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Separation of Variables for Riemannian Spaces of Constant Curvature

E G Kalnins
University of Waikato, New Zealand

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Preface

This book arose from the desire to give a compact and, in a somewhat restricted sense, complete treatment of the subject of separation of variables. The only book already available on the specific topic of separation of variables is that of Miller. This earlier work gives an excellent treatment of the relationships between the classical special functions of mathematical physics and Lie group theory.

The aim of the present work is to show how all the actual inequivalent separable coordinate systems can be computed for the Hamilton-Jacobi and Helmholtz equations on real positive definite Riemannian spaces of constant curvature. The results necessary for the solution of this problem are developed in the text. This allows the reader to obtain a feel for the subject without the necessity to read widely in the literature. It is in this spirit that the book has been written. Proofs that are central to the computation of all the inequivalent coordinate systems mentioned above are given in full; the more general results of the theory are often quoted, suitable references being given. We also, on occasion, appeal to the reader's intuition.

In Chapter 1 we give some introductory comments on the subject of separation of variables. Included here are the basic notions of additive and multiplicative separation of variables as well as an intuitive discussion of the basic problems of the associated theory of separation of variables.

Chapter 2 sketches the historical development of the theory of separation of variables, providing a useful summary and extracting, from the many contributions, the most significant results. It also provides an indication of the degrees of freedom available in the specification of a separable coordinate system.

In Chapters 3, 4 and 5 we give a solution of the central problem of this work, that is, we classify the separable coordinate systems on the real n-sphere S_n, on the real Euclidean n-space E_n and on the upper sheet of the double-sheeted hyperboloid H_n for the Hamilton-Jacobi and Helmholtz equations. The interplay between group theory and the constraints of separation of variables theory enables an elegant solution to be obtained. The resulting graphical calculus neatly summarizes the complete solution.

In Chapter 6 these methods are extended to the classification of all inequivalent separable coordinate systems for Laplace's equation and the null Hamilton-Jacobi equation on E_n. In Chapter 7, these ideas are further extended to the classification of all 'R-separable' coordinate systems for the heat equation on E_n.

In Chapter 8 other aspects of the theory of separation of variables are mentioned:

(a) the generalization of the classification of 'inequivalent' coordinate systems to complex Riemannian manifolds;

(b) the relationship between the special functions of mathematical physics and Lie group theory;

(c) the intrinsic characterization of separation of variables;

(d) the development of a mathematical theory for separation of variable techniques applied to the nonscalar valued equations of mathematical physics (e.g. Dirac equation, Maxwell's equations).

Much of this work is a consequence of a long-standing collaboration with my colleague Willard Miller Jr. Indeed chapters 3, 4, 5 and 6 are based on the following research reports co-authored with W. Miller Jr:

Separation of variables on n dimensional manifolds

1. The n sphere S_n and Euclidean n space R_n
2. The n dimensional hyperboloid H_n
3. Conformally Euclidean spaces

The first of these is to be published in the Journal of Mathematical Physics.

I would also like to acknowledge the influence and collaboration of Charles Boyer, Greg Reid and Pavel Winternitz. Finally, I thank my wife for her persistence in urging me to write this book.

Hamilton, New Zealand

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E. G. K.

1 Introduction

The method of separation of variables has its roots in the solution of many of the problems of classical physics [1]. In particular, we focus in this book on the solution of the Hamilton-Jacobi equation

$$H(p_1, \ldots, p_n; x^1, \ldots, x^n) = E, \quad p_i = \frac{\partial W}{\partial x^i}, \quad i=1,\ldots,n. \qquad (1.1)$$

by means of this technique. As is known from standard texts in classical mechanics [2], once a complete integral has been obtained the solution of the corresponding mechanical system can be achieved. A complete integral of (1.1) is any solution $W = W(x^1, \ldots, x^n; c_1, \ldots, c_n)$ such that $\Delta = \det(\partial^2 W/\partial x^i \partial c_j)_{n\times n} \neq 0$.

Most examples of the solution of the Hamilton-Jacobi equation are obtained by looking for solutions W which are additively separable, i.e. the solution has the form

$$W = \sum_{i=1}^{n} W_i(x^i; c_1, \ldots, c_n). \qquad (1.2)$$

This is the separation ansatz for an additive separation of variables. If H is a quadratic form in the canonical momenta p_i such that

$$H = \sum_{i,j=1}^{n} g^{ij} p_i p_j + V(x) = E, \qquad g^{ij} = g^{ji}; \; i,j=1,\ldots,n. \qquad (1.3)$$

then associated with this Hamiltonian is the Riemannian manifold having contravariant metric g^{ij}. As an example of additive separation of variables, consider the two-dimensional harmonic oscillator. The Hamiltonian is

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + \tfrac{1}{2}m(\omega_1^2 q_1^2 + \omega_2^2 q_2^2)$$

where $p_i = m\dot{q}_i$ $(i = 1, 2)$ are the momenta conjugate to the coordinates q_i. The Hamilton-Jacobi equation is

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q_1}\right)^2 + \left(\frac{\partial W}{\partial q_2}\right)^2\right] + \tfrac{1}{2}m(\omega_1^2 q_1^2 + \omega_2^2 q_2^2) = E \qquad (1.5)$$

Now, putting $W = W_1(q_1) + W_2(q_2)$, the separation equations are

$$\frac{1}{2m}\left(\frac{\partial W_1}{\partial q_1}\right)^2 + \tfrac{1}{2}m\omega_1^2 q_1^2 = \alpha_1 , \qquad (1.6)$$

$$\frac{1}{2m}\left(\frac{\partial W_2}{\partial q_2}\right)^2 + \tfrac{1}{2}m\omega_2^2 q_2^2 = E - \alpha_1 .$$

The corresponding solution is

$$W = \tfrac{1}{2}\sqrt{(2m\alpha_1 - m^2\omega_1^2 q_1^2)} + \frac{\alpha_1}{\omega_1}\sin^{-1}\left(\sqrt{\left(\frac{m}{2\alpha_1}\right)}\,\omega_1 q_1\right) \qquad (1.7)$$

$$+ \tfrac{1}{2}\sqrt{(2m(E-\alpha_1) - m^2\omega_2^2 q_2^2)} + \frac{(E-\alpha_2)}{\omega_2}\sin^{-1}\left(\sqrt{\left(\frac{m}{2(E-\alpha_1)}\right)}\,\omega_2 q_2\right).$$

The solution to the dynamical system is then obtained by solving

$$\beta_1 = \frac{\partial W}{\partial \alpha_1}, \quad t - t_0 = \frac{\partial W}{\partial E} \qquad (1.8)$$

or equivalently

$$\beta_1 = \frac{1}{\omega_1}\sin^{-1}\left(\sqrt{\left(\frac{m}{2\alpha_1}\right)}\,\omega_1 q_1\right) - \frac{1}{\omega_2}\sin^{-1}\left(\sqrt{\left(\frac{m}{2(E-\alpha_1)}\right)}\,\omega_2 q_2\right) \qquad (1.9)$$

$$t - t_0 = \frac{1}{\omega_2}\sin^{-1}\left(\sqrt{\left(\frac{m}{2(E-\alpha_1)}\right)}\,\omega_2 q_2\right)$$

which yields the solution

$$q_1 = \frac{1}{\omega_1}\sqrt{\left(\frac{\alpha_1}{m}\right)}\sin\left(\omega_1(t-t_0) + \frac{\beta_1\omega_1}{m}\right), \qquad (1.10)$$

$$q_2 = \frac{1}{\omega_2}\sqrt{\left(\frac{2(E-\alpha_1)}{m}\right)}\sin(\omega_2(t-t_0)) .$$

For a satisfactory theory of separation of variables of (1.1), three important problems are fundamental:

(I) Given a Riemannian manifold (e.g. Euclidean three space), how many 'inequivalent' coordinate systems does it permit which give a complete integral of (1.1) having the form (1.2)?

(II) How is it possible to characterize intrinsically (i.e. in a coordinate-free geometric way) the occurrence of additive separation of variables, given a Riemannian manifold M?

(III) What are the 'inequivalent' types of additive separation of variables that can occur on a Riemannian manifold of dimension n?

The main purpose of this book is to present a complete solution, for a class of Riemannian manifolds, to problem (I). This class of manifolds consists of the real positive definite Riemannian manifolds of constant curvature. These manifolds are most easily thought of as the n-dimensional real sphere S_n, real Euclidean n-space E_n and the upper sheet of the double sheeted n-dimensional hyperboloid H_n. More specifically, these manifolds can be defined as follows:

(a) S_n: the set of real vectors $(s_1, \ldots, s_{n+1})$ which satisfy $s_1^2 + \ldots + s_{n+1}^2 = 1$ and have infinitesimal distance $ds^2 = ds_1^2 + \ldots + ds_{n+1}^2$.

(b) E_n: the set of real vectors $(z_1, \ldots, z_n)$ with infinitesimal distance $ds^2 = dz_1^2 + \ldots + dz_n^2$.

(c) H_n: the set of real vectors $(v_0, v_1, \ldots, v_n)$ which satisfy $v_0^2 - v_1^2 - \ldots - v_n^2 = 1$, $v_0 > 1$, and have infinitesimal distance $ds^2 = dv_0^2 - dv_1^2 - \ldots - dv_n^2$.

In restricting ourselves to this problem we can give a complete treatment of a well defined mathematical problem for a class of manifolds that exhibit a lot of structure. It also serves as an introduction to the essential ingredients required in setting up a theory of separation of variables and, more specifically, in the solution of problem (I) in general.

In addition to the notion of additive separation, there is also the notion of product separation. This occurs for a Riemannian manifold when one is looking for solutions of the Helmholtz equation with a potential V:

$$\Delta_n\psi + V\psi = \sum_{i,j=1}^{n} \frac{1}{\sqrt{(g)}}\frac{1}{\partial x^i}\left(\sqrt{(g)}\,g^{ij}\frac{\partial\psi}{\partial x^j}\right) + V\psi = \lambda\psi ,$$

$$g = \det(g_{ij})_{n\times n} \qquad (1.11)$$

of the form $\psi = \Pi_{i=1}^{n} \psi_i(x^i; c_1, \ldots, c_n)$.

As an example of product separation, consider the two-dimensional (quantum mechanical) harmonic oscillator. The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right] + \frac{m}{2}\left[\omega_1^2 x^2 + \omega_2^2 y^2\right]\psi = E\psi . \qquad (1.12)$$

Putting $\psi = \psi_1(x)\psi_2(y)$, the functions ψ_i $(i = 1, 2)$ satisfy the separation equations

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_1}{\partial x^2} + (\tfrac{1}{2}m\omega_1^2 x^2 - \lambda_1)\psi_1 = 0, \qquad (1.13)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_2}{\partial y^2} + (\tfrac{1}{2}m\omega_2^2 y^2 - \lambda_2)\psi_2 = 0,$$

where $\lambda_1 + \lambda_2 = E$. A normalized set of solutions of these equations can be expressed in terms of the Hermite polynomials

$$\psi = N\exp\left[-\frac{m}{2\hbar}(\omega_1 x^2 + \omega_2 y^2)\right] H_{n_1}\left(\sqrt{\left(\frac{m\omega_1}{\hbar}\right)}x\right) H_{n_2}\left(\sqrt{\left(\frac{m\omega_2}{\hbar}\right)}y\right) \qquad (1.14)$$

where

$$N = \frac{2m}{h}\left[\frac{\omega_1\omega_2}{2^{n_1+n_2} n_1! n_2!}\right]^{\frac{1}{2}}$$

and $\lambda_i = \omega_i \hbar(n_i + \frac{1}{2})$, $i = 1, 2$. The energy E is thus quantized according to

$$E = \hbar\left[\omega_1(n_1 + \tfrac{1}{2}) + \omega_2(n_2 + \tfrac{1}{2})\right] \qquad (1.15)$$

for suitable integers n_1, n_2. The crucial observation, of course, is that (1.12) admits a solution via the product separation of variables ansatz $\psi = \psi_1(x)\psi_2(y)$.

The same three problems apply to product separation, i.e.

(I') Given a Riemannian manifold, how many 'inequivalent' coordinate systems does (1.11), with $V = 0$, permit that provide a solution by means of the separation of variables ansatz $\psi = \Pi_{i=1}^{n} \psi_i(x^i; c_1, \ldots, c_n)$?

(II') How is it possible to characterize intrinsically the occurrence of product separation, given a Riemannian manifold M?

(III') What are the 'inequivalent' types of product separation of variables that can occur on a Riemannian manifold of dimension n?