(5) **1.** What is $\lim_{x\to\infty} \dfrac{\sqrt{\pi}}{1+\pi e^{-2x}}$?

(a) 0
(b) $\dfrac{\sqrt{\pi}}{1+\pi}$
(c) $\dfrac{1}{\sqrt{\pi}}$
(d) ∞
(e) None of the above.

The answer is (e). $\lim_{x\to\infty} \dfrac{\sqrt{\pi}}{1+\pi e^{-2x}} = \dfrac{\sqrt{\pi}}{1+\pi e^{-2\cdot 0}} = \sqrt{\pi}$.

(5) **2.** What is $\lim_{x\to 0} \dfrac{\sin(2x)}{4x}$?

(a) 0
(b) $\frac{1}{4}$
(c) $\frac{1}{2}$
(d) ∞
(e) None of the above.

The answer is (c). $\lim_{x\to 0} \dfrac{\sin(2x)}{4x} = \lim_{x\to 0} \dfrac{\frac{d}{dx}\sin(2x)}{\frac{d}{dx}4x} = \lim_{x\to 0} \dfrac{2\cos 2x}{4} = \dfrac{2}{4} = \dfrac{1}{2}$.

(5) **3.** For which value of a is

$$f(x) = \begin{cases} 2x^2 & \text{for } x \le 1 \\ 3x+a & \text{for } x > 1 \end{cases}$$

continuous for all real x?

(a) -1
(b) 0
(c) 1
(d) $\frac{3}{2}$
(e) None of the above.

The answer is (a). $f(1) = 2$. $\lim_{x\to 1+} f(x) = 3 + a$. Therefore, $2 = 3 + a$, or $a = -1$.

(5) **4.** Suppose that the function $f(x)$ is differentiable and that $f(2)=1$, $f'(2)=\frac{1}{3}$, $f'(1)=5$. Let $g(x)=f^{-1}(x)$. What is $g'(2)$?

(a) $\frac{1}{5}$
(b) $\frac{1}{3}$
(c) 3
(d) 5
(e) Cannot be determined from the information given.

The answer is (e). $g'(2)=\dfrac{1}{f'(g(2))}$. Since $g(2)$ is not given, the answer cannot be determined.

(5) **5.** Suppose that the function $f(x)$ is differentiable and that $f(2)=1$, $f'(2)=\frac{1}{3}$, $f'(1)=5$. Let $g(x)=f^{-1}(x)$. What is $g'(1)$?

(a) $\frac{1}{5}$
(b) $\frac{1}{3}$
(c) 3
(d) 5
(e) Cannot be determined from the information given.

The answer is (c). $g'(1)=\dfrac{1}{f'(g(1))}=\dfrac{1}{f'(2)}=\dfrac{1}{\frac{1}{3}}=3$.

(5) **6.** On which of the following plots would a graph of the function $f(x)=3x^{6/5}$ (vertical axis) as a function of x (horizontal axis) appear as a straight line?

(a) linear scale on both the horizontal and vertical axes.
(b) logarithmic scale on both the horizontal and vertical axes.
(c) linear scale on the horizontal and logarithmic scale on the vertical axes.
(d) logarithmic scale on the horizontal and linear scale on the vertical axes.
(e) None of the above.

The answer is (b). $\log f(x)=\log(3x^{6/5})=\log 3+\dfrac{6}{5}\log x$, so $\log f(x)$ is a linear function of $\log x$.

(5) **7.** Denote the leaf size y of a plant by $y = f(x)$, where x is the age of the leaf. Assume that

$$f(x) = 4x^{5/2} \quad \text{for} \quad x \geq 0.$$

If you measure x with an accuracy of 5%, and if you compute y using the formula for $f(x)$, what is the percentage error of your computed y?

(a) $2\,\%$
(b) $4\,\%$
(c) $5\,\%$
(d) $12\frac{1}{2}\,\%$
(e) None of the above.

The answer is (d). Let Δx be the measurement error for the age x, and let Δy be the measurement error for the size y. Linear approximation yields

$$\Delta y \approx f'(x)\Delta x = 10x^{3/2}\Delta x.$$

Therefore, the percentage error $100\dfrac{\Delta y}{y}$ is related to the percentage error $100\dfrac{\Delta x}{x}$ by

$$100\frac{\Delta y}{y} \approx 100\frac{10x^{3/2}\Delta x}{y} = 100\frac{10x^{3/2}\Delta x}{4x^{5/2}} = 100\cdot\frac{5}{2}\cdot\frac{\Delta x}{x} = \frac{5}{2}\cdot\left(100\frac{\Delta x}{x}\right)$$

Measuring x with an accuracy of 5% means that $100\dfrac{\Delta x}{x} = 5$, which means that the percentage error of the computed y is $\dfrac{5}{2}\cdot 5 = 12.5$.

(5) **8.** If $\int_1^3 f(x)dx = 3$ and $\int_2^3 f(x)dx = 1$, what is $\int_1^2 f(x)dx$?

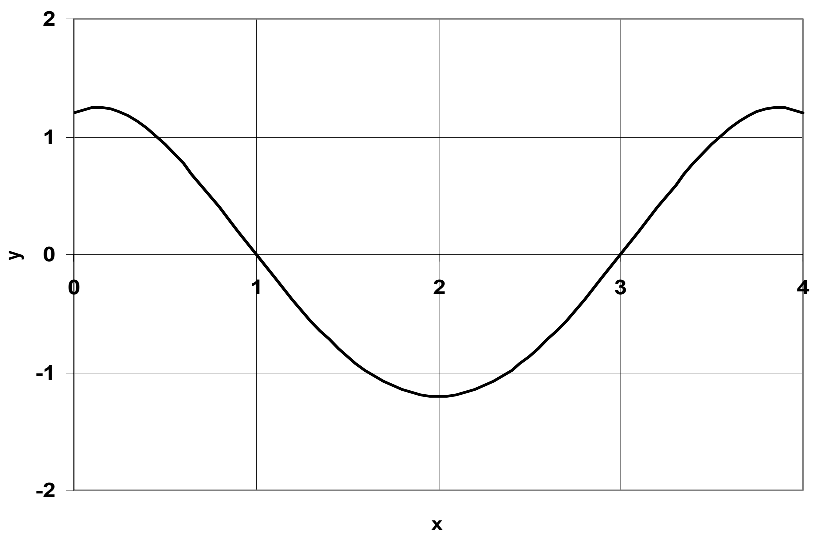
(a) -2
(b) -1
(c) 1
(d) 2
(e) None of the above.

The answer is (d).

$$\int_1^3 f(x)dx = \int_1^2 f(x)dx + \int_2^3 f(x)dx$$
$$\int_1^2 f(x)dx = \int_1^3 f(x)dx - \int_2^3 f(x)dx = 3 - 1 = 2$$

Problems 9 and 10 refer to the graph below, which is the graph of the ***derivative*** $f'(x)$ of a function $f(x)$.

Graph of $y = f'(x)$



(Yes, the graph shows $f'(x)$ (the derivative of $f(x)$), but the questions are about the function $f(x)$).

(5) **9.** At $x = 2$, the graph of $y = f(x)$ has a

(a) zero.
(b) local minimum.
(c) local maximum.
(d) inflection point.
(e) None of the above.

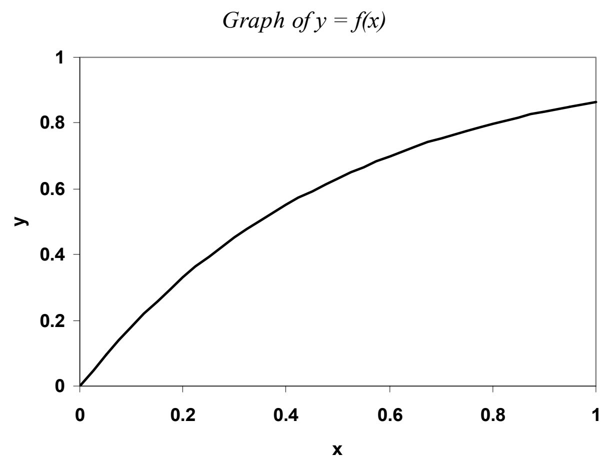
The answer is (d). The derivative changes at $x = 2$ from decreasing to increasing, so the function changes from concave down to concave up, which means that $x = 2$ is an inflection point

(5) **10.** At $x = 3$, the graph of $y = f(x)$ has a

(a) zero.
(b) local minimum.
(c) local maximum.
(d) inflection point.
(e) None of the above.

The answer is (b). At $x = 3$, the derivative changes from negative to positive, so the function changes from decreasing to increasing, which make $x = 3$ a local minimum.

(5) 11. Below is a graph of the function $y = f(x)$.



Which of the following statements is correct?

(a) $f'(x) > 0$ and $f''(x) > 0$.
(b) $f'(x) > 0$ and $f''(x) < 0$.
(c) $f'(x) < 0$ and $f''(x) > 0$.
(d) $f'(x) < 0$ and $f''(x) < 0$.
(e) None of the above.

The answer is (b). The function is increasing and concave down, so the derivative is positive and the second derivative is negative.

(5) 12. If the size of a population at time t is denoted by $N(t)$ and if the growth rate of the population is given by

$$g(N) = N(N-1)(3-N),$$

which of the following differential equations describes the per capita growth rate of this population?

(a) $\frac{1}{N}\frac{dN}{dt} = (N-1)(3-N)$
(b) $\frac{1}{N}\frac{dN}{dt} = g'(N)$
(c) $\frac{dN}{dt} = (N-1)(3-N)$
(d) $\frac{dN}{dt} = \log(g(N))$
(e) None of the above.

The answer is (a). The per capita growth rate is $\frac{1}{N}\frac{dN}{dt}$, which is given by the expression (a).

(20) 13. Compute the following limits. Give an exact answer and show your work.

(10) a. $\lim_{x\to 0}\dfrac{\sin x}{x+x^2}$

$$\lim_{x\to 0}\frac{\sin x}{x+x^2}=\lim_{x\to 0}\frac{\frac{d}{dx}\sin x}{\frac{d}{dx}(x+x^2)}=\lim_{x\to 0}\frac{\cos x}{1+2x}=\frac{1}{1+2\cdot 0}=1$$

(10) b. $\lim_{x\to\infty} xe^{-3x}$

$$\lim_{x\to\infty} xe^{-3x}=\lim_{x\to\infty}\frac{x}{e^{3x}}=\lim_{x\to\infty}\frac{\frac{d}{dx}x}{\frac{d}{dx}e^{3x}}=\lim_{x\to\infty}\frac{1}{3e^{3x}}=0$$

(20) 14. For each function, compute the derivative. Show your work.

(10) a. $f(x)=x^5+\dfrac{1}{x^2}-\sqrt[3]{x}$

$$f(x)=x^5+x^{-2}-x^{1/3}$$
$$f'(x)=5x^4-2x^{-3}-\tfrac{1}{3}x^{-2/3}$$
$$=5x^4-\frac{2}{x^3}-\frac{1}{3\sqrt[3]{x^2}}$$

(10) b. $f(x)=x^{-3}e^{2x}$

$$f'(x)=e^{2x}\frac{d\,x^{-3}}{dx}+x^{-3}\frac{d\,e^{2x}}{dx}=e^{2x}\left(-3x^{-4}\right)+x^{-3}\left(2e^{2x}\right)=e^{2x}\left(\frac{-3}{x^4}+\frac{2}{x^3}\right)$$
$$=\frac{e^{2x}(2x-3)}{x^4}$$

(30) 15. For each function, compute the derivative. Show your work.

(15) a. $f(x)=\dfrac{3x^2}{7+5x^2}$

$$f'(x)=\frac{(7+5x^2)\frac{d}{dx}3x^2-3x^2\frac{d}{dx}(7+5x^2)}{(7+5x^2)^2}$$
$$=\frac{(7+5x^2)(6x)-3x^2(10x)}{(7+5x^2)^2}=\frac{42x+30x^3-30x^3}{(7+5x^2)^2}$$
$$=\frac{42x}{(7+5x^2)^2}$$

(15) b. $f(x)=x^5+5^x$

$$f(x)=x^5+e^{x\ln 5}$$
$$f'(x)=5x^4+(\ln 5)e^{x\ln 5}=5x^4+(\ln 5)5^x$$

(30) 16. For each function, compute the derivative. Show your work.

(15) a. $f(x)=\exp(\tan(x^2+1))$

$$f'(x)=\frac{d}{dx}\left(\exp(\tan(x^2+1))\right)=\exp(\tan(x^2+1))\frac{d}{dx}(\tan(x^2+1))$$
$$=\exp(\tan(x^2+1))\sec^2(x^2+1)\frac{d}{dx}(x^2+1)$$
$$=2x\exp(\tan(x^2+1))\sec^2(x^2+1)$$

(15) b. $f(x)=\ln(\cos^2 x)$

$$f'(x)=\frac{d}{dx}\ln(\cos^2 x)=\frac{1}{\cos^2 x}\frac{d}{dx}\cos^2 x=\frac{1}{\cos^2 x}\left(2\cos x\frac{d}{dx}\cos x\right)$$
$$=\frac{2\cos x(-\sin x)}{\cos^2 x}=-2\frac{\sin x}{\cos x}$$
$$=-2\tan x$$

(10) 17. Find the derivative of

$$f(x) = g(-\ln x),$$

where $g(x)$ is a differentiable function.

$$f'(x) = \frac{d}{dx} g(-\ln x) = g'(-\ln x) \frac{d}{dx}(-\ln x) = g'(-\ln x)(-\frac{1}{x})$$
$$= -\frac{g'(-\ln x)}{x}$$

(15) 18. Find $\frac{dy}{dx}$ when

$$y = \ln(x + y^2)$$

Write your answer as $\frac{dy}{dx} = \cdots$, and simplify.

$$\frac{d}{dx} y = \frac{d}{dx} \ln(x + y^2)$$
$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x + y^2)}{x + y^2} = \frac{1 + \frac{d}{dx} y^2}{x + y^2} = \frac{1 + 2y\frac{dy}{dx}}{x + y^2}$$
$$(x + y^2)\frac{dy}{dx} = 1 + 2y\frac{dy}{dx}$$
$$(x + y^2 - 2y)\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{x + y^2 - 2y}$$

(15) 19. Consider the function

$$f(x) = (1+2x)^k$$

where k is a positive integer.

(10) a. Find the linear approximation of $f(x)$ near $x = 0$.

The linear approximation of $f(x)$ near $x = 0$ is given by $f(x) \approx f(0) + f'(0)(x-0)$. We compute: $f'(x) = k(1+2k)^{k-1} \cdot 2$, $f'(0) = 2k$, and $f(0) = 1$. Therefore, the linear approximation is

$$f(x) \approx 1 + 2kx$$

(5) b. Use your answer in part (a) to estimate $(1.002)^{50}$. Show your work and make sure that we can recognize that you used part (a).

In this case, $x = 0.001$ and $k = 50$. Therefore,

$$(1.002)^{50} \approx 1 + 2 \cdot 50 \cdot 0.001 = 1.1$$

(15) 20. Find all the asymptotes of

$$y = \frac{2x+1}{x+1}.$$

For each asymptote, state whether it is horizontal, vertical, or oblique, and justify your answer.

The denominator is zero at $x = -1$ while the numerator is 9, so the graph has a **vertical asymptote** at $x = -1$.

Now use long division to derive

$$y = \frac{2x+1}{x+1} = 2 - \frac{1}{x+1}$$

So, as $x \to \infty$, y approaches 2, *i.e.*, the graph has a **horizontal asymptote** at $y = 2$.

(20) 21. Let

$$f(x) = 7 + 5x - x^3$$

and note that $f'(x) = 5 - 3x^2$, and that $f''(x) = -6x$.

(10) a. Determine where $f(x)$ is increasing and where it is decreasing. Find all local extrema and identify whether they are local maxima or local minima.

The function $f(x)$ is increasing where $f'(x)$ is positive and decreasing where $f'(x)$ is negative. We compute:

$$f'(x) = 5 - 3x^2 > 0$$
$$3x^2 < 5$$
$$|x| < \sqrt{\tfrac{5}{3}}$$

Therefore, $f(x)$ is **increasing where** $|x| < \sqrt{\tfrac{5}{3}}$ and **decreasing where** $|x| > \sqrt{\tfrac{5}{3}}$.

The **local extrema** occur where $f'(x) = 0$, i.e., where $x = \pm\sqrt{\tfrac{5}{3}}$.

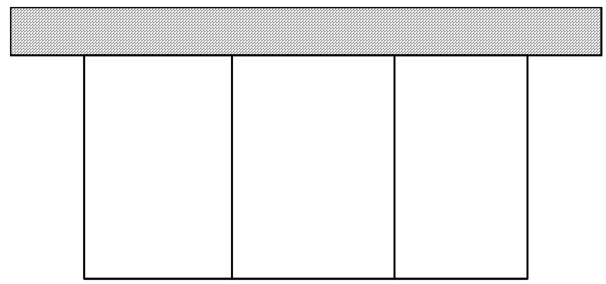
We compute that $f''(\sqrt{\tfrac{5}{3}}) = -6\sqrt{\tfrac{5}{3}} < 0$ and that $f''(-\sqrt{\tfrac{5}{3}}) = -6(-\sqrt{\tfrac{5}{3}}) > 0$, so $x = +\sqrt{\tfrac{5}{3}}$ **is a local maximum** and $x = -\sqrt{\tfrac{5}{3}}$ **is a local minimum**.

(10) b. Determine where $f(x)$ is concave up and where it is concave down. Find all inflection points.

The function $f(x)$ is concave up where $f''(x)$ is positive and concave down where $f''(x)$ is negative. Since $f''(x) = -6x$, it is positive for negative x and negative for positive x. Therefore, $f(x)$ **is concave up for** $x < 0$ and **concave down for** $x > 0$.

An inflection point occurs where the concavity changes, so $f(x)$ **has an inflection point at** $x = 0$

(15) 22. Sven and Ole want to fence in a rectangular plot of 1600 square feet against a solid wall. They also want to use additional fencing to build two internal dividers parallel to two outer boundary sections, as shown in the figure below. What is the minimum total length of fencing that this project will require? Verify that your answer yields the global minimum.



Let y be the length of the side parallel to the wall, and let x be the length of the sides perpendicular to the wall. The total area is $xy = 1600$, so we have

$$y = \frac{1600}{x}$$

The total length of fence is given by

$$p = 4x + y = 4x + \frac{1600}{x}.$$

Since the derivative $\frac{dp}{dx}$ is defined for all $x > 0$, the global minimum can occur only at the endpoints of the interval or at internal points where the derivative is zero. The interval of possible values of x is $(0, \infty)$, and $p \to \infty$ as $x \to 0$ or as $x \to \infty$. Therefore, the global minimum must occur at a point where the derivative is zero. We compute

$$\frac{dp}{dx} = \frac{d}{dx}\left(4x + \frac{1600}{x}\right) = 4 - \frac{1600}{x^2} = 0$$

Therefore, $x^2 = 400$, so $x = 20$, so the minimum total amount of fence is

$$p = 4 \cdot 20 + \frac{1600}{20} = 160 \text{ linear feet.}$$

(10) 23. The exponential growth function is given by

$$N(t) = N_0 e^{rt} \qquad \text{for} \quad t \geq 0,$$

where $N(t)$ denotes the amount of material left at time t and r is the growth rate ($r > 0$). Show that $N(t)$ satisfies the differential equation

$$\frac{dN}{dt} = rN.$$

$$\frac{dN}{dt} = \frac{d}{dt} N_0 e^{rt} = N_0 \frac{d}{dt} e^{rt} = N_0 r e^{rt} = r N_0 e^{rt} = rN$$

(20) 24. Solve the initial value problem

$$\frac{dy}{dx} = e^{-0.5x} \qquad \text{with} \qquad y(0) = 5.$$

Integrating the differential equation, we get

$$y = \int e^{-0.5x} dx = \frac{e^{-0.5x}}{-0.5} + C = -2e^{-0.5x} + C \;\; 5 = -2e^0 + C$$

Using the condition that $y = 5$ when $x = 0$, we have $5 = -2e^0 + C$, or $C = 7$.

Therefore, the solution is

$$y = -2e^{-0.5x} + 7.$$

(10) 25. Suppose that $N(t)$ is the population at time t. Assume that

$$\frac{dN}{dt} = 4N\left(1 - \frac{N}{10}\right) \qquad \text{for} \qquad t \geq 0,$$

and that $N(0) = 5$. Use a linear approximation to estimate $N(0.1)$.

Linear approximation applied to $N(t)$ yields

$$N(t) \approx N(0) + N'(0)(t - 0).$$

Since $N(0) = 5$, we have that $N'(0) = 4N(0)\left(1 - \frac{N(0)}{10}\right) = 4 \cdot 5 \cdot \left(1 - \frac{5}{10}\right) = 10$. Therefore,

$$N(t) \approx 5 + 10t$$

so

$$N(0.1) \approx 6$$

(10) 26. Evaluate $\int \sin(\frac{\pi}{4}x)dx$.

$$\int \sin(\frac{\pi}{4}x)dx = \frac{4}{\pi}\int \sin(\frac{\pi}{4}x)(\frac{\pi}{4}dx) = \frac{4}{\pi}(-\cos(\frac{\pi}{4}x)) + C$$
$$= -\frac{4}{\pi}\cos(\frac{\pi}{4}x) + C$$

(20) 27. Use a formula from geometry to find the **exact value** of the definite integral

$$\int_{-3}^{3}(\sqrt{9-x^2}-3)\,dx .$$

First separate the problem into two integrals as follows:

$$\int_{-3}^{3}(\sqrt{9-x^2}-3)\,dx = \int_{-3}^{3}\sqrt{9-x^2}\,dx - \int_{-3}^{3}3\,dx$$

The first integral is the area under the curve $y=\sqrt{9-x^2}$ from $x=-3$ to $x=3$, or the upper half of the circle $x^2+y^2=3^2$. This area is just half of the area of the circle of radius 3, or $\frac{9}{2}\pi$.

The second integral is the area of the rectangle with base 6 and height 3, or 18. Therefore,

$$\int_{-3}^{3}(\sqrt{9-x^2}-3)\,dx = \tfrac{9}{2}\pi - 18$$

(15) 28. Find the **exact value** of the definite integral $\int_0^4 e^{-4x}dx$. Show your work.

$$\int_0^4 e^{-4x}dx = \left.\frac{e^{-4x}}{-4}\right|_0^4 = \frac{e^{-16}}{-4} - \frac{e^0}{-4}$$
$$= \frac{1-e^{-16}}{4}$$

(15) 29. Find the function given by $f(x) = \frac{d}{dx}\int_1^{x^2}(1+u^2)du$. Simplify your answer.

$$f(x) = \frac{d}{dx}\int_1^{x^2}(1+u^2)du = \left(1+(x^2)^2\right)\frac{d}{dx}x^2$$
$$= 2x(1+x^4)$$

(15) 30. Find the **exact value** of the definite integral $\int_0^{\pi/8}\sec^2(2x)dx$. Show your work.

$$\int_0^{\pi/8}\sec^2(2x)dx = \left.\frac{\tan 2x}{2}\right|_0^{\pi/8} = \frac{\tan\left(2\cdot\frac{\pi}{8}\right)}{2} - \frac{\tan 0}{2} = \tfrac{1}{2}\tan\left(\frac{\pi}{4}\right)$$
$$= \tfrac{1}{2}$$

(20) 31. Evaluate the following indefinite integrals.

(10) a. $\int \frac{5}{x+4}dx$.

$$\int \frac{5}{x+4}dx = 5\int \frac{dx}{x+4}$$
$$= 5\ln|x+4| + C$$

(10) b. $\int 4xe^{x^2}dx$.

$$\int 4xe^{x^2}dx = 2\int e^{x^2}2xdx$$
$$= 2e^{x^2} + C$$

(15) 32. Set up but do not evaluate the integral for finding the area of the region bounded by curve $xy = 1$ and the line $x + 2y = 3$.

We first find the points of intersection of the curve $y = \frac{1}{x}$ and the line $y = -\frac{1}{2}x + \frac{3}{2}$ by setting $\frac{1}{x} = -\frac{1}{2}x + \frac{3}{2}$ and solving for x.

$$1 = -\tfrac{1}{2}x^2 + \tfrac{3}{2}x$$

$$-\tfrac{1}{2}x^2 + \tfrac{3}{2}x - 1 = 0$$

$$x^2 - 3x + 2 = 0$$

$$(x-1)(x-2) = 0$$

The points of intersection occur at $x = 1$ and $x = 2$.

The line $y = -\frac{1}{2}x + \frac{3}{2}$ is *above* the curve $y = \frac{1}{x}$ on the interval $1 \le x \le 2$. We can verify this by checking any point in the interior of the interval, *e.g.*, $x = \frac{3}{2}$. At $x = \frac{3}{2}$, the value of y on the line is $y = -\frac{1}{2} \cdot \frac{3}{2} + \frac{3}{2} = \frac{3}{4}$, while the value of y on the curve is $y = \frac{1}{\frac{3}{2}} = \frac{2}{3}$.

The integral for the area between the line and the curve is therefore

$$\int_1^2 \left(-\frac{x}{2} + \frac{3}{2} - \frac{1}{x} \right) dx$$