be chosen to be one of the types [23]:

$$\text{(i)}\quad L = I_{01} + b_2 I_{23} + \ldots + b_\nu I_{2\nu,2\nu+1}, \tag{6.23}$$

$$\text{(ii)}\quad L = L_{12} + b_2 I_{34} + \ldots + b_\nu I_{2\nu-1,2\nu},$$

$$\text{(iii)}\quad L = (I_{02} - I_{12}) + b_2 I_{34} + \ldots + b_\nu I_{2\nu-1,2\nu}.$$

If these elements correspond to a coordinate x^α, i.e., $L = p_{\alpha_1}$, then by local Lie theory the cartesian coordinates can be chosen to be

$$\text{(i)}\quad (y^1, \ldots, y^n) = e^{(x^{\alpha_1}+w_1)}(\rho_1\cos(b_2 x^{\alpha_1}+w_2),\ \rho_1\sin(b_2 x^{\alpha_1}+w_2), \ldots, \rho_\nu\cos(b_\nu x^{\alpha_1}+w_\nu),\ \rho_\nu\sin(b_\nu x^{\alpha_1}+w_\nu),\ v_{2\nu+2}, \ldots, v_n), \tag{6.24}$$

with $\rho_1^2 + \ldots + \rho_\nu^2 + v_{2\nu+2}^2 + \ldots + v_n^2 = 1$,

$$\text{(ii)}\quad (y^1, \ldots, y^n) = (1-\rho_1\cos(x^{\alpha_1}+w_1))^{-1}(\rho_1\sin(x^{\alpha_1}+w_1), \ldots, \rho_\nu\cos(b_\nu x^{\alpha_1}+w_\nu),\ \rho_\nu\sin(b_\nu x^{\alpha_1}+w_\nu),\ v_{2\nu+1}, \ldots, v_n),$$

with $\rho_1^2 + \ldots + \rho_\nu^2 + v_{2\nu+1}^2 + \ldots + v_n^2 = 1$,

$$\text{(iii)}\quad (y^1, \ldots, y^n) = (x^{\alpha_1}+w_1,\ \rho_2\cos(b_2 x^{\alpha_1}+w_1),\ \rho_2\sin(b_2 x^{\alpha_1}+w_1), \ldots, \rho_\nu\cos(b_\nu x^{\alpha_1}+w_\nu),\ \rho_\nu\sin(b_\nu x^{\alpha_1}+w_\nu),\ v_{2\nu+2}, \ldots, v_n).$$

Case (iii) need not be considered further as $L = P_1 + b_2 I_{23} + \dots + b_\nu I_{2\nu, 2\nu+1}$ which has already been considered in Chapter 4.

The metrics for cases (i) and (ii) are

$$\text{(i)}\quad ds^2 = e^{2(x^{\alpha_1}+w_1)}\big((dx^{\alpha_1}+dw_1)^2 + d\rho_1^2 + \rho_1^2(b_2\,dx^{\alpha_1}+dw_2)^2 + \dots + d\rho_\nu^2 + \rho_\nu^2(b_\nu\,dx^{\alpha_1}+dw_\nu)^2 + dv_{2\nu+2}^2 + \dots + dv_n^2\big), \tag{6.25}$$

$$\text{(ii)}\quad ds^2 = (1-\rho_1\cos(x^{\alpha_1}+w_1))^{-2}\big(d\rho_1^2+\rho_1^2(dx^{\alpha_1}+dw_1)^2 + \dots + d\rho_\nu^2 + \rho_\nu^2(b_\nu\,dx^{\alpha_1}+dw_\nu)^2 + dv_{2\nu+1}^2 + \dots + dv_n^2\big).$$

If there is only one ignorable coordinate α_1 then it must be orthogonal, which is only possible if $b_2 = \quad = b_\nu = 0$. Indeed, the requirements that the contravariant metric have the form (6.6) imply

$$\text{(i)}\quad dw_1 = -\sum_{j=2}^{\nu} \rho_{j-1}^2 b_j dw_j, \tag{6.26}$$

$$\text{(ii)}\quad dw_1 = -\sum_{j=2}^{\nu} \left(\frac{\rho_j^2}{\rho_1^2}\right) b_j dw_j.$$

Since the differentials $d\rho_j$, dw_j $(j \geq 2)$ must be independent and the only conditions on ρ_1^2 are those given in (2.3), then $d^2 w_1 = 0$ implies that $b_j = 0$, $j = 2, \dots, \nu$ and $dw_1 = 0$. By suitably redefining α_1 we can put $w_1 = 0$. Thus for one ignorable variable the conformal Lie symmetry p_{α_1} can be chosen as one of the three possibilities

$$\text{(i)}\quad p_{\alpha_1} = D, \tag{6.27}$$

$$\text{(ii)}\quad p_{\alpha_1} = P_1 + K_1 \text{ or } M_{12},$$

$$\text{(iii)}\quad p_{\alpha_1} = P_1.$$

Similarly, if we have q conformal Lie symmetries p_{α_i}, $i = 1, \dots, q$, we can repeat the arguments used in Chapter 5 with suitable modifications and show that these q ignorable variables must correspond to one of the three choices

(i) $p_{\alpha_1} = D$, $p_{\alpha_2} = M_{12}$, $p_{\alpha_3} = M_{34}$, $\dots$, $p_{\alpha_q} = M_{2q-3,2q-2}$,

(ii) (a) $p_{\alpha_1} = P_1 + K_1$, $p_{\alpha_2} = M_{12}$, $\dots$, $p_{\alpha_q} = M_{2q-3,2q-2}$,

(b) $p_{\alpha_1} = M_{12}$, $\dots$, $p_{\alpha_q} = M_{2q-1,2q}$, (6.28)

(iii) $p_{\alpha_1} = P_1$, $\dots$, $p_{\alpha_s} = P_s$, $p_{\alpha_{s+1}} = M_{s+1,s+2}$, $\dots$, $p_{\alpha_q} = M_{2q-s+3,2q-s+4}$.

The choices of coordinates for (6.28) (ii) (b) and (iii) correspond to cases already treated for the n-sphere S_n and Euclidean n-space, respectively. The other cases give rise to the coordinates

$$\text{(i)}\quad (y^1, \dots, y^n) = e^{x^{\alpha_1}}(\rho_1\cos x^{\alpha_2}, \rho_1 \sin x^{\alpha_2}, \dots, \rho_q\cos x^{\alpha_q}, \rho_q \sin x^{\alpha_q}, v_{2q+2}, \dots, v_n) \tag{6.29}$$

with $\rho_1^2 + \dots + \rho_q^2 + v_{2q+2}^2 + \dots + v_n^2 = 1$,

$$ds^2 = e^{2x^{\alpha_1}}\big[(dx^{\alpha_1})^2 + d\rho_1^2 + \rho_1^2(dx^{\alpha_2})^2 + \dots + d\rho_q^2 + \rho_q^2(dx^{\alpha_\beta})^2 + dv_{2q+2}^2 + \dots + dv_n^2\big],$$

$$\text{(ii)}\quad (y^1, \dots, y^n) = (1-\rho_1\cos x^{\alpha_1})^{-1}(\rho_1 \sin x^{\alpha_1}, \rho_2\cos x^{\alpha_2}, \rho_2 \sin x^{\alpha_2}, \dots, \rho_q \cos x^{\alpha_q}, \rho_q \sin x^{\alpha_q}, v_{2q+1}, \dots, v_n)$$

with $\rho_1^2 + \dots + \rho_q^2 + v_{2q+1}^2 + \dots + v_n^2 = 1$,

$$ds^2 = (1-\rho_1\cos x^{\alpha_1})^{-2}\big[d\rho_1^2 + \rho_1^2(dx^{\alpha_1})^2 + \dots + d\rho_q^2 + \rho_q^2(dx^{\alpha_q})^2 + dv_{2q+1}^2 + \dots + dv_n^2\big].$$

Thus all possible coordinate systems are orthogonal.

To solve our problem we have to find all coordinate systems $\{x^j\}$ such that the Riemannian space with infinitesimal distance,

$$ds^2 = \sum_{i=1}^{n} \frac{S}{S^{i1}}(dx^i)^2, \quad n \geq 4, \tag{6.30}$$

is conformally Euclidean, i.e. the conformal curvature tensor C_{lijk} vanishes. Here, as in Chapter 3, $S = \det(S_{ij}(x^i))_{n\times n}$ and S^{i1} is the i1 cofactor of the Stäckel matrix $(S_{ij}(x^i))_{n\times n}$. This problem has been solved for $n = 4$ [28] and by different methods for $n = 3$ [29]. The solution to the problem for $n = 4$ was non-trivial. Central to this was the result that it was always possible to find a function Q and Stäckel matrix $(\tilde{S}_{ij}(x^i))_{4\times 4}$ such that

$$ds^2 = Q\,d\tilde{s}^2 \qquad (6.31)$$

where

$$d\tilde{s}^2 = \sum_{i=1}^{4} \frac{\tilde{S}}{\tilde{S}^{i1}} (dx^i)^2 = \tilde{g}_{ij}\,dx^i dx^j$$

with Ricci tensor $\tilde{R}_{ij} = 0,\ i \neq j$.

This means that any four-dimensional Riemannian space that is in Stäckel form with respect to the coordinates $\{x^i\}$, and is conformally Euclidean, is itself conformal to a Riemannian space that is in Stäckel form and also satisfies the condition of product variable separation for the Helmholtz equation

$$\tilde{\Delta}_4 \psi = \lambda \psi\ , \qquad (6.32)$$

i.e., $\tilde{R}_{ij} = 0,\ i \neq j$. We show here that this result extends to n dimensions. To establish this result we note the following, due to Eisenhart [14].

A Riemannian space with infinitesimal distance $ds^2 = \Sigma_{i=1}^{n} H_i^2 (dx^i)^2$ for which the metric components are in Stäckel form is such that the components of the Riemann curvature tensor R_{jiik} for i, j, k distinct can be written

$$R_{jiik} = \frac{3}{4} H_i^2 \frac{\partial^2}{\partial x^j \partial x^k} \log H_i^2\ . \qquad (6.33)$$

The off-diagonal part of the Ricci tensor has components

$$R_{jk} = \sum_{l=1}^{n} H_l^{-2} R_{jllk} = \frac{3}{4} \frac{\partial^2}{\partial x^j \partial x^k} \log[\Pi' H_l^2\,],\ j \neq k, \qquad (6.34)$$

where the product Π' of the H_l^2 extends over all $l \neq j, k$. Another useful

result we need is Lemma 5 of reference [28]:

Lemma: Let $ds^2 = \Sigma_{j=1}^{n} H_j^2 (dx^j)^2$ be in Stäckel form. Then $d\tilde{s}^2 = e^{2\phi}\,ds^2$ is in Stäckel form with respect to the coordinates $\{x^i\}$ if and only if for $j \neq k$,

$$\frac{\partial^2 \phi}{\partial x^j \partial x^k} + 2\frac{\partial \phi}{\partial x^j}\frac{\partial \phi}{\partial x^k} + \frac{\partial \phi}{\partial x^j}\frac{\partial}{\partial x^k} \log H_j^2 + \frac{\partial \phi}{\partial x^k}\frac{\partial}{\partial x^j} \log H_k^2 = 0\ . \qquad (6.35)$$

Any function $e^{2\phi}$ that satisfies these requirements is called a <u>Stäckel multiplier.</u> As a corollary to this result we make the observation that $e^{2\phi} = H_j^{-2}$ is a Stäckel multiplier.

Theorem 6.3: Let $ds^2 = \Sigma_j H_j^2 (dx^j)^2$ be a conformally flat metric in Stäckel form. Then there exists a non-zero function q such that $ds^2 = q\,d\tilde{s}^2 = \Sigma_j\, q\tilde{H}_j^2 (dx^j)^2$ where the metric $d\tilde{s}^2$ is in Stäckel form and such that $\tilde{R}_{ij} = 0$ for $i \neq j$.

Proof: We prove this by explicit construction of the various possibilities. The conditions $C_{jiik} = 0\ \ (j \neq i \neq k)$ imply

$$R_{jiik} = \frac{1}{(n-2)} H_i^2 R_{jk} \qquad (6.36)$$

for the given metric $ds^2 = \Sigma_{j=1}^{n} H_j^2 (dx^j)^2$. If we take $q = H_1^{-2}$ then $d\tilde{s}^2 = \Sigma_{j=1}^{n} h_j^2 (dx^j)^2$, where $h_1^2 = 1,\ h_j^2 = H_j^2 H_1^{-2},\ j = 2, \ldots, n$, is in Stäckel form. For this we have

$$\frac{1}{(n-2)} R_{jk} = \frac{3}{4} \frac{\partial^2}{\partial x^j \partial x^k} \log h_i^2, \quad i \neq j \neq k. \qquad (6.37)$$

This implies that $R_{jk} = 0,\ j \neq k,\ j, k = 2, \ldots, n$, as we can always put $i = 1$ in (6.37). We can now apply Eisenhart's results (see Appendix) to the metric $(ds^*)^2 = h_2^2 (dx^2)^2 + \ldots + h_n^2 (dx^n)^2$ for fixed x^1. The conditions $R_{jk} = 0,\ j \neq k,\ j, k = 2, \ldots, n$, imply that

$$h_j^2 = X_{1j} [\prod_{k \neq j} (\sigma_{kj}^1 + \sigma_{jk}^1)\,],\ \ k = 2, \ldots, n, \qquad (6.37)$$

where $\sigma^1_{kj} = \sigma^1_{kj}(x^1, x^k)$, $X_{1j} = X_{1j}(x^1, x^j)$. The various degenerate forms of this metric have been given by Eisenhart; we work through the various possibilities. For the most general metric the coefficients h_j^2 assume the form

$$h_j^2 = X_{1j}\left[\prod_{k\neq j}(\sigma^1_k - \sigma^1_j)\right], \quad j = 2, \ldots, n, \; \partial_k \sigma^1_k \neq 0, \quad \forall k. \tag{6.38}$$

From Lemma 4 of reference [28],

$$h_j^2 = \tilde{X}_{1j}\left[\prod_{k\neq j}(\psi_{ik}\psi_{ij}\psi_{jk})\right] \tag{6.39}$$

where $\psi_{kl} = \psi_{kl}(x^k, x^l)$, $\tilde{X}_{1j} = \tilde{X}_{1j}(x^1, x^j)$ and $\psi_{kl} = -\psi_{lk}$. Consequently we can take

$$\sigma^1_l - \sigma^1_j = \psi_{1l}\psi_{1j}\psi_{lj}. \tag{6.40}$$

The form of the functions σ^1_l can be easily obtained. Putting $x^1 = y^1$ (fixed) in (6.40) we have, in obvious notation,

$$\psi_{lj} = \phi_l \phi_j (\sigma_l - \sigma_j) \tag{6.41}$$

where $\phi_l = [\psi_{1l}(y^1, x^l)]^{-1}$, $\sigma^1_l = \sigma_l(y^1, x^l)$. Absorbing the ϕ_l functions into the ψ_{1l} functions, we have

$$\sigma^1_l - \sigma^1_j = \tilde{\psi}_{1l}\tilde{\psi}_{1j}(\sigma_l - \sigma_j). \tag{6.42}$$

This equation has only two solutions:

$$\text{(a)} \quad \sigma^1_l = \frac{f(x^1)}{(\sigma_1 - \sigma_l)} + g(x^1), \quad l = 2, \ldots, n, \tag{6.43}$$

$$\text{(b)} \quad \sigma^1_l = f(x^1)\sigma_l + g(x^1), \quad l = 2, \ldots, n.$$

For case (a) the metric coefficients h_j^2 have the form

$$h_j^2 = X_{1j}\left[\prod_{k\neq j}(\sigma_k - \sigma_j)\right] / \left[\left\{\prod_{l\neq j}(\sigma_1 - \sigma_l)\right\}(\sigma_1 - \sigma_j)^{n-2}\right], \quad j \neq 1.$$

Now from the Stäckel conditions (2.52) for $i = j$, $k = 1$, we have

$$\frac{\partial^2}{\partial x^1 \partial x^j} \log h_j^2 = 0, \tag{6.44}$$

i.e. $X_{1j}(\sigma_1 - \sigma_j)^{2-n} = X_j q_j(x^1)$. From the same conditions with $k = 1$ we can deduce that $q_j(x^1) = q(x^1)$, i.e. the functions $q_j(x^1)$ are independent of j. Multiplying the metric $d\tilde{s}^2$ by $[q(x^1)]^{-1}\prod_{l\neq 1}(\sigma_1 - \sigma_l)$, we obtain a metric $d\hat{s}^2 = \Sigma_{j=1}^n \hat{h}_j^2 (dx^j)^2$ whose coefficients have the form

$$\hat{h}_j^2 = X_j\left[\prod_{k\neq j}(\sigma_k - \sigma_j)\right], \quad j = 1, \ldots, n. \tag{6.45}$$

This metric satisfies $\hat{R}_{ij} = 0$ for all $i \neq j$, i, $j = 1, \ldots, n$.

For case (b) the metric coefficients can be reduced to

$$h_j^2 = X_{1j}\left[\prod_{k\neq j}(\sigma_k - \sigma_j)\right]. \tag{6.46}$$

The Stäckel conditions (2.52) with $k = 1$ imply that the metric coefficients have the form

$$h_1^2 = 1, \quad h_j^2 = X_j q(x^1) \prod_{k\neq j}(\sigma_k - \sigma_j). \tag{6.47}$$

After multiplication by $[q(x^1)]^{-1}$ to obtain the metric $d\hat{s}^2 = \Sigma_{j=1}^n \hat{h}_j^2 (dx^j)^2$, the coefficients take the form

$$\hat{h}_1^2 = X_1, \quad \hat{h}_j^2 = X_j \prod_{k\neq j}(\sigma_k - \sigma_j), \quad j, k = 2, \ldots, n. \tag{6.48}$$

We next consider the possibility that some of the functions σ^1_{kj} do not depend on x^k. Proceeding as we did for the n-sphere, we find that the metric coefficients can assume one of the following forms:

$$\text{(a)} \quad h_1^2 = 1, \quad h_j^2 = \left[X_{1j}\prod_{k\neq j}(\sigma^1_k - \sigma^1_j)\right]\left(\prod_{a\in N_2}(\sigma^1_a + g_a)\right), \tag{6.49}$$

$$h_a^2 = X_{1a}\prod_{b\neq a}(\sigma^1_b - \sigma^1_a),$$

where i, j, ..., $\in N_1$, a, b, ..., $\in N_2$ and $g_l = g_l(x^1)$, and the functions $\sigma^1_n = \sigma^1_m(x^1, x^m)$ are such that $(\partial_m)\sigma^1_m \neq 0$. N_1, N_2 are mutually exclusive subsets of $\{2, \ldots, n\}$.

(b) $h_1^2 = 1, \quad h_j^2 = X_{1j} \prod_{k \neq j} (\sigma_k^1 - \sigma_j^1), \quad h_a^2 = X_{1a} \prod_{b \neq a} (\sigma_b^1 - \sigma_a^1).$ (6.50)

For case (a) we can take $g_l = 0$ by suitably redefining the functions σ_l^1, $l \in N_2$. From the conditions (6.34) we see that

$$R_{1j} = \frac{3}{4} \frac{\partial^2}{\partial x^1 \partial x^j} \log h_a^2 = 0, \quad j \in N_1 . \tag{6.51}$$

Therefore the metric $(d\hat{s})^2 = (dx^1)^2 + \Sigma_{j \in N_1} h_j^2 (dx^j)^2$ for fixed x^a, $a \in N_2$, is in Stäckel form and satisfies the conditions $R_{ijjk} = 0$, for i, j, $k \in N_1$ and distinct

$$h_j^2 = [X_j \prod_{k \neq j} (\sigma_k - \sigma_j)] (\prod_{a \in N_2} \sigma_a^1) . \tag{6.52}$$

Using Lemma 4 as we did for (6.37), we see that the functions σ_a^1 have the forms

(a) $\sigma_a^1 = \dfrac{f(x^1)}{(\sigma_1 - \sigma_a)}, \quad a \in N_2,$ (6.53)

(b) $\sigma_a^1 = f(x^1)(\sigma_1 - \sigma_a).$

Using the Stäckel conditions just as we did for the metric (6.37), we can readily see that each metric of this type is conformal to a metric $d\hat{s}^2 = \Sigma_{j=1}^n \hat{h}_j^2 (dx^j)^2$ where

(i) $\hat{h}_l^2 = X_l \prod_{m \neq l} (\sigma_m - \sigma_l), \quad l, m \in N_2 \cup \{1\},$ (6.54)

$\hat{h}_j^2 = [X_j \prod_{k \neq j} (\sigma_k - \sigma_j)] (\prod_{l \in N_2 \cup \{1\}} \sigma_l),$

(ii) $\hat{h}_1^2 = X_1, \quad \hat{h}_j^2 = [X_j \prod_{k \neq j} (\sigma_k - \sigma_j)] (\prod_{l \in N_2} \sigma_l),$ (6.55)

$\hat{h}_a^2 = X_a \prod_{b \neq a} (\sigma_b - \sigma_a).$

These metrics both satisfy the conditions

$$\hat{R}_{ij} = 0, \quad i \neq j . \tag{6.56}$$

For case (b) it is easy to see that $R_{1m} = 0$ for all m. This follows from

(6.36). Consequently this metric already satisfies the conditions (6.56) and the metric coefficients can be taken as

(iii) $\hat{h}_1^2 = X_1, \quad \hat{h}_j^2 = X_j \prod_{k \neq j} (\sigma_k - \sigma_j), \quad \hat{h}_a^2 = X_a \prod_{b \neq a} (\sigma_b - \sigma_a).$ (6.57)

We can now treat the general case. The infinitesimal distance can be written

$$ds^2 = (dx^1)^2 + \sum_{I=1}^{p} \{ \sum_{i \in N_I} (h_{i1}^I)^2 (dx^i)^2 \} \{ \prod_{l \in N_{p+1}} (\sigma_l^1 + \sigma_{l1}) \} + \sum_{j \in N_{p+1}} (h_{j1}^{p+1})^2 (dx^j)^2 , \tag{6.58}$$

where $\{N_1, \ldots, N_{p+1}\}$ is a pattern of the integers 2, ..., n into mutually exclusive sets. We also have that $\partial_j h_{i1}^I = 0$ if $j \notin N_I$, and the coefficients $(h_{j1}^{p+1})^2$ have the form

$$(h_{j1}^{p+1})^2 = X_{ij} \prod_{\substack{i \neq j \\ i \in N_{p+1}}} (\sigma_i^1 - \sigma_j^1), \quad j \in N_{p+1} . \tag{6.59}$$

From this form and the relations (6.36) we see that $R_{ij} = 0$, provided that $j \notin N_{p+1}$. As a consequence of this and the Stäckel conditions (2.52), the metric can be written

$$ds^2 = (dx^1)^2 + \sum_{I=1}^{P} q_I(x^1) \{ \sum_{i \in N_I} (\tilde{h}_i^I)^2 (dx^i)^2 \} \times \{ \prod_{l \in N_{p+1}} (\sigma_l^1 + \alpha_{l1}) \} + \sum_{j \in N_{p+1}} X_{ij} \prod_{\substack{i \neq j \\ i \in N_{p+1}}} (\sigma_i^1 - \sigma_j^1)(dx^j)^2 , \tag{6.60}$$

where $(\partial_1) \tilde{h}_i^I = 0, \quad \alpha_{i1} = \alpha_{i1}(x^1)$. Applying the results of previous special cases, we can readily show that σ_j^1, $j \in N_{p+1}$ have the solutions

(a) $\sigma_j^1 = \dfrac{f(x^1)}{\sigma_1 - \sigma_j}, \quad j \in N_{p+1},$ (6.61)

(b) $\sigma_j^1 = f(x^1)\sigma_j, \quad j \in N_{p+1}.$ (6.62)

The Stäckel conditions with $j = 1$, $i \in N_I$ and $k \in N_{p+1}$ imply that

$$\frac{\partial^2}{\partial x^1 \partial x^k} \log(\sigma_k^1 + \alpha_{I1}) \tag{6.63}$$

$$- \frac{\partial}{\partial x^I} \log[q_I \prod_{l \in N_{p+1}} (\sigma_l^1 + \alpha_{I1})] \frac{\partial}{\partial x^k} \log(\sigma_k^1 + \alpha_{I1})$$

$$+ \frac{\partial}{\partial x^k} \log(\sigma_k^1 + \alpha_{I1}) \frac{\partial}{\partial x^I} \log[X_{ik} \prod_{l \neq k} (\sigma_l^1 - \sigma_k^1)] = 0 .$$

Differentiating this expression with respect to x^l, we obtain

$$\frac{\partial}{\partial x^k} \log(\sigma_k^1 + \alpha_{I1}) \frac{\partial^2}{\partial x^1 \partial x^l} \log\left[\frac{\sigma_l^1 - \sigma_k^1}{\sigma_l^1 + \alpha_{I1}}\right] = 0. \tag{6.64}$$

These conditions imply

$$\frac{\partial^2}{\partial x^1 \partial x^l} \log\left[\frac{\sigma_l^1 - \sigma_k^1}{\sigma_l^1 + \alpha_{I1}}\right] = 0, \tag{6.65}$$

as $\partial_k \sigma_k^1 \neq 0$. For the first possible choice (6.61) of the functions σ_j^1, $j \in N_{p+1}$ we choose

$$\sigma_l^1 + \alpha_{I1} = \alpha_{I1}\left[\frac{A_{I1} - \sigma_l}{\sigma_1 - \sigma_l}\right].$$

The conditions (6.65) imply $(\partial^2/\partial x^1 \partial x l) \log[A_{I1} - \sigma_l] = 0$, hence $\partial_1 A_{I1}(x^1) = 0$, i.e., A_{I1} is a constant. Similarly, the second choice (6.62) requires that α_{I1} is a constant. These conditions, when fed back into the Stäckel form conditions (2.52), imply

$$q_I(x^1) = q(x^1), \quad I = 1, \ldots, p+1 \tag{6.66}$$

and

(a) $$h_j^2 = q_{p+1}(x^1) X_j \prod_{i \neq j} (\sigma_i - \sigma_j) / \prod_{l \neq j} (\sigma_1 - \sigma_l), \quad j \in N_{p+1}, \tag{6.67}$$

(b) $$h_j^2 = q_{p+1}(x^1) X_j \prod_{i \neq j} (\sigma_i - \sigma_j) . \tag{6.68}$$

By multiplying by suitable functions we can always reduce these metrics:

(i) $$ds^2 = \sum_{l \in N_{p+1} \cup \{1\}} X_l [\prod_{m \neq l} (\sigma_m - \sigma_l)] (dx^l)^2 \tag{6.69}$$

$$+ \sum_{I=1}^{P} [\prod_{l \in N_{p+1} \cup \{1\}} (\sigma_l^1 + \alpha_I)] d\omega_I^2 ,$$

(ii) $$ds^2 = (dx^1)^2 + \sum_{l \in N_{p+1}} X_l [\prod_{m \neq l} (\sigma_m - \sigma_l)] (dx^l)^2 \tag{6.70}$$

$$+ \sum_{I=1}^{P} [\prod_{l \in N_{p+1}} (\sigma_l^1 + \alpha_I)] d\omega_I^2 ,$$

where each infinitesimal distance $d\omega_I^2$ corresponds to a Riemannian space which has a metric in Stäckel form and satisfies the condition $R_{ij}^I = 0$, $i, j \neq$, $i, j \in N_I$. We note here that the form is not the most general. In fact the general infinitesimal distance can be written

$$ds^2 = (dx^1)^2 + \sum_{j \in Q} h_{j1}^2 (dx^j)^2 \tag{6.71}$$

$$+ \sum_{I=1}^{P} \{ \sum_{i \in N_I} (h_{i1}^I)^2 (dx^i)^2 \} \{ \prod_{l \in N_{p+1}} (\sigma_l^1 + \alpha_{I1}) \}$$

$$+ \sum_{j \in N_{p+1}} (h_j^{p+1})^2 (dx^j)^2 ,$$

where $\partial_k h_{j1}^2 = 0$ unless $j \in Q \cup \{1\}$. Here $Q, N_1, \ldots, N_{p+1}$ form a partition of $\{2, \ldots, n\}$ into mutually exclusive sets. The remainder of the metric has the same significance as in (6.60), i.e., $\partial_j h_{i1}^I = 0$ unless $j \in N_I$. However, we notice that for a metric of this type $R_{ik} = 0$ for $i \neq k$ for all $i, k = 1, \ldots, n$. This is easy to see from (6.37), as it is always possible to find an $i \neq j, k$ such that $(\partial^2/\partial x^j \partial x^k) \log h_i^2 = 0$. Consequently, a metric of this type trivially satisfies the requirements of the theorem.

3. CONSTRUCTION OF SEPARABLE COORDINATES FOR CONFORMALLY EUCLIDEAN SPACES

To compute all the possible coordinate systems that are in Stäckel form and conformally flat we need to impose the conditions $C_{ijji} = 0, \ i \neq j$. Here

$$C_{ijji} = R_{ijji} - \frac{1}{(n-2)}\left[H_j^2 R_{ii} + H_i^2 R_{jj}\right] + \frac{1}{(n-1)(n-2)} H_i^2 H_j^2 R = 0 . \tag{6.72}$$

Now $R_{ii} = \Sigma_{k\neq i} H_k^{-2} R_{ikki}$ and if we define

$$B_{ijji} = H_i^{-2} H_j^{-2} R_{ijji} \tag{6.73}$$

we can write (6.72) in the form

$$B_{ijji} - \frac{1}{(n-2)}\left[\sum_k (B_{ikki} + B_{jkkj})\right] + \frac{1}{(n-1)(n-2)} R = 0. \tag{6.74}$$

If i, j, k, ℓ are four distinct indices, then from these conditions we observe that

$$B_{ijji} + B_{k\ell\ell k} = B_{ikki} + B_{j\ell\ell j} = B_{i\ell\ell i} + B_{jkkj} . \tag{6.75}$$

It is these relations that help us to find all possible metrics that we need. For the most general metric (6.45) we can take $\sigma_i = x^i$, $i = 1, \ldots, n$, and the metric coefficients as

$$h_j^2 = X_j\left[\prod_{i\neq j} (x^i - x^j)\right] \quad i, j = 1, \ldots, n. \tag{6.76}$$

In terms of these coordinates, the quantities B_{ijji} have the form

$$B_{ijji} = \left[\prod_{\ell\neq j} (x^j - x^\ell)\right]^{-1} \left\{\frac{-2}{(x^i - x^j)^2}\left(\frac{1}{X_j}\right) - \frac{1}{(x^i - x^j)}\left(\frac{1}{X_j}\right)'\right\}$$

$$+ \left[\prod_{\ell\neq i} (x^i - x^\ell)\right]^{-1} \left\{\frac{-2}{(x^i - x^j)^2}\left(\frac{1}{X_i}\right) - \frac{1}{(x^j - x^i)}\left(\frac{1}{X_i}\right)'\right\}$$

$$+ \sum_{\ell\neq i,j} \frac{1}{X_\ell (x^\ell - x^i)(x^\ell - x^j)\prod_{k\neq\ell}(x^\ell - x^k)} \tag{6.77}$$

and, if we substitute these expressions back into conditions (6.75), the resulting equations have the general solution $1/X_i = \Sigma_{m=0}^{n+2} a_m (x^i)^m$. If $a_{n+2} \neq 0$ then it is always possible to choose $a_0 = 0$ and change coordinates according to $x^i \to (x^{-i})^{-1}$. We see that the metrics of this type are conformal to a metric of the form

$$ds^2 = \frac{1}{4}\sum_{i=1}^{n} \frac{\prod_{j\neq i}(x^j - x^i)}{\prod_{\ell=1}^{n+1}(x^i - e_\ell)} (dx^i)^2 . \tag{6.78}$$

This form can be recognized as that of elliptic coordinates on the n-sphere S_n or type A coordinates on the hyperboloid H_n. In fact, if we had not performed this transformation, the metric would have assumed the form

$$ds^2 = \sum_{i=1}^{n} \frac{\prod_{j\neq i}(x^j - x^i)}{\prod_{\ell=1}^{n+2}(x^i - e_\ell)} (dx^i)^2 \tag{6.79}$$

which is the form for general cyclidic coordinates [26]. Bôcher has given an extensive account of such systems. We need to determine how many coordinate systems of this type occur. Note that any set of coordinates is conformally equivalent to one in which the coordinates and e_i's have been changed according to the birational transformation

$$e_i' = \frac{\alpha e_i + \beta}{\gamma e_i + \delta}, \quad i = 1, \ldots, n+2, \tag{6.80}$$

$$x_i' = \frac{\alpha x_i + \beta}{\gamma x_i + \delta}, \quad i = 1, \ldots, n, \quad \alpha\delta - \beta\gamma \neq 0.$$

The corresponding conformally related metric then assumes the form (6.79) in the primed indices. Using such transformations it is not difficult to find all inequivalent conformally flat metrics of this type. We recall that our spaces are positive definite, i.e., the coordinates $\{x^j\}$ must stay in ranges so that the metric is positive definite. We see that the metric of the form (6.78) corresponds to the most general cyclidic type of Bôcher but with $e_{n+2} = \infty$. If we take these degrees of freedom into consideration, there are only two elliptic types of coordinates. They are

(i) $e_1 < e_2 < x^1 < e_3 < x^2 < \ldots < e_{n+1} < x^n < e_{n+2} = \infty$,

(ii) $x^1 < f_1 < x^2 < \ldots < f_{n-1} < x_n < f_n = \infty$, (6.81)

where $e_1 = \alpha + i\beta$, $e_2 = \alpha - i\beta$, $e_3 = f_1, \ldots, e_{n+2} = f_n$. As we saw in Chapter 5, the remaining metrics of type (6.78) that arise on H_n occur when there is one double or triple root in the polynomial $\Pi_{\ell=1}^{n+2}(x^i - e_\ell)$. Upon transformation of this double or triple root to ∞ via a birational transformation of type (6.80), the metric assumes the form

$$ds^2 = \frac{1}{4}\sum_{i=1}^{n} \frac{\{\Pi_{j\neq i}(x^j - x^i)\}}{\prod_{\ell=1}^{p}(x^i - e_\ell)}(dx^i)^2 \tag{6.82}$$

where $p = n$ or $n-1$. This says that all such coordinate systems are conformally equivalent to the two nondegenerate coordinate systems on E_n, i.e., elliptic and parabolic coordinates. For the remaining conformally flat metrics we can now make a crucial observation. Each conformally flat metric can be written in the form

$$ds^2 = \sum_{I=1}^{P} dw_I^2 \tag{6.83}$$

where $P \geq 2$ and each form dw_I^2 is the metric of a Riemannian space. We assume that the integers $\{1, \ldots, n\}$ are subdivided into mutually exclusive subsets N_I, $I = 1, \ldots, p$, $N_I \cap N_J = \phi$ and $dw_I^2 = \Sigma_{i\in N_I} H_i^2(dx^i)^2$, $\partial_j H_i^2 = 0$ unless $i, j \in N_I$. If $\dim N_I \geq 2$ and $p > 2$ then the relations (6.75) become

$$B_{ijji} + B_{k\ell\ell k} = 0, \quad i, j \in N_I, \; k, \ell \in N_J, \tag{6.84}$$

as $B_{ikki} = 0$ if $i \in N_I$, $j \in N_J$; $I \neq J$. These conditions imply $B_{ijji} = 0$, $i, j \in N_I$, since if $m, n \in N_k$ then

$$B_{ijji} + B_{mnnm} = 0, \quad B_{k\ell\ell k} + B_{mnnm} = 0. \tag{6.85}$$

The corresponding coordinates in this case can be identified with those on E_n, i.e., $B_{ijji} = 0$ for $i, j \in N_I$, $I = 1, \ldots, p$. If we still require $\dim N_I \geq 2$ but $p = 2$, then the only restriction we have is

$$B_{ijji} + B_{k\ell\ell k} = 0, \quad i, j \in N_1, \; k, \ell \in N_2, \tag{6.86}$$

which implies $B_{ijji} = 1$, $B_{k\ell\ell k} = -1$ with suitable renormalizations. The metric dw_1^2 corresponds to that of a separable coordinate system on H_{p_1} and ds_2^2 to a coordinate system on S_{p_2} where $p_i = \dim N_i$, $i = 1, 2$. Finally we mention the case when $\dim N_1 = 1$. If $p > 2$ then we can reason as before that $B_{ijji} = 0$ for all i, j, i.e., the metric is flat. If $p = 2$ the only conditions we obtain are

$$B_{ijji} = B_{kjjk}, \quad i, j, k \in N_2. \tag{6.87}$$

These conditions imply that $B_{ijji} = 1$ for all i, j. We can now summarize these results.

<u>Theorem 6.4:</u> The Hamilton-Jacobi equation (6.3) admits a separation of variables in orthogonal coordinates only. The metric associated with each such coordinate system $\{x^j\}$ can be written as $ds^2 = Q\,d\hat{s}^2 = Q(\Sigma_{i=1}^n h_i^2(dx^i)^2)$ where $d\hat{s}^2$ admits a true separation for the Helmholtz equation $\Delta_n \psi = E\psi$ on one of the manifolds

(i) E_n,

(ii) $S_p \times H_q$, $p + q = n$, $p, q \geq 1$,

(iii) H_n (this case includes S_n).

All separable systems on these manifolds permit R-separation of the Laplace equation (6.4).

The last statement in Theorem (6.3) has not as yet been proved. To do this we discuss the various types of coordinate system and show that we do obtain the correct result. For the two coordinate systems (6.81) (i) and (ii), a suitable choice of pentaspherical coordinates is $Y_0 = \overset{L}{v}_0$, $Y_i = \overset{L}{v}_i$ $(i = 1, \ldots, n)$, $Y_{n+1} = 1$, where $L = A, B$ and we are using the notation of Chapter 5. The operators that describe this system are obtained from those we already know for H_n with the Casimir invariant excluded via the correspondences (6.10) and (6.11). The metric has the form

$$ds^2 = \frac{-1}{(\frac{L}{v_0} + 1)^2} \left[-dv_1^2 \dots -dv_n^2 + dv_0^2 \right]. \qquad (6.88)$$

From the general theory of confocal cyclides and their relation to orthogonal separable coordinate systems, Bôcher [26] has shown that general cyclidic coordinates can always be chosen such that the Laplace equation (6.4) admits an R-separation of variables with

$$R(x) = (y_0 + y_{n+1})^{(n-2)/2} . \qquad (6.89)$$

The function $\phi = R^{-1}\psi$ satisfies the equation

$$\left[\sum_{i=1}^{n} \left(\frac{1}{\Pi_{j\neq 1}(x^i - x^j)} \right) \Theta(x^i) \frac{\partial}{\partial x^i} \left(\Theta(x^i) \frac{\partial}{\partial x^i} \right) \right) \qquad (6.90)$$

$$+ \left(\frac{n^2-4}{4} \sum_{i=1}^{n} x^i + \frac{1}{4} n(2-n) \sum_{i=1}^{n+2} e_i \right) \Big] \phi = 0$$

where $\Theta(x) = 2[\Pi_{i=1}^{n+2} (x - e_i)]^{\frac{1}{2}}$. In particular, if $e_{n+2} = \infty$ these equations have the form

$$\left[\sum_{i=1}^{n} \left(\frac{1}{\Pi_{j\neq 1}(x^i - x^j)} \right. Q(x^i) \frac{\partial}{\partial x^i} \left(Q(x^i) \frac{\partial}{\partial x^i} \right) \right) - \frac{1}{4} n(n-2) \Big] \phi = 0$$

$$(6.91)$$

where

$$Q(x^1) = 2\left[\prod_{i=1}^{n+1} (x - e_i) \right]^{\frac{1}{2}}, \quad R = \left(\frac{L}{v_0} + 1\right)^{(n-2)/2},$$

which can be recognized as the Helmholtz equation on H_n with eigenvalue $\frac{1}{4}n(n-2)$ for the corresponding Laplacian. For the remaining coordinate systems of type (ii) the metric has the form

$$ds^2 = \frac{-1}{(v_0 + v_1)^2} \left[dv_0^2 - dv_1^2 - \dots - dv_p^2 - du_1^2 - \dots - du_{q+1}^2 \right] \quad (6.92)$$

where $v_0^2 - \Sigma_{i=1}^{p} v_i^2 = 1$, $\Sigma_{j=1}^{q+1} u_j^2 = 1$, $p + q = n$ and the separation equations have the form

$$(\Delta_{H_p} + \Delta_{S_q})\psi = \frac{1}{4}n(n-2)\psi \qquad (6.93)$$

where Δ_{H_p}, Δ_{S_q} are the respective Laplace operators on H_p and S_q and the R function is $R = (v_0 + v_1)^{\frac{1}{2}(n-2)}$. We see from these results that there is no need to develop further graphical techniques for solutions of (6.3) and (6.4) and that the last statement of the theorem readily follows from these considerations.

7 Separation of variables for the heat equation

In this chapter we use the results of Chapters 3 and 4 and extend them to give a complete treatment of the problem of classifying all separable coordinate systems for the heat equation in Euclidean n-space

$$\Delta_n \psi + \kappa \psi_t = 0 \qquad \text{III}'$$

and the corresponding Hamilton-Jacobi equation

$$\sum_{i,j=1}^{n} g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j} + \kappa \frac{\partial W}{\partial t} = 0 \qquad \text{III}$$

where Δ_n is the Laplacian operator on E_n and, of course, $R_{ijk\ell} = 0$.

We note that III' can also be thought of as a form of Schrödinger's equation if we write $\kappa = ik$ (k real). This, of course, does not affect any of the separability properties. The solution to the classification problem of III is due principally to Reid [30], [31].

The trick in dealing with III is to consider instead the (n+2)-dimensional Minkowski space Hamilton-Jacobi equation

$$H = p_1^2 + \ldots + p_{n+1}^2 - p_{n+2}^2 = E, \qquad p_i = \frac{\partial W}{\partial z_i}. \qquad (7.1)$$

If we then look for solutions of $H = E$ for which $W = W' + \varepsilon(z_{n+1} + z_{n+2})$, then W' satisfies the Hamilton-Jacobi equation

$$H' = p_1'^2 - \ldots - p_n'^2 - \varepsilon p_t' = E \qquad (7.2)$$

where $p_i = p_i' = \partial W'/\partial x^i$, $p_t' = \partial W'/\partial t$ and $t = z_{n+1} - z_{n+2}$. Thus solutions of the form $W = W' + \varepsilon(z_{n+1} + z_{n+2})$ are solutions of the 'heat type' Hamilton-Jacobi equation III. In exactly an analogous way, if we consider solutions ψ of the Laplace equation in (n+2)-dimensional Minkowski space

$$\left[\left(\frac{\partial}{\partial z_1}\right)^2 + \ldots + \left(\frac{\partial}{\partial z_{n+1}}\right)^2 - \left(\frac{\partial}{\partial z_{n+2}}\right)^2\right] \psi = E\psi \qquad (7.3)$$

then we can look for solutions of the form

$$\psi = e^{\varepsilon(z_{n+1} + z_{n+2})} \psi',$$

where ψ' satisfies the associated heat equation

$$\left(\left(\frac{\partial}{\partial x^1}\right)^2 + \ldots + \left(\frac{\partial}{\partial x^n}\right)^2 + \varepsilon \partial_t\right) \psi' = E\psi'. \qquad (7.4)$$

In dealing with these equations for which $E \neq 0$ we are also effectively dealing with the case $E = 0$. The relationship between solutions of (7.4) and (7.3) are summarized in the following result [30].

<u>Theorem 7.1</u> (Reid): To every separable solution of (7.3) there corresponds an R-separable solution of (7.4) and vice versa. This statement also holds in the case of the corresponding Hamilton-Jacobi equations.

To construct a mapping from (7.4) to (7.3), suppose that $\{y^j\}$ is an R-separable coordinate system for (7.4), i.e.

$$x^i = x^i(y^1, \ldots, y^{n+1}), \qquad (7.5)$$
$$t = t(y^1, \ldots, y^{n+1}).$$

Then there are functions $\psi_j(x^j)$, $j = 1, \ldots, n+1$, R, such that a solution of (7.4) has the form

$$\psi = e^R \prod_{j=1}^{n+1} \psi_j(y^j). \qquad (7.6)$$

Consider now the coordinate system $\{\tilde{y}^j\}$:

$$z_j = x^j(\tilde{y}^1, \ldots, \tilde{y}^n), \quad j = 1, \ldots, n, \qquad (7.7)$$
$$z_{n+1} - z_{n+2} = 2t(\tilde{y}^1, \ldots, \tilde{y}^n),$$

$$z_{n+1} + z_{n+2} = \tilde{y}^{n+2} - R/\varepsilon .$$

This is a coordinate system in Minkowski space with an ignorable variable $\tilde{y}^{n+2}$ corresponding to the symmetry operator

$$L = \frac{\partial}{\partial_y n+2} = \tfrac{1}{2}\left(\frac{\partial}{\partial z_{n+1}} + \frac{\partial}{\partial z_{n+2}}\right) \qquad (7.8)$$

whose eigenvalue we shall specify as ε. Let

$$\overline{\psi} = \exp[\varepsilon \tilde{y}^{n+1}]\, e^{R} \prod_{j=1}^{n+1} \psi_j(\tilde{y}^j) .$$

As ψ satisfies (7.4), $\overline{\psi}$ is easily shown to satisfy (7.3). In other words, $\{\tilde{y}^j\}$ is a separable coordinate system for the Helmholtz equation (7.3). If $\{\tilde{y}^j\}$ is a separable coordinate system for (7.4) with symmetry operator (7.3), then

$$z_j = x^j(\tilde{y}^1, \ldots, \tilde{y}^{n+1}), \qquad (7.9)$$

$$z_{n+1} - z_{n+2} = 2t(\tilde{y}^1, \ldots, \tilde{y}^{n+1}),$$

$$z_{n+1} + z_{n+2} = \tilde{y}^{n+2} + f(\tilde{y}^1, \ldots, \tilde{y}^{n+1}),$$

which, if we let $f = -R/\varepsilon$, is the image of (7.7) and the theorem is proved.

If we define $\tilde{\psi} = e^{-Et/2\varepsilon}\psi$ then, on substituting, $\tilde{\psi}$ is an R-separable solution of (7.4) with $E = 0$ iff ψ is an R-separable solution of (7.4). Hence we may as well consider separable coordinate systems for which $E \neq 0$.

If is found that the additive variable separation ansatz is insufficient to deal with all the relevant solutions of (7.2). In fact, we need the additive analogue of R-separation, i.e., solutions of the form

$$W = R + \sum_i W_i(x^i; c_1, \ldots, c_n) . \qquad (7.10)$$

However, if we look for solutions of (7.1) then the usual additive ansatz is sufficient. The classification problem that needs to be solved is then the finding of all separable solutions of (7.1) of the form

$$W = \sum_{i=1}^{n+1} W_i(x^i) + \varepsilon(z_{n+1} + z_{n+2}) ,$$

i.e., in which $z_{n+1} + z_{n+2}$ is an ignorable variable. The following result derivable from a theorem due to Benenti [17] is crucial to this classification.

Theorem 7.2 (Reid): All Hamilton-Jacobi separable coordinate systems for $H = \sum_{i=1}^{n+1} p_i^2 - p_{n+2}^2 = E$ of the form $W = W' + \varepsilon(z_{n+1} + z_{n+2})$ are equivalent to a coordinate system associated with the Hamilton-Jacobi equation

$$\sum_{i,j=1}^{n} g^{ij} p_i p_j + 2p_{n+1}p_{n+2} + g^{n+2,n+2} p_{n+2}^2 = E \qquad (7.11)$$

where

$$g^{ij} = \begin{array}{cc} \longleftarrow n_1 \longrightarrow & \longleftarrow n_2 \longrightarrow \\ \left[\begin{array}{c|c} H_a^{-2}\delta_{ab} & 0 \\ \hline 0 & g^{\alpha\beta} \end{array}\right. & \left.\right] \end{array}$$

with

$$H_a^{-2} = \frac{S^{a1}}{S} ,$$

$$g^{\alpha\beta} = \sum_a A_a^{\alpha\beta}(x^a)\frac{S^{a1}}{S} ,$$

and

$$g^{n+1,n+1} = \sum_a v_a(x^a)\frac{S^{a1}}{S} .$$

Here S and S^{a1} are the determinant and (a, 1)-cofactor of a Stäckel matrix $S = (S_{ij}(x^i))_{n_1 \times n_1}$. The cartesian coordinates z_i can be expressed in terms of the x^i by means of the equations

$$z_i = z_i(x^1, \ldots, x^n, t), \qquad i = 1, \ldots, n, \qquad (7.12)$$

$$z_{n+1} - z_{n+2} = 2t = 2x^{n+1} ,$$

$$z_{n+1} + z_{n+2} = x^{n+2} + f(x^1, \ldots, x^n, t) .$$

Using this result and the associated group theory, a complete solution can be obtained. We first need to observe that the Lie algebra of Lie symmetries of (7.1) is $E(n+1, 1)$, consisting of a basis

$$I_{n+2,j} = z_{n+2}p_j + z_j p_{n+2}, \qquad j = 1, \ldots, n+1, \tag{7.13}$$

$$I_{jk} = z_j p_k - z_k p_j, \qquad j, k = 1, \ldots, n+1,$$

$$P_j = p_i, \qquad i = 1, \ldots, n+2.$$

The non-zero commutation relations for this basis are

$$[I_{ij}, I_{kl}] = \delta_{kj} I_{il} + \delta_{il} I_{jk} + \delta_{lj} I_{ik} + \delta_{ik} I_{lj}, \qquad (7.140$$

$$[I_{ij}, I_{n+2k}] = \delta_{jk} I_{n+2i} - \delta_{ik} I_{n+2j},$$

$$[I_{n+2j}, I_{n+2k}] = I_{jk},$$

$$[P_i, I_{jk}] = \delta_{ij} P_k - \delta_{ki} P_j,$$

$$[P_i, I_{n+2j}] = \delta_{in+2} P_j + \delta_{ij} P_{n+2}.$$

As we are seeking solutions of the form

$$W = \sum_a W_a(x^a) + \sum_\alpha c_\alpha x^\alpha + \varepsilon(z_{n+1} + z_{n+2}),$$

any of the ignorable variables x^α that occur in a separable coordinate system of this type must correspond to the Lie symmetries p_α such that $[p_\alpha, p_{n+1} + p_{n+2}] = 0$. This defines a subalgebra of $E(n+1, 1)$ for which a suitable basis is

$$\varepsilon = \tfrac{1}{2}(p_{z_{n+1}} + p_{z_{n+2}}), \tag{7.15}$$

$$P_u, \quad M_{uv} = z_u p_{z_v} - z_v p_{z_u},$$

$$B_u = z_u \varepsilon - \tfrac{1}{2}(z_{n+1} - z_{n+2}) p_{z_u}, \qquad u, v = 1, \ldots, n.$$

This is the Galilean algebra whose non-zero commutation relations are

$$[M_{uv}, M_{rs}] = \delta_{rv} M_{us} + \delta_{us} M_{vr} + \delta_{sv} M_{ur} + \delta_{ur} M_{sv}, \tag{7.16}$$

$$[P_u, M_{rs}] = \delta_{ur} P_s - \delta_{su} P_r,$$

$$[P_u, B_v] = \delta_{uv} \varepsilon,$$

$$[M_{uv}, B_w] = \delta_{vw} B_u - \delta_{wu} B_v.$$

The key to the computation of all the inequivalent coordinate systems for (7.1) is the following result.

Theorem 7.3 (Reid): All Hamilton-Jacobi separable coordinate systems for $H = \Sigma_{i=1}^{n+1} p_i^2 - p_{n+2}^2 = E$ of the form

$$W = \sum_i^{i=1} W_i(x^i; c_1, \ldots, c_n, \varepsilon) + \varepsilon(z_{n+1} + z_{n+2})$$

are equivalent to a coordinate system associated with the Hamilton-Jacobi equation

$$\sum_{i=1}^n g^{ii} p_i^2 + 2p_{n+1}p_{n+2} + g^{n+2,n+2} p_{n+2}^2 = E. \tag{7.17}$$

The metric tensor has only one non-diagonal term. Furthermore, there exists a partition $\{1, \ldots, n\} = \cup_{l=1}^L B_l$, where $B_l = \{l_1, \ldots, l_{n_l}\}$, $B_l \cap B_m = \emptyset$ and $n_l \geq 1$, such that the metric coefficients have the form

$$g^{ll} = \frac{1}{\sigma_l} \tilde{g}^{ll}, \qquad l \in B_l, \tag{7.18}$$

$$g^{n+2,n+2} = \sum_{l=1}^L \frac{V_l}{\sigma_l}$$

where $\tilde{g}^{ll} = \tilde{g}^{ll}(x^m)$, $V_l = V_l(x^m)$, $m \in B_l$, and $\sigma_m = \sigma_m(x^{n+1})$.

Proof: The proof proceeds in two parts. First we use group theory. If the coordinate system $\{x^i\}$ has $q+1$ ignorable coordinates x^α then the standard cartesian coordinates are obtained by means of the usual methods of local Lie theory [32].

For each ignorable the corresponding Lie symmetry p_α has the form

$$p_\alpha = \rho^u_\alpha P_u + m^{uv}_\alpha M_{uv} + \beta^u_\alpha B_u, \quad \alpha \neq n+1. \tag{7.19}$$

(Note: we cannot have a term $\kappa_\alpha K_2$, as this would imply that there are two orthogonal vectors which are both null.) To deduce the forms that the p_α must have, we proceed as follows. Let $x^{n+1} = c$ and $x^{n+2} = k$. The corresponding Hamilton-Jacobi equation is then

$$\sum_u P_u^2 = E. \tag{7.20}$$

Thus the choice of coordinates

$$\bar{z}_i = z_i(x^1, \ldots, x^n, c), \quad i = 1, \ldots, n, \tag{7.21}$$

on this reduced Hamilton-Jacobi equation can be solved by means of separation of variables in E_n. This system of coordinates is equivalent to a coordinate system $\{\bar{x}^i\}$ on E_n for which

$$\bar{x}^u = T^u(x^v; c) \tag{7.22}$$

and the corresponding Lie symmetries are

$$\bar{p}_{\alpha_1} = \bar{I}_{12}, \ldots, \bar{p}_{\alpha_p} = \bar{I}_{2p-1,2p},$$
$$\bar{p}_{\alpha_{p+1}} = \bar{P}_{2p+1}, \ldots, \bar{p}_{\alpha_r} = \bar{P}_{p+r}. \tag{7.23}$$

Now consider

$$\bar{x}^u = T^u(x^v; c{=}0), \quad \bar{x}^{n-1} = x^{n-1}, \quad \bar{x}^n = x^n,$$

which is an equivalent separable system for (7.1). Setting $x^{n-1} = x^n = 0$, then

$$\bar{p}_\alpha = \bar{\rho}^u_\alpha \bar{P}_u + m^{uv}_s \bar{M}_{uv}, \quad \alpha \neq n-1, n. \tag{7.24}$$

Comparing coefficients $\bar{\rho}^u_\alpha$ and m^{uv}_s with (7.23), we see that, with the restriction $x^{n-1} = x^n = 0$ removed,

$$p_{\alpha_1} = M_{12} + \beta^u_1 B_u, \ldots, p_{\alpha_p} = M_{2p-1,2p} + \beta^u_p B_u, \tag{7.25}$$
$$p_{\alpha_{p+1}} = P_{2p+1} + \beta^u_{p+1} B_u, \ldots, p_{\alpha_r} = P_{p+r} + \beta^u_r B_u$$

for $\alpha_s \neq n-1, n$.

We can assume that $\beta^u_s = 0$, $s = p+1, \ldots, q$, and $\beta^v_s \neq 0$ for some v, $s = q+1, \ldots, r$, and we show that $\beta^u_s = 0$, $s = 1, \ldots, p$. Consider first the Lie symmetry $p_{\alpha_1} = M_{12} + \beta^u_1 B_u$. Using the adjoint action of the B_u's it is possible to choose $\beta^1_1 = \beta^2_1 = 0$. Now, using rotations independent of z_1 and z_2, we can take $p_{\alpha_1} = M_{12} + \beta B_3$. Again setting $x^{n-1} = c$ and $x^n = k$, $p_{\alpha_1} \rightarrow M_{12} - \beta_c P_3$, which must be a Lie symmetry for a separable system on E_n. This is possible only if $\beta = 0$. We can now show that $\beta^u_s = 0$, $u \neq s+p$, $q+1 \leq s \leq r$. Using rotations independent of z_{s+p},

$$p_{\alpha_s} \rightarrow P_{s+p} + \beta^{s+p}_s B_{s+p} + \beta B_v \tag{7.26}$$

for some $v \neq s+p$ where

$$\beta = [\sum_{u \neq s+p} (\beta^u_s)^2]^{\frac{1}{2}}.$$

Using a further rotation about z_{s+p}, z_v axes, we obtain

$$p_{\alpha_s} \rightarrow \frac{1}{\kappa}[\beta^{s+p}_s P_{s+p} - \beta P_v] + \kappa B_{s+p} \tag{7.27}$$

where

$$\kappa = [\beta^2 + (\beta^{s+p}_s)^2]^{\frac{1}{2}} \neq 0.$$

Applying the adjoint action of K_{-2},

$$p_{\alpha_s} \rightarrow B_{s+p} - \bar{\beta} P_v \tag{7.28}$$

by suitably redefining x^{α_s} and taking $\bar{\beta} = \beta / \kappa^2$. The corresponding cartesian coordinates for this as a Lie symmetry are calculated from

$$dx^{\alpha_s} = \frac{dz_{s+p}}{\frac{1}{2}(z_{n+2} - z_{n+1})} = \frac{dz_v}{-\bar{\beta}} = \frac{dz_w}{0} = \frac{dz_{n+1}}{\frac{1}{2}z_{s+p}} = \frac{dz_{n+2}}{\frac{1}{2}z_{s+p}},$$

$$w \neq v,\ s+p. \tag{7.29}$$

This implies coordinates that can be written as

$$z_{s+p} = -x^{n+1}x^{\alpha_s} + A_{s+p}, \tag{7.30}$$

$$z_v = -\bar{\beta}x^{\alpha_s} + A_v,$$

$$z_{n+1} + z_{n+2} = x^{n+2} - \tfrac{1}{2}x^{n+1}(x^{\alpha_s})^2 + A_{s+p}x^{\alpha_s} + D,$$

$$z_{n+1} - z_{n+2} = 2x^{n+1},$$

where the $A_1, \ldots, A_n$ and D do not depend on x^{α_s} and x^{n+1}. If $\bar{\beta} \neq 0$ and $x^{n+1} = c$ then, from the arguments we used in the proof of Theorem 7.3, A_{s+p} and A_v are functions of x^{n+1} alone. This is clearly not possible, as z_{s+p}, z_v, $z_{n+1} - z_{n+2}$ would then be linearly dependent. Therefore $\bar{\beta} = 0$. Consequently, the ignorable variables can always be chosen as

$$p_{\alpha_1} = I_{12},\ p_{\alpha_2} = I_{34},\ \ldots,\ p_{\alpha_p} = I_{2p-1,2p}, \tag{7.31}$$

$$p_{\alpha_{p+1}} = P_{2p+1}, \ldots, p_{\alpha_q} = P_{p+q}$$

$$p_{\alpha_{q+1}} = B_{p+q+1} - \rho^{p+q+1}P_{p+q+1}, \ldots, p_{\alpha_r} = B_{p+r} - \rho^{p+r}P_{p+r}.$$

Indeed, if these are the Lie symmetries that correspond to a given separable coordinate system then the coordinates z_i can be chosen as

$$z_{2s-1} = A_{2s-1}\cos(x^{\alpha_s} + A_{2s}), \tag{7.32}$$

$$z_{2s} = A_{2s-1}\sin(x^{\alpha_s} + A_{2s}), \quad 1 \leq s \leq p,$$

$$z_{p+s} = x^{\alpha_s} + A_{p+s}, \quad p+1 \leq s \leq q,$$

$$z_{p+s} = [\rho_{p+s} - x^{n-1}]x^{\alpha_s} + A_{p+s}; \quad q+1 \leq s \leq r,$$

$$z_s = A_s, \qquad r+1 \leq s \leq n,$$

$$z_{n+1} + z_{n+2} = x^{n+2} + \sum_{s=q+1}^{r} (\tfrac{1}{2}[\rho_{p+s} - x^{n-1}]((x^{\alpha_s})^2 + A_{p+s}x^{\alpha_s}) + D,$$

$$z_{n+1} - z_{n+2} = 2x^{n+1},$$

where $A_1, \ldots, A_n$, D are functions of the non-ignorable variables x^a including x^n. If we take x^{n+1} to be a constant then the resulting coordinate system is separable on E_n and consequently A_{2s}, $1 \leq s \leq p+1$, A_{p+s}, $p+1 \leq s \leq r$, are not functions of the non-ignorable variables x^a, $a \neq n+1$ and are consequently a function of x^{n+1} only.

Using the transformations

$$x^{\alpha_s} + A_{2s} \rightarrow x^{\alpha_s}, \quad 1 \leq s \leq p, \tag{7.33}$$

$$x^{\alpha_s} + A_{p+s} \rightarrow x^{\alpha_s}, \quad p+1 \leq s \leq q,$$

$$x^{\alpha_s} \rightarrow x^{\alpha_s} - \frac{A_{p+s}}{[\rho_{p+s} - x^{n+1}]}, \quad q+1 \leq s \leq r,$$

we can assume that

$$A_{2s} = 0, \qquad 1 \leq s \leq p, \tag{7.34}$$

$$A_{p+s} = 0, \qquad p+1 \leq s \leq q.$$

With this choice of ignorable variables we see that it is always possible to take

$$g^{ij} = g^{ii}\delta_{ij}, \quad i, j = 1, \ldots, n. \tag{7.35}$$

To establish the second part of the theorem we proceed as follows.

It follows from the necessary and sufficient conditions for Stäckel form (2.52) that

$$\partial_{v\,n+1}\log g_{uu} = \partial_v \log g_{uu}\,\partial_{n+1}\log g_{uu} - \partial_v \log g_{uu}\,\partial_{n+1}\log g_{vv}. \tag{7.36}$$

We can also invoke the curvature condition

$$R_{vuu\,n+1} = \tfrac{1}{2}g_{uu}\,\partial_{v\,n+1}\log g_{uu} + \frac{1}{4}g_{uu}\left[\partial_v \log(g_{uu})\,\partial_{n+1}\log(g_{uu})\right] - \partial_v \log g_{uu}\,\partial_{n+1}\log g_{vv} = 0. \tag{7.37}$$

Combining these results, we obtain that

$$\partial_{v\,n+1}\log g_{uu} = 0 \tag{7.38}$$

i.e.

$$g_{uu} = \sigma_u(x^{n+1})\,\bar{g}_{uu},$$

where $\partial_{n+1}\bar{g}_{uu} = 0$. From the conditions (7.36) and (7.37) we deduce that

$$\partial_v \log(g_{uu})\,\partial_{n+1}\log(\sigma_u/\sigma_v) = 0\,. \tag{7.39}$$

We can now group together coordinates whose σ functions are proportional. Reordering the indices if necessary, we see that we can write the portion $\Sigma_{i=1}^{n} g^{ii} p_i^2$ in the form

$$\sum_{l=1}^{L} \frac{1}{\sigma_l}\left(\sum_{l \in B_l} \tilde{g}^{ll} p_l^2\right).$$

If we now substitute the particular form (7.38) of the separable solution into the Hamilton-Jacobi equation, we obtain

$$\sum_{l=1}^{L} \frac{1}{\sigma_l}\left(\sum_{l \in B_l} \tilde{g}^{ll}\left(\frac{\partial W_l}{\partial x^l}\right)^2\right) + 2\frac{\partial W_{n+1}}{\partial x^{n+1}}\varepsilon + \varepsilon^2 g^{n+2\,n+2} = 0. \tag{7.40}$$

From this equation we see that $g^{n+2\,n+2}$ has the form

$$g^{n+2\,n+2} = \sum_{l=1}^{L}\frac{V_l}{\sigma_l} + h(x^{n+1})\,. \tag{7.41}$$

By choosing an equivalent ignorable coordinate $\bar{x}^{n+2}$ by means of a transformation of the form

$$\bar{x}^{n+2} = x^{n+2} - \tfrac{1}{2}\int h(x^{n+1})\,dx^{n+1}, \quad \bar{x}^j = x^j \quad (j = 1, \ldots, n+1),$$

then

$$\bar{g}^{n+2\,n+2} = \sum_{l=1}^{L}\frac{V_l}{\sigma_l} \quad \text{with } V_l = V_l(x^m), \quad m \in B_l\,.$$

This completes the proof of the theorem.

For the completion of our classification we need to impose the curvature conditions $R_{ijkl} = 0$ to find the possible forms of the functions σ_l, V_l. It follows from Theorem 7.2 that the functions $\tilde{g}^{ll}$, $l \in B_m$, must be in Stäckel form, i.e., there exists a Stäckel matrix

$$\tilde{S}^{(m)} = (S_{ln}(x^n))_{n_m \times n_m}$$

such that $\tilde{g}^{ll} = S^{(m)l1}/S^{(m)}$ in obvious notation. The curvature conditions $R_{ijkl} = 0$ imply that $\bar{R}_{ijkl} = 0$ for i, j, k, $l \in B_m$ and some m fixed. Furthermore, from the requirement that the underlying Riemannian manifold have the signature + ... +, the corresponding quadratic form $\bar{H}_{(m)} = \Sigma_{l \in B_m} \bar{g}^{ll} p_l^2$ must correspond to a separable coordinate system on E_{n_m} of the type classified in Chapter 4.

Further, we note from (7.40) that each V_l function has the form

$$V_l = \sum_{m \in B_l} v_m(x^m)\frac{S^{(l)m1}}{S^{(l)}} \tag{7.42}$$

The remaining question to answer is: given a choice of coordinate system x^l, $l \in B_l$, what possible choice of functions V_l and σ_l can be made so as to correspond to Minkowski space of dimensions $n+2$? A useful result to determine the functions V_l is the following.

<u>Theorem 7.4</u> (Reid): If $\{x^i\}$ is a separable coordinate system on E_{n+2} for which the Hamilton-Jacobi equation has the form (7.17), then the functions V_l must satisfy

$$\partial_m \partial_n V_l = 0, \qquad m \neq n. \tag{7.43}$$

Proof: This result is readily proved by noting that a function V_l of the form (7.42) must satisfy the conditions

$$\partial_c \partial_d V_l - (\partial_c V_l)(\partial_d \log \tilde{g}^{cc}) - (\partial_d V_l)(\partial_c \log \tilde{g}^{dd}) = 0 \tag{7.44}$$

These conditions can readily be derived from the Eisenhart conditions of Stäckel form (2.52) applied to the Hamiltonian

$$\tilde{H} = \sum_{l \in B_l} \tilde{g}^{ll} p_l^2 + V_l p_{r+1}^2 .$$

Further, we can compute the curvature condition $R_{c\,n+1\,n+1\,d} = 0$ and obtain

$$2\partial_c \partial_d V_l + (\partial_c V_l)(\partial_d \log \tilde{g}^{cc}) + (\partial_d V_l)(\partial_c \log \tilde{g}^{dd}) = 0. \tag{7.45}$$

The result then follows, i.e. $\partial_c \partial_d V_l = 0$. Furthermore, this result is actually independent of the choice of coordinates x^l, $l \in B_l$. We could, in fact, find suitable cartesian coordinates z^i, $i \in B_l$, such that

$$\tilde{H} = \sum_{i=1}^{n_l} p_i^2 + V_l p_{n+1}^2$$

and deduce similarly that

$$\partial_i \partial_j V_l = 0. \tag{7.46}$$

For the determination of the functions σ_l we have the curvature conditions $R_{n+1\,c\,c\,n+1} = 0$ which are equivalent to

$$\tfrac{1}{2}\sigma_l'' - \left(\frac{\sigma_l'}{4\sigma_l}\right)^2 - \tfrac{1}{2} \sum_{m=1}^{L} \frac{\partial_c^2 V_m}{\sigma_m} = 0 \tag{7.47}$$

where $\sigma_l' = \partial_{n+1} \sigma_l$. These conditions together with (7.46) imply that

$$V_m = \sum_{l \in B_m} \left(\tfrac{1}{4} \zeta_m (z^l)^2 + \gamma_l z^l\right) + \delta \tag{7.48}$$

and $2\sigma_l \sigma_l'' - \sigma_l'^2 = \zeta_l$, $\sigma_l > 0$, ζ_l, δ real. There are four distinct solutions of this last equation which represent inequivalent coordinate systems. These are summarized as follows.

Possible values for the functions $\sigma(x^{n+1})$

Type	σ_q	ζ_q	γ_q
I	1	0	arbitrary
II	$(x^{n+1} + v_q)^2$	0	arbitrary
III	$(x^{n+1} + v_q)$	-1	0
IV	$\lvert (x^{n+1} + v_q)^2 \pm w_q^2 \rvert$	$\pm 4w_q^2$, $w_q \neq 0$	0

A knowledge of how the lower-dimensional coordinate systems are constructed provides the clue as to the general solution. If we consider the class of Hamiltonians

$$\hat{H}_l = \frac{1}{\sigma_l}(p_1^2 + V_l p_3^2) + 2p_2 p_3, \qquad l = \text{I, II, III, IV} \pm, \tag{7.49}$$

then we can choose cartesian coordinates z_i, $i = 1, 2, 3$, such that

$$z_1 = x^1 \sigma^{\frac{1}{2}} + \tfrac{1}{2}\gamma \int\!\!\int \sigma^{-3/2}, \tag{7.50}$$
$$z_2 + z_3 = x^3 - \frac{1}{4}\sigma'(x^1)^2 - \tfrac{1}{2}\gamma x^1 \sigma^{\frac{1}{2}} \int \sigma^{-3/2} - \frac{\gamma^2}{8} \int (\int \sigma^{-3/2})^2,$$
$$z_2 - z_3 = 2x^2,$$

where $\int f(y) = \int_{y=y_0}^{x^2} f(y)\,dy$. If we now define

$$F_{uq}(x^u, x^{n+1}) = y^u \sigma_q^{\frac{1}{2}} + \tfrac{1}{2}\gamma_u \int\!\!\int \sigma_q^{-3/2} \tag{7.51}$$

and

$$G_{uq}(x^u, x^{n+1}) = \frac{1}{4}\sigma_q'(x^u)^2 + \tfrac{1}{2}\gamma_u x^u \sigma_q^{\frac{1}{2}} \int \sigma_q^{-3/2} + \frac{1}{8}\gamma_u^2 \int (\int \sigma_q^{-3/2})^2 \tag{7.52}$$

then the cartesian coordinates for the Hamiltonian

$$H = \sum_{l=1}^{L} \frac{1}{\sigma_l}\left(\sum_{m \in B_l} p_m^2 + V_l p_{n+1}^2\right) + 2p_{n+1}p_{n+2} \qquad (7.53)$$

can be obtained by means of the formulas

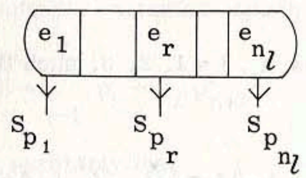
$$z_u = F_{uq}, \qquad (7.54)$$

$$z_{n+1} + z_{n+2} = x^{n+2} - \sum_{l=1}^{L} \sum_{m \in B_L} G_{ml},$$

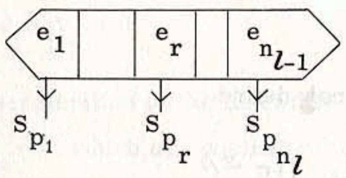
$$z_{n+1} - z_{n+2} = 2x^{n+1}.$$

It can readily be verified that this choice of coordinates does correspond to a Hamiltonian of the form (7.53). The remaining problem is to find the forms of the functions V_l compatible with a given choice of separable coordinates $z^i = z^i(x^j)$, $i, j \in B_l$.

To find all possible coordinate systems we can generalize the graphical calculus developed for E_n separable coordinate systems. Quite generally such coordinate systems can be represented by sums of two types of graph:

(A) 

Elliptic

(B) 

Parabolic

We will say that E_n splits into subspaces E_{n_r}. Cartesian coordinates on E_{n_r} for case (A) are given by

$$y^{iq} = ({}_{N_l}w_i)(p_i S_q), \quad 1 \le i \le N_l, \quad 1 \le q \le p_i + 1, \qquad (7.55)$$

where $n_l = \frac{1}{2}\sum_{i=1}^{N_l} p_i(p_i + 1)$, and for case (B) by

$$y^1 = {}_{N_l-1}w_1, \quad y^{iq} = ({}_{N_l-1}w_i)(p_i S_q) \qquad (7.56)$$

where $2 \le i \le N_l - 1$, $1 \le q \le p_{i+1}$, $n_l = \frac{1}{2}\sum_{i=2}^{N_l-1} p_i(p_i+1) + 1$.

We say that the sphere S_{p_j} is attached to the e_j in particular if $p_j = 0$, S_{p_j} is the zero-dimensional or trivial sphere and we define ${}_0S_1 = 1$ (i.e., no coordinate is attached).

Each separable coordinate system on E_n can be represented as a sum of graphs of type (A) or (B). If we now permit some of the σ_l functions to be equal we may assume that each set of coordinates z^i, $i \in B_l$, can be chosen to correspond to a connected graph of one of these forms. From the explicit forms of the coordinates and the criterion (7.43) we can determine the V_l for $n_l > 1$.

Suppose that $\zeta_l \ne 0$, then we can take

$$V_l = \sum_{i=1}^{n_l} \frac{1}{4}\zeta_l (y^i)^2 = \frac{1}{4}\sum_{i=1}^{n_l} \zeta_l ({}_{n_l}w_i)^2.$$

From the explicit form of coordinates corresponding to types (A) and (B) we can readily verify that for case (A)

$$V_l = \frac{\zeta_l}{4} c_l^2 \left(\sum_{i=1}^{n_l} x^i - \sum_{i=1}^{n_l} e_i\right). \qquad (7.57)$$

For case (B) it is never possible to satisfy conditions (7.43). Now consider when $\zeta_l = 0$ and suppose that at least one of the γ_i is not zero. We can in fact (by a suitable rotation) assume that only $\gamma_1 \ne 0$. Then, examining the explicit form of the coordinates, we can verify that type (B) is the only compatible coordinate system. In this case

$$V_l = \tfrac{1}{2}\gamma_1 c_r (x^1 + \ldots + x^{n_l} + e_1 + \ldots + e_{n_l - 1}). \qquad (7.58)$$

If $\gamma_q = \zeta_l = 0$ then both systems of type (A) and (B) are possible. If $N_l = 1$ then clearly all cases of σ_l and corresponding V_l are possible.

It remains to develop a graphical calculus to describe all possible separable coordinate systems. The notation chosen is to represent a given coordinate system by the graphical representation

$$\overset{L_1}{\boxed{G_1}} \quad \overset{L_q}{\boxed{G_q}}$$

where G_q represents a given separable coordinate system on E_{n_q} and the box indicates that there is one function, σ_q, associated with this graph in the manner we have derived in equations (7.53) and (7.54). The symbol L_q is a latin number (I, II, III or IV) indicating the type of σ_q function associated with the graph. In general, we take each G_q to correspond to a connected graph of a separable coordinate system on E_{n_q} and allow for the possibility that some σ_q functions may be identical. We note by looking at the possibilities that in general the parameters occurring in the σ_q function are needed in addition to the symbol L_q when specifying a given coordinate system. This is necessary because a given coordinate system may have several L_q's specifying the same type of σ_q function but with different parameters. The procedure for constructing all coordinate systems now emerges.

(A) Construct the graphs representing a separable system on E_n as a sum of disjoint components G_q having a typical representation

$$G_1 \quad G_2 \quad \ldots\ldots \quad G_q .$$

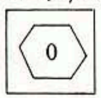
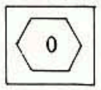
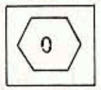
(B) Construct all possible separable systems associated with this graph by considering those which have the representation

$$\overset{L_1}{\boxed{G_1}} \quad \overset{L_q}{\boxed{G_q}}$$

where the L_q correspond to those σ_q functions that are compatible with the corresponding graph G_q. The complete list of possible coordinate

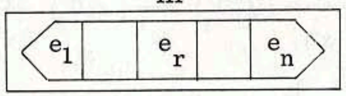
systems for $n = 1$ is given in Table 7.1.

Table 7.1

(1)	I 	$z_1 = y^1 + \gamma(x^2)^2/4$
(2)	II ($v \neq 0$) 	$z_1 = \sigma_{II}^{\frac{1}{2}} y^1 + \dfrac{\gamma}{4(x^2 + v)}$
(3)	III ($v \neq 0$) 	$z_1 = \sigma_{III}^{\frac{1}{2}} y^1$
(4)	IV± ($v \neq 0, \omega^2 \neq 1$) 	$z_1 = \sigma_{IV\pm}^{\frac{1}{2}} y^1$

It is relatively straightforward to compute all the properties of a separable system once its graph has been specified. Indeed, given a general graph, the corresponding details of the coordinate system can readily be deduced. As an example, consider the graph

III



The choice of coordinates is

$$z_j^2 = c^2 \frac{\Pi_{x=1}^{n} (x^i - e_j)}{\Pi_{i \neq j} (e_i - e_j)} (x^{n+1} + v_3), \qquad j = 1, \ldots, n, \qquad (7.59)$$

$$z_{n+1} + z_{n+2} = x^{n+2} - \frac{c^2}{4}\left[\sum_{i=1}^{n} (x^i - e_i)\right],$$

$$z_{n+1} - z_{n+2} = 2x^{n+1}.$$

The Hamilton-Jacobi equation assumes the form

$$\frac{1}{(x^{n+1} + v_3)}\left[\frac{1}{c^2}\sum \frac{1}{[\Pi_{j\neq 1}(x^i - x^j)]} p_i^2 + \frac{c^2}{4}\left(\sum_{i=1}^{n}(x^i - e_i)\right)p_{n+2}^2\right]$$

$$+ 2p_{n+1}p_{n+2} \tag{7.60}$$

where $p_i = \sqrt{[\Pi_{j=1}^{n}(x^i - e_j)]}\,\frac{\partial W}{\partial x^i}$.

This equation admits the separable solution of the form

$$W = \sum_{i=1}^{n+1} W_i(x^i) + \varepsilon x^{n+2} \tag{7.61}$$

where the separation equations are

$$c^{-2}\left[\prod_{i=1}^{n}(x^i - e_j)\right]\left(\frac{dW_i}{dx^i}\right)^2 + \frac{1}{4}c^2\varepsilon^2 (x^i)^n \tag{7.62}$$

$$+ \left(\overline{E} - \frac{\varepsilon^2 c^2}{4}\left(\sum_{i=1}^{n} e_i\right)\right)(x^i)^{n-1} + \sum_{j=2}^{n}\lambda_j (x^i)^{n-j} = 0, \quad i = 1, \dots, n,$$

$$\frac{1}{(x^{n+1} + v_3)}\overline{E} + 2\varepsilon\frac{dW_{n+1}}{dx^{n+1}} = E.$$

This separable system provides a 'separable' solution of the corresponding heat equation using Theorem 7.1. The corresponding Helmholtz equation has the form

$$\frac{1}{(x^{n+1}+v_3)}\left[c^{-2}\sum\frac{1}{[\Pi_{j\neq 1}(x^i - x^j)]}\left\{\sqrt{(\mathcal{P}_i)}\,\frac{\partial}{\partial x^i}\left(\sqrt{(\mathcal{P}_i)}\,\frac{\partial\psi}{\partial x^i}\right)\right\}\right.$$

$$\left. + \frac{c^2}{4}\left(\sum_{i=1}^{n}(x^i - e_i)\right)\frac{\partial^2\psi}{(\partial x^{n+2})^2}\right] + 2\frac{\partial^2\psi}{\partial x^{n+1}\partial x^{n+2}} = \lambda\psi. \tag{7.63}$$

This equation admits product separable solutions of the form

$$\psi = e^{\varepsilon x^{n+2}}\prod_{i=1}^{n+1}\psi_i(x^i)$$

with corresponding separation equations

$$\overline{c}^2\left\{\sqrt{(\mathcal{P}_i)}\,\frac{d}{dx^i}\left(\sqrt{(\mathcal{P}_i)}\,\frac{d\psi_i}{dx^i}\right)\right\} + \left[\frac{1}{4}c^2\varepsilon^2(x^i)^n\right. \tag{7.64}$$

$$\left. + \left(\overline{\lambda} - \frac{\varepsilon^2 c^2}{4}\left(\sum_{i=1}^{n} e_i\right)\right)(x^i)^{n-1} + \sum_{j=2}^{n}\lambda_j(x^i)^{n-j}\right]\psi_i = 0,$$

$$i = 1, \dots, n.$$

We see from (7.59) that this particular choice of coordinate system does not lead to R-separable solutions of the corresponding heat equation or Hamilton-Jacobi equation.

The constants of the motion that are equal to the separation constants are readily computed from those for the corresponding Euclidean coordinates inside the boxes of the given representation of the coordinate system. If we work with the corresponding heat equation the rules can be summarized as in Table 7.2. For instance, returning to the coordinate system (7.59), the

Table 7.2: Images of Euclidean operators

	L_q I	II	III	IV±
P_b^2	$P_b^2 + \gamma_b \varepsilon B_b$	$(v_q P_b - B_b)^2 + \gamma_b \varepsilon P_b$	$v_q P_b^2 - \{P_b, B_b\}_+$	$(v_q P_b - B_b)^2 \pm w_q^2 P_b^2$
M_{ab}^2	M_{ab}^2	M_{ab}^2	M_{ab}^2	M_{ab}^2
$\{M_{,b}, P_b\}$	$\{M_{qb}, P_b\}_+ - \gamma_q B_b^2/4$	$\{M_{qb}, v_q P_b - B_b\}_+ - \gamma_q P_b^2/4$	Does not occur	Does not occur

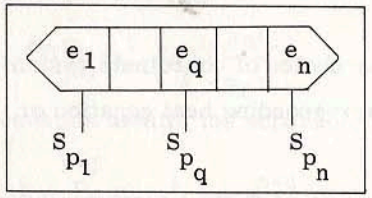
corresponding operators that are in the enveloping algebra of the Galilean algebra are obtained from those of the corresponding E_n graph by replacing

P_b^2 in the expressions (4.33) everywhere by $v_q P_b^2 - \{P_b, B_b\}_+$.

In particular, we notice that for a general E_n graph the operators that arise from the attached spheres of a given connected component remain unchanged and it is only those operators that arise from the block at the base of the given graph that are modified.

The general form of a contribution corresponding to a type III σ-function is a graphical component of the form

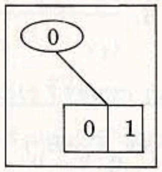
III



The same rules apply to components of this type. We should, of course, realise that all the results that are valid for a given connected component of a graphical representation of a coordinate system combine to give the complete solution to any graph.

To complete the discussion, let us examine one example in detail:

II $(v_q = 0)$ (7.65)



A suitable choice of coordinates is

$$z_1 = x^4 \tfrac{1}{2}(x^1 - x^2) - \frac{\gamma}{4x^4}, \tag{7.66}$$

$$z_2 = x^4 \sqrt{(x^1 x^2)} \cos x^3,$$

$$z_3 = x^4 \sqrt{(x^1 x^2)} \sin x^3,$$

$$z_4 + z_5 = x^5 - \tfrac{1}{2}x^4 \frac{1}{4}(x^1 + x^2)^2 - \frac{\gamma}{4x^4} \tfrac{1}{2}(x^1 - x^2) + \frac{\gamma^2}{96}(x^4)^{-3},$$

$$z_4 - z_5 = 2x^4 .$$

The Hamilton-Jacobi equation in this coordinate system is

$$\frac{1}{(x^4)^2}\left[\frac{1}{(x^1 - x^2)}\left(\left(\frac{\partial W}{\partial x^1}\right)^2 - \left(\frac{\partial W}{\partial x^2}\right)^2\right) + \frac{1}{x^1 x^2}\left(\frac{\partial W}{\partial x^3}\right)^2\right] \tag{7.67}$$
$$+ 2\frac{\partial W}{\partial x^4}\frac{\partial W}{\partial x^5} + \frac{1}{4}(x^1 + x^2)\frac{1}{(x^4)^2}\left(\frac{\partial W}{\partial x^5}\right)^2 = E.$$

The separation equations are

$$\left(\frac{dW_i}{dx^i}\right)^2 - \frac{1}{x^i}\lambda_3^2 + \frac{1}{4}(x^i)^2\varepsilon^2 - \lambda_2 x^i + \lambda_1 = 0, \quad i = 1, 2, \tag{7.68}$$

$$2\varepsilon\frac{dW_4}{dx^4} + \lambda_2/(x^4)^2 = E,$$

$$\frac{dW_3}{dx^3} = \lambda_3, \quad \frac{dW_5}{dx^5} = \varepsilon .$$

The operators that describe this separation are

$$L_1 = B_1^2 + B_2^2 + B_3^2 + \frac{\gamma}{2}(P_4 - P_5)P_1 , \tag{7.69}$$

$$L_2 = \tfrac{1}{2}\{I_{12}, B_2\}_+ - \frac{1}{4}\gamma P_2^2 ,$$

$$L_3 = I_{12}, \quad L_4 = \tfrac{1}{2}(P_4 - P_5) .$$

The corresponding Helmholtz equation in this coordinate system is

$$\frac{1}{(x^4)^2}\left[\frac{1}{(x^1 - x^2)}\left\{\frac{1}{(x^1)^{\frac{1}{2}}}\frac{\partial}{\partial x^1}\left((x^1)^{\frac{1}{2}}\frac{\partial\psi}{\partial x^1}\right) - \frac{1}{(x^2)^{\frac{1}{2}}}\frac{\partial}{\partial x^2}\left((x^2)^{\frac{1}{2}}\frac{\partial\psi}{\partial x^2}\right)\right\}\right.$$
$$\left. + \frac{1}{x^1 x^2}\frac{\partial^2\psi}{(\partial x^3)^2} + \frac{1}{4}(x^1 + x^2)\frac{\partial^2\psi}{(\partial x^5)^2}\right] + 2\frac{\partial^2\psi}{\partial x^4 \partial x^5} = \lambda\psi \tag{7.70}$$

with corresponding separation equations

$$\frac{1}{(x^i)^{\frac{1}{2}}}\frac{d}{dx^i}\left((x^i)^{\frac{1}{2}}\frac{d\psi_i}{dx^i}\right) + \left(-\frac{\lambda_3^2}{x^i} + \frac{1}{4}(x^i)^2\varepsilon^2 - \lambda_2 x^i + \lambda_1\right)\psi_i = 0$$
$$i = 1, 2, \tag{7.71}$$

$$2\varepsilon\frac{d\psi_4}{dx^4} + \left(\frac{\lambda_2}{(x^4)^2} - \lambda\right)\psi_4 = 0,$$

$$\frac{d\psi_3}{dx^3} = \lambda_3\psi_3, \quad \frac{d\psi_5}{dx^5} = \varepsilon\psi_5 .$$

If, instead, we regard this as a separable coordinate system for the corresponding heat equation then the corresponding solutions are R-separable and can be written in the form

$$\psi = e^{R} \prod_{i=1}^{4} \psi_i(x^i)$$

where

$$R = -\frac{1}{8}x^4(x^1 + x^2)^2 - \frac{1}{8}\frac{\gamma}{x^4}(x^1 - x^2) + \frac{\gamma^2}{96(x^4)^3} \qquad (7.72)$$

and the ψ_i satisfy the above separation equations.

8 Other aspects of variable separation

Evidently we have in this book followed a very deliberate path of development. We have basically given a complete solution to Problem I for the Riemannian manifolds S_n, E_n, H_n. In addition, we have made complete statements about Problem I when applied to Laplace's equation on E_n and the heat equation on E_n. A natural extension of these problems is to consider the classifying of all 'inequivalent' separable coordinate systems on the corresponding complex Riemannian manifolds.

(i) $S_{n\mathbb{C}}$, the complex n-sphere consisting of complex $n+1$ vectors $(s_1, \ldots, s_{n+1})$ that satisfy $s_1^2 + \ldots + s_{n+1}^2 = 1$. Clearly, the complexification of H_n is identical with $S_{n\mathbb{C}}$. The infinitesimal distance is $ds^2 = ds_1^2 + \ldots + ds_{n+1}^2$.

(ii) $E_{n\mathbb{C}}$, the complex Euclidean n-space, parametrized by the complex vectors $(z_1, \ldots, z_n)$ with infinitesimal distance $ds^2 = dz_1^2 + \ldots + dz_n^2$.

These problems have not been completely solved and they involve considerably greater complexity than in the real positive definite case. The major reason for this is that in general there are three types of variables involved in a separable coordinate system. In fact, Benenti [17] has proved:

Theorem (Benenti) : For the Hamilton-Jacobi equation

$$\sum_{i,j=1}^{n} g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^i} = E$$

to admit a complete integral obtained via an additive separation of variables $W = \Sigma_{i=1}^{n} W_i(x^i; c_1, \ldots, c_n)$ on a complex Riemannian manifold M the contravariant metric can be taken to be in the form

$$
\begin{array}{c}
\begin{array}{ccc} \longleftarrow n_1 \longrightarrow & \longleftarrow n_2 \longrightarrow & \longleftarrow n_3 \longrightarrow \end{array} \\
(g^{ij}) = \left[\begin{array}{c|c|c} \delta_{ab} H_a^{-2} & 0 & 0 \\ \hline 0 & 0 & g^{r\alpha} \\ \hline 0 & g^{\alpha r} & g^{\alpha\beta} \end{array}\right]
\end{array}
$$

where $H_a^{-2} = S^{a1}/S$, $g^{r\alpha} = A^{r\alpha}(x^r)(S^{r1}/S)$ and $g^{\alpha\beta} = \Sigma_i A^{\alpha\beta}_i (x^i)(S^{i1}/S)$.

Here $\tilde{S} = (S_{ij}(x^i))_{n_1+n_2) \times (n_1+n_2)}$ is a Stäckel matrix with $S = \det \tilde{S}$ and S^{a1} the a1 cofactor of $\tilde{S}$.

The variables x^a are second class coordinates, the variables x^r are called first order coordinates and the variables x^α are first class coordinates. We see that if $n_2 = 0$ then we obtain the form of the contravariant metric given in theorem (3.1). The main difficulty in the complex case is the presence of the first order variables x^r.

The complete solution of the conditions $R_{ijk\ell} = 0$ from the general form of the contravariant metric seems hopelessly difficult. Only recently the complete solution has been obtained for the case $n_2 = 0$ for $S_{n\mathbb{C}}$ and $E_{n\mathbb{C}}$, albeit by rather indirect means [33]. The complete solution of this problem is being actively pursued at present.

Other aspects of variable separation that we have not discussed are the following.

1. <u>The intimate relation between the special functions of mathematical physics and Lie group theory</u> [21], [32]. The Hilbert space structure that underlies, e.g., the Helmholtz equation in E_n leads to identities between the various separated solutions. A good account of this is given by Miller [34].

2. <u>The intrinsic (geometric) characterization of separation of variables.</u> This has been solved completely for Hamilton-Jacobi, Helmholtz and Laplace equations. The theorems that do this characterize the symmetries that describe a given separable coordinate system using algebraic criteria. It is in this sense that the solution is complete. We refer the interested reader to [35], [36] for details.

3. <u>The solution of the linear nonscalar equations of mathematical physics using the techniques of separation of variables</u> [37]. Only scattered results are known for this type of equation [38], [39] and a systematic approach to these equations is currently being pursued. This has been stimulated to some extent by studies of the separability of the wave equations for massless fields of spin $\frac{1}{2}$, 1 and 2 in non-flat Riemannian backgrounds [40]. In addition, the separability of the Dirac equation in a Kerr space-time background has recently been established by Chandrasekhar [41]. For a systematic theory of solutions of these equations obtainable by means of a 'separation of variables' procedure, all these examples need to fit into a suitable mathematical framework.