CHAPTER 3

Lie Theory and Bessel Functions

The machinery constructed in Chapters 1 and 2 will here be applied to the study of Bessel functions. As was demonstrated in Section 2-7 these functions are related to the representation theory of $\mathscr{G}(0, 0)$. The Bessel functions appear in two distinct ways: as matrix elements of local irreducible representations of $G(0, 0)$ and as basis functions for irreducible representations of $\mathscr{G}(0, 0)$. The first relationship will yield addition theorems, and the second will yield generating functions and recursion relations for Bessel functions.

Since $\mathscr{G}(0, 0) \cong \mathscr{T}_3 \oplus (\mathscr{E})$, the nontrivial part of the representation theory of $\mathscr{G}(0, 0)$ is concerned with the subalgebra $\mathscr{T}_3$. For a theory of Bessel functions, it is sufficient to study the representation theory of $\mathscr{T}_3$ and the local Lie group T_3. Thus, we restrict ourselves to $\mathscr{T}_3$ throughout this chapter.

The Euclidean group in the plane E_3 is a real 3-parameter global Lie group whose Lie algebra is a real form of $\mathscr{T}_3$. The faithful irreducible unitary representations of E_3 are well known as is the fact that with respect to a suitable basis, the matrix elements of these representations are proportional to Bessel functions of integral order (Vilenkin [1], Wigner [2]). In Section 3-4 we will study the relationship between local representations of T_3 and unitary representations of E_3.

The computations involved in this chapter are rather simple and will serve as an introduction to the much more complicated theory of hypergeometric and confluent hypergeometric functions.

3-1 The Representations $Q(\omega, m_o)$

Each irreducible representation of $\mathscr{G}(0, 0)$ classified in Theorem 2.1 is designated by the symbol $Q^\mu(\omega, m_o)$ where μ, ω, m_o are complex constants such that $\omega \neq 0$ and $0 \leqslant \operatorname{Re} m_o < 1$. The spectrum S of this representation is the set $\{m_o + n\colon n \text{ an integer}\}$ and the representation space V has a basis $\{f_m\colon m \in S\}$, such that

$$\begin{aligned} &J^3 f_m = m f_m\,, \qquad E f_m = \mu f_m\,, \qquad J^+ f_m = \omega f_{m+1}\,, \\ &J^- f_m = \omega f_{m-1}\,, \qquad C_{0,0} f_m = (J^+ J^-) f_m = \omega^2 f_m\,. \end{aligned} \tag{3.1}$$

The operators J^+, J^-, J^3, E satisfy the commutation relations

$$[J^3, J^\pm] = \pm J^\pm, \qquad [J^+, J^-] = 0, \qquad [J^\pm, E] = [J^3, E] = 0.$$

We will try to find a realization of the algebraic representation $Q^\mu(\omega, m_o)$ in terms of linear differential operators acting on a vector space of functions of one complex variable. In particular we will realize $Q^\mu(\omega, m_o)$ in such a way that the differential operators take the form (2.38). This can be accomplished as follows: Let $\mathscr{V}_1$ be the complex vector space consisting of all finite linear combinations of the functions $h_n(z) = z^n$, $n = 0, \pm 1, \pm 2, \ldots$. In Eqs. (2.38) set $\lambda = m_o$, $c_1 = \omega$, i.e.,

$$J^3 = m_0 + z\frac{d}{dz}, \qquad E = \mu, \qquad J^+ = \omega z, \qquad J^- = \frac{\omega}{z}. \tag{3.2}$$

Define the basis vectors f_m of $\mathscr{V}_1$ by $f_m(z) = h_n(z)$ where $m = m_o + n$ and n runs over the integers. Then,

$$\begin{aligned} J^3 f_m &= \left(m_o + z\frac{d}{dz}\right) z^n = (m_o + n) z^n = m f_m\,, \\ J^+ f_m &= (\omega z) z^n = \omega z^{n+1} = \omega f_{m+1}\,, \\ J^- f_m &= (\omega/z) z^n = \omega z^{n-1} = \omega f_{m-1}\,, \\ E f_m &= \mu f_m \end{aligned} \tag{3.3}$$

for all $m \in S$. These equations coincide with (3.1) and yield a realization of $Q^\mu(\omega, m_o)$.

Now that we have realized the abstract representation $Q^\mu(\omega, m_o)$ in terms of differential operators acting on analytic functions, we can apply the Lie theory of transformation groups to obtain a multiplier representation of the local Lie group whose Lie algebra is $\mathscr{G}(0, 0)$. However, before beginning this computation it will be convenient to simplify the problem

slightly. Since $\mathscr{G}(0, 0) \cong \mathscr{T}_3 \oplus (\mathscr{E})$ and $\rho(\mathscr{E}) = \mathrm{E}$ is a multiple of the identity operator for every irreducible representation ρ, the nontrivial part of the representation theory of $\mathscr{G}(0, 0)$ is given by the action of ρ on $\mathscr{T}_3$. We will lose nothing of importance for special function theory if we study only those representations $Q^\mu(\omega, m_o)$ for which $\mathrm{E} = \mu = 0$. Moreover, the representations $Q^0(\omega, m_o)$ induce an irreducible representation $Q(\omega, m_o)$ of the 3-dimensional subalgebra $\mathscr{T}_3$. The action of $Q(\omega, m_o)$ is obtained explicitly from Eqs. (3.1)-(3.3) by suppressing the operator E.

It was shown in Section 1-2 that $\mathscr{T}_3$ is the Lie algebra of the local Lie group T_3, a multiplicative matrix group with elements

$$g = \begin{pmatrix} 1 & 0 & 0 & \tau \\ 0 & e^{-\tau} & 0 & c \\ 0 & 0 & e^{\tau} & b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad b, c, \tau \in \mathbb{C}. \tag{3.4}$$

We will extend the representation $Q(\omega, m_o)$ of $\mathscr{T}_3$, defined on $\mathscr{V}_1$, to a multiplier representation of T_3. Let $\mathscr{A}_1$ be the vector space of all functions of z analytic in some neighborhood of the point $z = 1$. Clearly $\mathscr{V}_1 \subset \mathscr{A}_1$. According to Theorem 1.10 the differential operators

$$\mathrm{J}^3 = m_o + z\frac{d}{dz}, \qquad \mathrm{J}^+ = \omega z, \qquad \mathrm{J}^- = \frac{\omega}{z} \tag{3.5}$$

generate a Lie algebra which is the algebra of generalized Lie derivatives of a multiplier representation A of T_3 acting on $\mathscr{A}_1$. We will verify this fact and compute the multiplier ν. The action of the group element $\exp \tau \mathscr{J}^3$ on $\mathscr{A}_1$, $\tau \in \mathbb{C}$, is obtained by solving the differential equations

$$\frac{dz}{d\tau} = z, \qquad \frac{d}{d\tau}\nu(z^o, \exp \tau \mathscr{J}^3) = m_o \nu(z^o, \exp \tau \mathscr{J}^3)$$

with initial conditions $z(0) = z^o$, $\nu(z^o, \mathbf{e}) \equiv 1$. (See Section 1-2 for the definition of $\exp \tau \mathscr{J}^3$.) The solution of these equations is

$$z(\tau) = z^o e^\tau, \qquad \nu(z^o, \exp \tau \mathscr{J}^3) = e^{m_o \tau}.$$

Thus, if $f \in \mathscr{A}_1$ is analytic in a neighborhood of z^o then

$$[\mathbf{A}(\exp \tau \mathscr{J}^3) f](z^o) = [\exp \tau \mathrm{J}^3] f(z^o) = e^{m_o \tau} f(z^o e^\tau)$$

for $|\tau|$ sufficiently small. Similar computations yield

$$[\mathbf{A}(\exp b \mathscr{J}^+) f](z^o) = \exp(b\omega z^o) f(z^o),$$

$$[\mathbf{A}(\exp c \mathscr{J}^-) f](z^o) = \exp(c\omega / z^o) f(z^o).$$

If $g \in T_3$ is given by (3.4) we find

$$g = (\exp b\mathscr{J}^+)(\exp c\mathscr{J}^-)(\exp \tau\mathscr{J}^3).$$

Hence, for $|\tau|$, $|b|$, $|c|$ sufficiently small the operator $\mathbf{A}(g)$ acting on $f \in \mathscr{A}_1$ takes the form

$$\begin{aligned}[\mathbf{A}(g)f](z) &= [\mathbf{A}(\exp b\mathscr{J}^+)\{\mathbf{A}(\exp c\mathscr{J}^-)\,\mathbf{A}(\exp \tau\mathscr{J}^3)f\}](z)\\ &= \exp(b\omega z)[\mathbf{A}(\exp c\mathscr{J}^-)\{\mathbf{A}(\exp \tau\mathscr{J}^3)f\}](z)\\ &= \exp \omega\left(bz + \frac{c}{z}\right)[\mathbf{A}(\exp \tau\mathscr{J}^3)f](z)\\ &= \exp\left(\omega bz + \frac{\omega c}{z} + m_o\tau\right)f(e^\tau z), \qquad (3.6)\end{aligned}$$

defined for z in a sufficiently small neighborhood of $z = 1$. The multiplier ν is given by $\nu(z, g) = \exp(\omega bz + \omega c/z + m_o\tau)$. Conversely, (3.6) defines a multiplier representation of T_3 whose generalized Lie derivatives are the differential operators (3.5).

The elements of $\mathscr{V}_1$ are analytic for all values of $z \neq 0$, so if $f \in \mathscr{V}_1$ and $z \neq 0$ the expression

$$[\mathbf{A}(g)f](z) = \exp\left(\omega bz + \frac{\omega c}{z} + m_o\tau\right)f(e^\tau z)$$

defines an element of $\mathscr{A}_1$ for all $g \in T_3$, not just for g in a sufficiently small neighborhood of the identity. We define the vector subspace $\bar{\mathscr{V}}_1$ of $\mathscr{A}_1$ (the **completion** of $\mathscr{V}_1$, see Section 2-2) as the space of all finite linear combinations of functions of the form $\mathbf{A}(g)f$ for all $g \in T_3$, $f \in \mathscr{V}_1$. By definition, $\bar{\mathscr{V}}_1$ is invariant under A:

$$\mathbf{A}(g)f \in \bar{\mathscr{V}}_1 \qquad \text{for all} \quad f \in \bar{\mathscr{V}}_1, \qquad g \in T_3.$$

Furthermore, if $f \in \bar{\mathscr{V}}_1$, $g_1, g_2 \in T_3$, then

$$\mathbf{A}(g_1g_2)f = \mathbf{A}(g_1)[\mathbf{A}(g_2)f]. \qquad (3.7)$$

By restricting from $\mathscr{A}_1$ to $\bar{\mathscr{V}}_1$ the space on which the local multiplier representation A acts, we have been able to extend A over the whole group T_3. Thus, the representation $Q(\omega, m_o)$ of $\mathscr{T}_3$ has been "exponentiated" to yield a global representation of T_3.

The basis functions $f_m(z) = h_n(z) = z^n$, $n = 0, \pm 1, \ldots, m = m_o + n$, for $\mathscr{V}_1$ form an analytic basis for $\bar{\mathscr{V}}_1$ (see Section 2-2). This is true because every element f in $\mathscr{V}_1$ is analytic at all points $z \neq 0$ in $\mathbb{C}$; hence, f has a unique Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad a_n \in \mathbb{C},$$

which converges for all $z \neq 0$. Thus, we can define the matrix elements $A_{lk}(g)$ of the operators $\mathbf{A}(g)$ on $\mathscr{V}_1$ by

$$[\mathbf{A}(g)h_k](z) = \sum_{l=-\infty}^{\infty} A_{lk}(g)\, h_l(z), \qquad g \in T_3, \quad k = 0, \pm 1, \pm 2, \ldots . \tag{3.8}$$

Using (3.6) we obtain

$$\exp\left(\omega b z + \frac{\omega c}{z} + (k + m_o)\tau\right) z^k = \sum_{l=-\infty}^{\infty} A_{lk}(g) z^l \tag{3.9}$$

which can be regarded as a generating function for the matrix elements. From $\mathbf{A}(g_1 g_2)h_k = \mathbf{A}(g_1)[\mathbf{A}(g_2)h_k]$ there follows the identity

$$\begin{aligned}\mathbf{A}(g_1 g_2)h_k &= \sum_{l=-\infty}^{\infty} A_{lk}(g_1 g_2)h_l = \mathbf{A}(g_1)\left[\sum_{j=-\infty}^{\infty} A_{jk}(g_2)h_j\right] \\ &= \sum_{j=-\infty}^{\infty} A_{jk}(g_2)\,\mathbf{A}(g_1)h_j = \sum_{j=-\infty}^{\infty} A_{jk}(g_2) \sum_{l=-\infty}^{\infty} A_{lj}(g_1)h_l \\ &= \sum_{l=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} A_{lj}(g_1)\, A_{jk}(g_2)h_l\,.\end{aligned}$$

Comparing coefficients of $h_l(z) = z^l$ we obtain the addition theorem

$$A_{lk}(g_1 g_2) = \sum_{j=-\infty}^{\infty} A_{lj}(g_1)\, A_{jk}(g_2), \qquad l, k = 0, \pm 1, \pm 2, \ldots, \tag{3.10}$$

valid for all $g_1, g_2 \in T_3$. To find an explicit expression for the matrix elements $A_{lk}(g)$ it is sufficient to compute the coefficient of z^l on the left-hand side of (3.9). Thus,

$$\begin{aligned}\exp\left(\omega b z + \frac{\omega c}{z} + (k + m_o)\tau\right) z^k &= \sum_{s=0}^{\infty} \frac{(\omega b z)^s}{s!} \sum_{j=0}^{\infty} \frac{(\omega c z^{-1})^j}{j!}\, e^{(k+m_o)\tau} z^k \\ &= e^{(k+m_o)\tau} \sum_{l=-\infty}^{\infty} z^l \sum_s \frac{(\omega b)^s (c\omega)^{s+k-l}}{s!(s+k-l)!}\,,\end{aligned}$$

so

$$A_{lk}(g) = e^{(k+m_o)\tau}(c\omega)^{k-l} \sum_s \frac{(\omega^2 bc)^s}{s!(s+k-l)!}\,, \qquad l, k = 0, \pm 1, \pm 2, \ldots, \tag{3.11}$$

where the sum extends over all nonnegative integral values of s such that the summand is defined. Specifically,

$$\begin{aligned} A_{lk}(g) &= \frac{e^{(m_0+k)\tau}(c\omega)^{k-l}}{(k-l)!}\,{}_0F_1(k-l+1;\omega^2bc) \qquad \text{if} \quad k \geqslant l, \\ A_{lk}(g) &= \frac{e^{(m_0+k)\tau}(b\omega)^{l-k}}{(l-k)!}\,{}_0F_1(l-k+1,\omega^2bc) \qquad \text{if} \quad l \geqslant k, \end{aligned} \tag{3.12}$$

where the function ${}_0F_1$ is defined by its power series expansion (A. 21). These two cases can be combined into the single expression

$$A_{lk}(g) = \frac{e^{(m_0+k)\tau}(c\omega)^{(k-l+|k-l|)/2}\,(b\omega)^{(l-k+|k-l|)/2}}{|\,k-l\,|!}\,{}_0F_1(|\,k-l\,|+1;\omega^2bc) \tag{3.12$'$}$$

valid for all integral values of l, k.

If $bc \neq 0$ the matrix elements are closely related to Bessel functions. To see this we introduce new group parameters r, v in place of b, c by

$$r = 2|\,bc\,|^{1/2}\,e^{i(\arg b+\arg c+\pi)/2}, \qquad v = \left|\frac{b}{c}\right|^{1/2} e^{i(\arg b-\arg c-\pi)/2} \tag{3.13}$$

where $-\pi < \arg b, \arg c \leqslant \pi$, or briefly, $r = (ibc)^{1/2}$ and $v = (b/ic)^{1/2}$. With these definitions it follows that $b = rv/2$, $c = -rv^{-1}/2$, and this coordinate change is one-to-one if $bc \neq 0$. In terms of the coordinates r, v the matrix elements (3.11) can be expressed as

$$A_{lk}(g) = e^{(m_0+k)\tau}(-v)^{l-k}\,J_{l-k}(-\omega r), \qquad k = 0, \pm 1, \pm 2, \ldots \tag{3.14}$$

[see (A. 20)]. The functions J_l are Bessel functions of integral order. In particular, for $\tau = 0$, $v = -1$, $\omega = -1$, $k = 0$, substitution of (3.14) into (3.9) yields the well-known generating function

$$\exp\left[\frac{r}{2}(z - z^{-1})\right] = \sum_{l=-\infty}^{\infty} J_l(r)z^l. \tag{3.15}$$

for Bessel functions.

We can obtain addition theorems for the functions ${}_0F_1$ by substituting (3.12) into (3.10). After some simplification one finds

$$\begin{aligned} &\frac{(b_1+b_2)^n}{n!}\,{}_0F_1(n+1;(b_1+b_2)(c_1+c_1)) \\ &\quad = \sum_{j=-\infty}^{\infty} \frac{b_1^{(j+|j|)/2}\,c_1^{(-j+|j|)/2}\,b_2^{(n-j+|n-j|)/2}\,c_2^{(j-n+|n-j|)/2}}{|\,j\,|!\,|\,n-j\,|!} \\ &\qquad \cdot\,{}_0F_1(|\,j\,|+1;b_1c_1)\,{}_0F_1(|\,n-j\,|+1;b_2c_2) \end{aligned} \tag{3.16}$$

where n is a nonnegative integer and b_1, b_2, c_1, $c_2 \in \mathbb{C}$. Some special cases of this formula are of interest. If $b_1 = 1$, $b_2 = c_1 = 0$, $c_2 = x$, (3.16) reduces to

$$ {}_0F_1(n+1; x) = \sum_{j=0}^{\infty} \frac{n!\, x^j}{(n+j)!\, j!}, $$

which is the power series expansion for ${}_0F_1$. (We have used the fact that ${}_0F_1(k; 0) = 1$.) If $b_1 = 1$, $b_2 = 0$, $c_1 = x$, $c_2 = y$, then

$$ {}_0F_1(n+1; x+y) = n! \sum_{j=0}^{\infty} \frac{y^j\, {}_0F_1(n+j+1; x)}{(j+n)!\, j!}. $$

If $b_1 = x$, $b_2 = 0$, $c_1 = -c_2 = 1$ there follows

$$ 1 = \sum_{j=0}^{\infty} \frac{n!\, x^j(-1)^j}{(j+n)!\, j!}\, {}_0F_1(n+j+1; x). $$

For $b_1 = c_1 = x$, $b_2 = c_2 = y$ we obtain

$$ \frac{(x+y)^n}{n!}\, {}_0F_1(n+1; (x+y)^2) = \sum_{j=-\infty}^{\infty} \frac{x^{|j|} y^{|n-j|}}{|j|!\, |n-j|!} \cdot {}_0F_1(|j|+1; x^2)\, {}_0F_1(|n-j|+1; y^2). $$

The addition theorem (3.10) can also be used to derive identities relating Bessel functions of integral order. However, it will be more convenient to derive these results in Section 3-3 as special cases of identities relating Bessel functions of arbitrary order.

3-2 Recursion Relations for the Matrix Elements

The matrix elements $A_{lk}(g)$, $g \in T_3$, computed in (3.11) are entire analytic functions of the group parameters τ, b, c. Hence they can be considered as analytic functions on the group manifold. Denoting by $\mathcal{A}(T_3)$ the space of all functions on T_3 which are analytic in some neighborhood of the identity element, we see that $A_{lk} \in \mathcal{A}(T_3)$ for all integers l, k, where the A_{lk} are matrix elements corresponding to the representation $Q(\omega, m_o)$. There is a natural action P of T_3 on $\mathcal{A}(T_3)$ as a local transformation group. P is defined by

$$ [\mathbf{P}(g')f](g) = f(gg') \tag{3.17} $$

for all $f \in \mathcal{A}(T_3)$ and all g', g in a sufficiently small neighborhood of $\mathbf{e} \in T_3$ (the neighborhood depends on f). From (3.17) it is evident that

$$\mathbf{P}(g_1 g_2) f = \mathbf{P}(g_1)[\mathbf{P}(g_2) f],$$

if g_1, g_2, and $g_1 g_2$ are in the domain of f. Thus, P defines T_3 as an effective local transformation group on the 3-parameter group manifold of T_3.

The action of P on the matrix elements A_{jk} is easily determined:

$$[\mathbf{P}(g') A_{jk}](g) = A_{jk}(gg') = \sum_{l=-\infty}^{\infty} A_{lk}(g') A_{jl}(g) \tag{3.18}$$

for all $g, g' \in T_3$, $j, k = 0, \pm 1, \pm 2, \ldots$. Comparing this expression with (3.8) we see that for fixed j the functions $\{A_{jk}, k = 0, \pm 1, \pm 2, \ldots\}$, form a basis for the representation $Q(\omega, m_o)$ of T_3. The Lie derivatives J^+, J^-, J^3 defined by

$$\begin{aligned} \mathrm{J}^+ f(g) &= \frac{d}{db}[\mathbf{P}(\exp b \mathscr{J}^+) f](g)\Big|_{b=0}, \\ \mathrm{J}^- f(g) &= \frac{d}{dc}[\mathbf{P}(\exp c \mathscr{J}^-) f](g)\Big|_{c=0}, \\ \mathrm{J}^3 f(g) &= \frac{d}{d\tau}[\mathbf{P}(\exp \tau \mathscr{J}^3) f](g)\Big|_{\tau=0} \end{aligned} \tag{3.19}$$

for all $f \in \mathcal{A}(T_3)$, must, therefore, satisfy the commutation relations

$$[\mathrm{J}^3, \mathrm{J}^\pm] = \pm \mathrm{J}^\pm, \qquad [\mathrm{J}^+, \mathrm{J}^-] = 0$$

and act on the basis vectors A_{jk} as follows:

$$\begin{aligned} &\mathrm{J}^3 A_{jk}(g) = (m_o + k) A_{jk}(g), \\ &\mathrm{J}^+ A_{jk}(g) = \omega A_{j\,k+1}(g), \qquad \mathrm{J}^- A_{jk}(g) = \omega A_{j\,k-1}(g), \\ &\mathrm{C}_{0,0} A_{jk}(g) = \mathrm{J}^+ \mathrm{J}^- A_{jk}(g) = \omega^2 A_{jk}(g), \qquad j, k = 0, \pm 1, \pm 2, \ldots . \end{aligned} \tag{3.20}$$

These expressions give recursion relations and differential equations for the matrix elements A_{jk}. To evaluate them it is necessary to compute the Lie derivatives defined by (3.19). The elements of $\mathcal{A}(T_3)$ can be considered as analytic functions of the group coordinates b, c, τ, so the Lie derivatives are linear differential operators in these three complex variables. However, if we wish to relate our results to Bessel functions it is more convenient to use the group coordinates r, v, τ, (3.13). As we

have seen, in terms of the r, v coordinate system the matrix elements are given by (3.14):

$$A_{lk}(g) = e^{(m_o+k)\tau}(-v)^{l-k}\, J_{l-k}(-\omega r).$$

This coordinate system is not defined over the whole group but only for those group elements such that $bc \neq 0$. If g has coordinates (r, v, τ) and

$$g' = \begin{pmatrix} 1 & 0 & 0 & \tau' \\ 0 & e^{-\tau'} & 0 & c' \\ 0 & 0 & e^{\tau'} & b' \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

a simple computation shows that the coordinates of gg' are given by

$$\left(r\left[1 - 2\frac{e^{-\tau}c'v}{r} + \frac{2e^{\tau}b'}{rv} - 4\frac{b'c'}{r^2}\right]^{1/2}, v\left[\frac{1 + 2e^{\tau}b'/rv}{1 - 2e^{-\tau}c'v/r}\right]^{1/2}, \tau + \tau'\right),$$

for $|b'|$, $|c'|$ sufficiently small. From this result and the definition (3.19) of the Lie derivatives there follows

$$\mathrm{J}^+ = e^{\tau}\left(\frac{1}{v}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial v}\right), \qquad \mathrm{J}^- = e^{-\tau}\left(-v\frac{\partial}{\partial r} + \frac{v^2}{r}\frac{\partial}{\partial v}\right), \qquad \mathrm{J}^3 = \frac{\partial}{\partial \tau}. \tag{3.21}$$

Substituting (3.14) and (3.21) into (3.20) we obtain, after factoring out the dependence on v and τ,

$$\begin{gathered} \left[\frac{d}{dr} + \frac{n}{r}\right] J_n(r) = J_{n-1}(r), \qquad \left[-\frac{d}{dr} + \frac{n}{r}\right] J_n(r) = J_{n+1}(r), \\ \left[-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{n^2}{r^2}\right] J_n(r) = J_n(r) \end{gathered} \tag{3.22}$$

where n is an integer.

3-3 Realizations of $Q(\omega, m_o)$ in Two Variables

In the previous sections realizations of the representation $Q(\omega, m_o)$ of $\mathscr{T}_3$ have been determined on spaces of functions of one and three complex variables, respectively. We will now find realizations of this representation on spaces of functions of two complex variables, x and y. In particular, we look for functions $f_m(x, y) = Z_m(x)\, e^{my}$, such that

$$\begin{gathered} \mathrm{J}^3 f_m = m f_m, \qquad \mathrm{J}^+ f_m = \omega f_{m+1}, \qquad \mathrm{J}^- f_m = \omega f_{m-1}, \\ \mathrm{C}_{0,0} f_m = \mathrm{J}^+\mathrm{J}^- f_m = \omega^2 f_m, \qquad \omega \neq 0, \end{gathered} \tag{3.23}$$

for all $m \in S = \{m_o + k: k \text{ an integer}\}$, where the differential operators $J^{\pm}$, J^3 are given by

$$J^3 = \frac{\partial}{\partial y}, \qquad J^{\pm} = e^{\pm y}\left(\pm \frac{\partial}{\partial x} - \frac{1}{x}\frac{\partial}{\partial y}\right). \tag{3.24}$$

These operators are the *type C″* operators classified in Section 2-7. (The constants p, q occurring in the expression for the *type C″* operators can be set equal to 0 without any loss of generality.)

The content of (3.23) is, thus, a series of equations relating the functions $Z_m(x)$:

$$\left[\frac{d}{dx} - \frac{m}{x}\right] Z_m(x) = \omega Z_{m+1}(x), \qquad \left[-\frac{d}{dx} - \frac{m}{x}\right] Z_m(x) = \omega Z_{m-1}(x)$$
$$\left[-\frac{d^2}{dx^2} - \frac{1}{x}\frac{d}{dx} + \frac{m^2}{x^2}\right] Z_m(x) = \omega^2 Z_m(x), \qquad m \in S. \tag{3.25}$$

The complex constant ω in these equations is clearly nonessential; we could remove it by making the change of variable $x' = \omega x$. Hence, we will assume $\omega = -1$. (For this choice of ω, Eqs. (3.25) agree with the conventional recursion relations for cylindrical functions, (A. 23).) Nonzero solutions of these equations are well known. If $m_o = 0$ so that the elements of S are integers, we see from (3.22) that $Z_m(x) = J_m(x)$ is a solution for all $m \in S$. Furthermore, for arbitrary complex values of m_o we can obtain the solutions $Z_m(x) = J_m(x)$, $J_{-m}(x)$, $H_m^{(1)}(x)$, $H_m^{(2)}(x)$, $N_m(x)$, for all $m \in S$ (Magnus *et al.* [1]). Thus, the Bessel functions of first and second kind, the Hankel functions of first and second kind, and the Neuman functions each satisfy the recursion relations (3.25) for $\omega = -1$. Each of these functions is analytic for all values of x except $x = 0$.

If, conversely, functions $Z_m(x)$ defined for each $m \in S$ satisfy (3.25) for $\omega = -1$, then the vectors $f_m(x, y) = Z_m(x)\, e^{my}$ form a basis for a realization of the representation $Q(-1, m_o)$ of $\mathscr{T}_3$. This Lie algebra representation induces a local representation of T_3. In fact if $\mathscr{A}$ is the space of all functions analytic in some neighborhood of the point $(x^o, y^o) = (1, 0)$, the Lie derivatives

$$J^3 = \frac{\partial}{\partial y}, \qquad J^{\pm} = e^{\pm y}\left(\pm \frac{\partial}{\partial x} - \frac{1}{x}\frac{\partial}{\partial y}\right)$$

define a local multiplier representation T of T_3 on $\mathscr{A}$. From Theorem 1.10,

$$[\mathbf{T}(\exp b\mathscr{J}^+)f](x, e^y) = f(x(b), e^{y(b)}), \qquad b \in \mathscr{C}, \quad f \in \mathscr{A},$$

where

$$\frac{dx}{db}(b) = e^{y(b)}, \qquad \frac{dy}{db}(b) = -\frac{e^{y(b)}}{x(b)}, \qquad x(0) = x, \quad y(0) = y.$$

The solution of these equations is

$$[\mathbf{T}(\exp b\mathscr{J}^+)f](x, t) = f\left[x\left(1 - \frac{2bt}{x}\right)^{1/2}, t\left(1 + \frac{2bt}{x}\right)^{-1/2}\right]$$

where for convenience we use the new variable $t = e^y$. Similar computations yield

$$[\mathbf{T}(\exp c\mathscr{J}^-)f](x, t) = f\left[x\left(1 - \frac{2c}{xt}\right)^{1/2}, t\left(1 - \frac{2c}{xt}\right)^{1/2}\right],$$

$$[\mathbf{T}(\exp \tau\mathscr{J}^3)f](x, t) = f(x, te^\tau).$$

If

$$g = \begin{pmatrix} 1 & 0 & 0 & \tau \\ 0 & e^{-\tau} & 0 & c \\ 0 & 0 & e^{\tau} & b \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then $g = (\exp b\mathscr{J}^+)(\exp c\mathscr{J}^-)(\exp \tau\mathscr{J}^3)$. So, for $f \in \mathscr{A}$ and g in a sufficiently small neighborhood of the identity we have

$$[\mathbf{T}(g)f](x, t) = [\mathbf{T}(\exp b\mathscr{J}^+)\,\mathbf{T}(\exp c\mathscr{J}^-)\,\mathbf{T}(\exp \tau\mathscr{J}^3)f](x, t).$$

An explicit computation gives

$$[\mathbf{T}(g)f](x, t) = f\left[x\left(1 + \frac{2bt}{x}\right)^{1/2}\left(1 - \frac{2c}{xt}\right)^{1/2}, te^\tau\left(\frac{1 - \dfrac{2c}{xt}}{1 + \dfrac{2bt}{x}}\right)^{1/2}\right] \tag{3.26}$$

defined for $|2bt/x| < 1$, $|2c/xt| < 1$. According to Section 2-2, our realization of the representation $Q(-1, m_o)$ of $\mathscr{T}_3$ on the space generated by the functions $f_m(x, y) = Z_m(x)\, e^{my}$, $m \in S$, can be extended to a local representation of T_3 where the group action is given by (3.26).

In particular, the functions $f_m(x, y)$ form an analytic basis for this space, as can easily be seen from (2.2) and the y-dependence e^{my} of the basis vectors. Thus, the matrix elements of this local representation

with respect to the basis f_m are uniquely determined by $Q(-1, m_o)$ and without computation we immediately obtain the relations

$$[\mathbf{T}(g) f_{m_o+k}](x, t) = \sum_{l=-\infty}^{\infty} A_{lk}(g) f_{m_o+l}(x, t), \qquad k = 0, \pm 1, \pm 2, \ldots, \tag{3.27}$$

$$Z_m \left[x\left(1 + \frac{2bt}{x}\right)^{1/2}\left(1 - \frac{2c}{xt}\right)^{1/2}\right] \cdot \left[te^{\tau}\left(\frac{1 - \dfrac{2c}{xt}}{1 + \dfrac{2bt}{x}}\right)^{1/2}\right]^m$$

$$= \sum_{l=-\infty}^{\infty} A_{l,m-m_o}(g)\, Z_{m_o+l}(x)\, t^{m_o+l} \tag{3.28}$$

where the matrix elements $A_{lk}(g)$ are given by (3.12) ($\omega = -1$). The region of convergence of these relations is determined by examining the singularities of the functions on the left-hand side of (3.28). Since Z_m, $m \in \mathbb{C}$, is analytic in x for all nonzero values of x, it follows that the infinite series (3.28) converges absolutely for $|2bt/x| < 1$; $|2c/xt| < 1$. Explicitly,

$$Z_m \left[x\left(1 + \frac{2bt}{x}\right)^{1/2}\left(1 - \frac{2c}{xt}\right)^{1/2}\right]\left(\frac{1 - \dfrac{2c}{xt}}{1 + \dfrac{2bt}{x}}\right)^{m/2}$$

$$= \sum_{n=-\infty}^{\infty} \frac{(-1)^{|n|}}{|n|!}\, c^{(-n+|n|)/2}\, b^{(n+|n|)/2}\, {}_0F_1(|n| + 1; bc)\, Z_{m+n}(x) t^n,$$

$$m \in \mathbb{C}, \quad \left|\frac{2bt}{x}\right| < 1, \quad \left|\frac{2c}{xt}\right| < 1. \tag{3.29}$$

Several of the fundamental identities for cylindrical functions are special cases of this formula. If $c = 0$, $t = 1$, Eq. (3.29) becomes

$$Z_m \left[x\left(1 + \frac{2b}{x}\right)^{1/2}\right]\left(1 + \frac{2b}{x}\right)^{-m/2} = \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} Z_{m+n}(x), \qquad \left|\frac{2b}{x}\right| < 1. \tag{3.30}$$

If $b = 0$, $t = 1$, one obtains

$$Z_m \left[x\left(1 + \frac{2c}{xt}\right)^{1/2}\right]\left(1 + \frac{2c}{xt}\right)^{m/2} = \sum_{n=0}^{\infty} \frac{c^n}{n!} Z_{m-n}(x), \qquad \left|\frac{2c}{x}\right| < 1. \tag{3.31}$$

In the case where $Z_m \equiv J_m$, (3.30) and (3.31) are known as the formulas of Lommel.

If $bc \neq 0$ we can introduce the coordinates r, v defined by (3.13), such that $b = rv/2$, $c = -r/2v$. In this case

$$A_{lk}(g) = e^{(m_0+k)\tau}(-v)^{l-k} J_{l-k}(r)$$

and Eq. (3.29) simplifies to

$$Z_m\left[x\left(1+\frac{rv}{x}\right)^{1/2}\left(1+\frac{r}{vx}\right)^{1/2}\right]\left(\frac{1+\dfrac{r}{vx}}{1+\dfrac{rv}{x}}\right)^{m/2} = \sum_{n=-\infty}^{\infty}(-v)^n J_n(r)\, Z_{m+n}(x),$$

$$\left|\frac{rv}{x}\right| < 1, \quad \left|\frac{r}{vx}\right| < 1. \tag{3.32}$$

For $Z_m \equiv J_m$ this is a generalization of Graf's addition theorem (Erdélyi *et al.* [1], Vol. II, p. 44).

3-4 Weisner's Method for Bessel Functions

Expressions (3.27) are valid only for group elements g in a sufficiently small neighborhood of the identity element of T_3. However, we can also use the *type C″* operators (3.24) to derive identities for cylindrical functions associated with group elements bounded away from the identity. The following remarks are pertinent. If $f(x, t)$ is a solution of the equation $\mathrm{C}_{0,0} f = \omega^2 f$, i.e.,

$$\left(-\frac{\partial^2}{\partial x^2} - \frac{1}{x}\frac{\partial}{\partial x} + \frac{t^2}{x^2}\frac{\partial^2}{\partial t^2}\right) f(x, t) = \omega^2 f(x, t), \tag{3.33}$$

then the function $\mathbf{T}(g) f$ given formally by

$$[\mathbf{T}(g)f](x, t) = f\left(\left[(x+2bt)\left(x-\frac{2c}{t}\right)\right]^{1/2}, te^{\tau}\left(\frac{1-\dfrac{2c}{xt}}{1+\dfrac{2bt}{x}}\right)^{1/2}\right)$$

satisfies the equation $\mathrm{C}_{0,0}(\mathbf{T}(g)f) = \omega^2(\mathbf{T}(g)f)$ whenever $\mathbf{T}(g) f$ can be defined. This follows from the fact that $\mathrm{C}_{0,0}$ commutes with the differential operators J^+, J^-, J^3. Furthermore, if f is a solution of the equation

$$(x_1\mathrm{J}^+ + x_2\mathrm{J}^- + x_3\mathrm{J}^3) f(x, t) = \lambda f(x, t) \tag{3.34}$$

for constants x_1, x_2, x_3, λ, then $\mathbf{T}(g) f$ is a solution of the equation

$$[\mathbf{T}(g)(x_1\mathrm{J}^+ + x_2\mathrm{J}^- + x_3\mathrm{J}^3)\,\mathbf{T}(g^{-1})][\mathbf{T}(g)f] = \lambda[\mathbf{T}(g)f]$$

where

$$\mathbf{T}(g)(x_1\mathrm{J}^+ + x_2\mathrm{J}^- + x_3\mathrm{J}^3)\,\mathbf{T}(g^{-1}) = (x_1e^{\tau} - bx_3)\mathrm{J}^+ + (x_2e^{-\tau} + cx_3)\mathrm{J}^- + x_3\mathrm{J}^3. \tag{3.35}$$

This is a consequence of Eq. (1.43).

As an example of the application of these remarks consider the function $f(x, t) = J_m(x)t^m$, $m \in \mathbb{C}$. Here $\mathrm{C}_{0,0}f = f$, $\mathrm{J}^3 f = mf$, so the function

$$[\mathbf{T}(g)f](x,t) = J_m\left[\left((x+2bt)\left(x - \frac{2c}{t}\right)\right)^{1/2}\right] e^{m\tau}\left(\frac{t^2 - \dfrac{2ct}{x}}{1 + \dfrac{2bt}{x}}\right)^{m/2} \tag{3.36}$$

satisfies the equations

$$\mathrm{C}_{0,0}[\mathbf{T}(g)f] = \mathbf{T}(g)f, \qquad (-b\mathrm{J}^+ + c\mathrm{J}^- + \mathrm{J}^3)[\mathbf{T}(g)f] = m[\mathbf{T}(g)f]. \tag{3.37}$$

If $\tau = b = 0$, $c = -1$, we can write (3.36) in the form

$$h(x,t) = \left(x^2 + \frac{2x}{t}\right)^{-m/2} J_m\left[\left(x^2 + \frac{2x}{t}\right)^{1/2}\right](2 + xt)^m.$$

Since $x^{-m}J_m(x)$ is an entire function of x, h has a Laurent expansion about $t = 0$:

$$h(x,t) = \sum_{n=-\infty}^{\infty} h_n(x)\,t^n, \qquad |xt| < 2.$$

Substituting this expansion in the first equation (3.37) we see that $h_n(x)$ is a solution of Bessel's equation

$$\left(-\frac{d^2}{dx^2} - \frac{1}{x}\frac{d}{dx} + \frac{n^2}{x^2}\right)h_n(x) = h_n(x)$$

for each integer n. The function $h(x, t)$ is bounded for $x = 0$, so we must have $h_n(x) = c_nJ_n(x)$, $c_n \in \mathbb{C}$ [see (A.22)]. Thus,

$$h(x,t) = \sum_{n=-\infty}^{\infty} c_nJ_n(x)\,t^n.$$

From the second equation (3.37), $(-\mathrm{J}^- + \mathrm{J}^3)\,h(x,t) = mh(x,t)$, it follows that $c_{n+1} = (m-n)c_n$. The constant c_0 can be determined

directly by setting $x = 0$ in $h(x, t)$. The result is $c_0 = 1/\Gamma(m + 1)$, whence $c_n = 1/\Gamma(m - n + 1)$. Thus, we obtain the identity

$$\left(x^2 + \frac{2x}{t}\right)^{-m/2} J_m\left[\left(x^2 + \frac{2x}{t}\right)^{1/2}\right](2 + xt)^m = \sum_{n=-\infty}^{\infty} \frac{J_n(x)\, t^n}{\Gamma(m - n + 1)}, \quad |xt| < 2. \tag{3.38}$$

Equation (3.38) is obviously not a special case of (3.29). For other examples of generating functions derived by this method see Weisner [3].

The above results were all obtained by using *type C″* operators. The *type D″* operators lead to no new results for special functions so we will not consider them here.

3-5 The Real Euclidean Group E_3

The Euclidean group in the plane, E_3, can be defined as the real 3-parameter group of matrices

$$g(\theta, x_1, x_2) = \begin{pmatrix} \cos\theta & -\sin\theta & x_1 \\ \sin\theta & \cos\theta & x_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.39}$$

where x_1, x_2 are real and θ is a real variable determined up to a multiple of 2π. As is well known, E_3 acts as a transformation group in the plane. If $\mathbf{y} = (y_1, y_2)$ is a point in the plane R^2 the action of E_3 is given by

$$\begin{aligned} \mathbf{y} \to g^{-1}\mathbf{y} = ((y_1 - x_1)\cos\theta + (y_2 - x_2)\sin\theta, \\ -(y_1 - x_1)\sin\theta + (y_2 - x_2)\cos\theta). \end{aligned}$$

This action corresponds to a translation $(y_1, y_2) \to (y_1 - x_1, y_2 - x_2)$, followed by a rotation through the angle θ in a clockwise direction about the point (0, 0). It is easily verified in terms of the group parameters that the group multiplication is given by

$$\begin{aligned} &g(\theta, x_1, x_2)\, g(\theta', x_1', x_2') \\ &\quad = g(\theta + \theta', x_1' \cos\theta - x_2' \sin\theta + x_1, x_1' \sin\theta + x_2' \cos\theta + x_2). \end{aligned} \tag{3.40}$$

The identity element of E_3 is the identity matrix $g(0, 0, 0)$ and the inverse of $g(\theta, x_1, x_2)$ is

$$g^{-1}(\theta, x_1, x_2) = g(-\theta, -x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta - x_2 \cos\theta),$$

since

$$g(\theta, x_1, x_2)\, g(-\theta, -x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta - x_2 \cos\theta) = g(0, 0, 0).$$

As a basis for the real Lie algebra $\mathscr{E}_3$ of E_3 we choose the elements

$$\mathscr{J}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathscr{J}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathscr{J}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.41}$$

with commutation relations

$$[\mathscr{J}_1, \mathscr{J}_2] = \mathscr{O}, \qquad [\mathscr{J}_3, \mathscr{J}_1] = \mathscr{J}_2, \qquad [\mathscr{J}_3, \mathscr{J}_2] = -\mathscr{J}_1.$$

The complex matrices $\mathscr{J}^+ = -\mathscr{J}_2 + i\mathscr{J}_1$, $\mathscr{J}^- = \mathscr{J}_2 + i\mathscr{J}_1$, $\mathscr{J}^3 = i\mathscr{J}_3$, $i = \sqrt{-1}$, satisfy the commutation relations

$$[\mathscr{J}^3, \mathscr{J}^+] = \mathscr{J}^+, \qquad [\mathscr{J}^3, \mathscr{J}^-] = -\mathscr{J}^-, \qquad [\mathscr{J}^+, \mathscr{J}^-] = \mathscr{O}$$

which are identical with the relations (1.41) for a basis of $\mathscr{T}_3$. Thus, the complex Lie algebra generated by the basis elements (3.41) is $\mathscr{T}_3$. We say that $\mathscr{T}_3$ is the **complexification** of $\mathscr{E}_3$ and $\mathscr{E}_3$ is a **real form** of $\mathscr{T}_3$ (See Helgason [1], p. 152). Due to this relationship between $\mathscr{T}_3$ and $\mathscr{E}_3$, the abstract irreducible representation $Q(\omega, m_o)$ of $\mathscr{T}_3$ induces an irreducible representation of $\mathscr{E}_3$.

There is another matrix realization of E_3 which we will find useful. Namely, we define matrices

$$g(\theta, x_1, x_2) = \begin{pmatrix} e^{i\theta} & 0 & (x_2 - ix_1)/2 \\ 0 & e^{-i\theta} & (-x_2 - ix_1)/2 \\ 0 & 0 & 1 \end{pmatrix},$$
$$x_1, x_2 \in R, \quad 0 \leqslant \theta \leqslant 2\pi, \quad \bmod 2\pi. \tag{3.42}$$

These matrices form a realization of E_3 isomorphic to the realization by the matrices (3.39). In fact,

$$\begin{aligned} &g(\theta, x_1, x_2)\, g(\theta', x_1', x_2') \\ &\quad = g(\theta + \theta', x_1' \cos\theta - x_2' \sin\theta + x_1, x_1' \sin\theta + x_2' \cos\theta + x_2) \end{aligned}$$

in agreement with (3.40).

It will sometimes be convenient to use, instead of the coordinates x_1, x_2, the polar coordinates $r \geqslant 0$ and φ defined by

$$x_1 + ix_2 = re^{i\varphi},$$

where we assume that $re^{i\varphi} = 0$ implies $r = \varphi = 0$. In terms of these new coordinates the matrices (3.42) take the form

$$g[\theta, r, \varphi] = \begin{pmatrix} e^{i\theta} & 0 & -(i/2)\, re^{i\varphi} \\ 0 & e^{-i\theta} & -(i/2)\, re^{-i\varphi} \\ 0 & 0 & 1 \end{pmatrix}.$$

If g' has coordinates $[\theta', r', \varphi']$ and g'' has coordinates $[\theta'', r'', \varphi'']$ then an easy computation shows $g'g''$ has coordinates $[\theta, r, \varphi]$ where

$$\theta = \theta' + \theta'', \qquad re^{i\varphi} = r'e^{i\varphi'} + r''e^{i(\varphi''+\theta')}. \tag{3.43}$$

In particular, the coordinates of g'^{-1} are $[-\theta', r', \varphi' - \theta' + \pi]$.

3-6 Unitary Representations of Lie Groups

We shall be interested in the connection between the local multiplier representations of T_3 and (global) **unitary** representations of E_3 on a Hilbert space. Hence, we make a brief digression to discuss unitary representations. The basic concepts of the theory of unitary representations of (global) Lie groups are presented in several standard references (Naimark [1, 2]; Helgason [1]). Here, a few of the fundamental definitions and results in this theory will be listed which are useful for the study of special functions. It is assumed that the reader is familiar with the basic concepts of Hilbert space theory.

Let $\mathscr{H}$ be a complex Hilbert space with the inner product of two vectors f, f' in $\mathscr{H}$ denoted by $\langle f, f'\rangle$ and the norm of f by $|f| = (\langle f, f\rangle)^{1/2}$. We assume $\langle f, af'\rangle = a\langle f, f'\rangle$, $\langle af, f'\rangle = \bar{a}\langle f, f'\rangle$ for every $a \in \mathbb{C}$, i.e., the inner product is linear in the second argument, conjugate linear in the first argument. A sequence of vectors $\{f_n\}$, $n = 1, 2,\ldots$, is said to converge to $f \in \mathscr{H}$, $(f_n \to f)$, if $\lim_{n\to\infty} |f_n - f| = 0$. Let $\mathscr{D}$ be a vector subspace of $\mathscr{H}$. $\mathscr{D}$ is **dense** in $\mathscr{H}$ if for every $f \in \mathscr{H}$ there is a sequence of vectors $f_n \in \mathscr{D}$ such that $f_n \to f$. $\mathscr{D}$ is **closed** if every convergent sequence of vectors in $\mathscr{D}$ converges to an element in $\mathscr{D}$. $\mathscr{H}$ is closed by definition.

If $\mathscr{B}$ is a subspace of $\mathscr{H}$, $\bar{\mathscr{B}}$, the **closure** of $\mathscr{B}$ is defined to be the intersection of all closed subspaces of $\mathscr{H}$ containing $\mathscr{B}$. It is easy to see that $\bar{\mathscr{B}}$ is closed.

A linear operator $\mathbf{U}$ on $\mathscr{H}$ is **unitary** if $\langle \mathbf{U}f, \mathbf{U}f'\rangle = \langle f, f'\rangle$ for every $f, f' \in \mathscr{H}$. If $\mathbf{U}$ is unitary, so is $\mathbf{U}^{-1}$ where $\mathbf{U}^{-1}$ is the unique operator such that $\mathbf{U}\mathbf{U}^{-1} = \mathbf{I}$, ($\mathbf{I}$ is the identity operator).

Let G be a real (global) Lie group. A **unitary representation** of G

on $\mathscr{H}$ consists of a family of unitary operators $\mathbf{U}(g)$ on $\mathscr{H}$, defined for every $g \in G$, with the properties:

(1) $\mathbf{U}(gg') = \mathbf{U}(g)\,\mathbf{U}(g')$ for all $g, g' \in G$,

(2) if $g_n \to g$ in the topology of G then $\mathbf{U}(g_n)f \to \mathbf{U}(g)f$ in $\mathscr{H}$, for all $f \in \mathscr{H}$.

Since the unitary operators have unique inverses, it follows easily from (1) that $\mathbf{U}(\mathbf{e}) = \mathbf{I}$ and $\mathbf{U}(g^{-1}) = \mathbf{U}(g)^{-1}$. Property (2) states that the operators $\mathbf{U}(g)$ are **strongly continuous** as functions of g.

A unitary representation of G on $\mathscr{H}$ is **irreducible** if there is no proper closed subspace of $\mathscr{H}$ which is invariant under all the operators $\mathbf{U}(g)$, i.e., if there is no proper closed subspace $\mathscr{H}'$ of $\mathscr{H}$ such that $\mathbf{U}(g)f \in \mathscr{H}'$ for all $f \in \mathscr{H}'$, $g \in G$.

Let $\mathscr{G}$ be the Lie algebra of G and $\exp \alpha$, $\alpha \in \mathscr{G}$, the exponential map from $\mathscr{G}$ to G. If the operators $\mathbf{U}(g)$ form a unitary representation of G, then in analogy with Section 1-3 we might try to construct a representation of $\mathscr{G}$ in terms of linear operators L_α on $\mathscr{H}$ defined by

$$L_\alpha f = \frac{d}{dt}\mathbf{U}(\exp t\alpha)f\Big|_{t=0} = \lim_{t\to 0}\left[\frac{\mathbf{U}(\exp t\alpha) - \mathbf{I}}{t}\right]f, \qquad f \in \mathscr{H}, \quad \alpha \in \mathscr{G}, \tag{3.44}$$

where the limit exists in the sense of the norm of $\mathscr{H}$. However, the limit may not exist for all vectors f so L_α may not be well defined as a linear operator on $\mathscr{H}$. Denote by $\mathscr{D}(L_\alpha)$ the set of vectors f for which this limit exists. Obviously, $\mathscr{D}(L_\alpha)$ is a subspace of $\mathscr{H}$.

Suppose there is a dense subspace $\mathscr{D}$ of $\mathscr{H}$ with the properties:

(1) $\mathscr{D} \subset \mathscr{D}(L_\alpha)$ for all $\alpha \in \mathscr{G}$,

(2) $\mathscr{D}$ is invariant under all the operators $\mathbf{U}(g)$,

(3) $\mathscr{D}$ is invariant under all the operators L_α, (3.45)

(4) For any $f \in \mathscr{D}$ the vector $\mathbf{U}(g)f$ is an analytic function on G.

We say a vector-valued function h of G on $\mathscr{H}$ is **analytic** at $g_o \in G$ if there exists a coordinate system $\{y_1, \ldots, y_m\}$ on a neighborhood W of g_o such that $y_1(g_o) = \cdots = y_m(g_o) = 0$ and

$$h(g) = \sum_{n_1,\ldots,n_m \geq 0} a_{n_1\cdots n_m} y_1(g)^{n_1} \cdots y_m(g)^{n_m}, \qquad g \in W.$$

Here, the coefficients $a_{n_1\cdots n_m}$ belong to $\mathscr{H}$ and $\sum_{n_1,\ldots,n_m \geq 0} | a_{n_1\cdots n_m} | < \infty$ (Helgason [1], p. 440).

If such a subspace $\mathscr{D}$ exists we can immediately carry over the standard methods in Chapter 1 relating Lie groups to Lie algebras. On the dense

subspace $\mathscr{D}$ Theorems 1.8–1.11 relating local multiplier representations to algebras of Lie derivatives have exact analogs. Thus, for $f \in \mathscr{D}$ we have

$$\begin{aligned}
&(1)\quad L_{a\alpha+b\beta}f = aL_\alpha f + bL_\beta f, \qquad a, b \in \mathscr{C}, \quad \alpha, \beta \in \mathscr{G},\\
&(2)\quad L_{[\alpha,\beta]}f = [L_\alpha, L_\beta]f = L_\alpha L_\beta f - L_\beta L_\alpha f,\\
&(3)\quad \mathbf{U}(\exp t\alpha)f = \sum_{n=0}^{\infty} \frac{(tL_\alpha)^n}{n!} f,\\
&(4)\quad \frac{d}{dt}\mathbf{U}(\exp t\alpha)f = L_\alpha(\mathbf{U}(\exp t\alpha)f).
\end{aligned} \tag{3.46}$$

These results follow from the fact that $\mathbf{U}(g)f$ is analytic in g (Naimark [2], Chapter 3). Convergence in (3) is in the sense of the norm.

Equations (3.46) show that on $\mathscr{D}$ the **infinitesimal operators** L_α completely determine the unitary operators $\mathbf{U}(g)$ for g in a neighborhood of the identity. However, since $\mathscr{D}$ is dense in $\mathscr{H}$ and $\mathbf{U}(g)$ is a bounded operator it follows from a standard argument that $\mathbf{U}(g)$ is uniquely determined on $\mathscr{H}$ for all g in the connected component of the identity element in G (Naimark [2], p. 100).

It can be shown that such a subspace $\mathscr{D}$ described above actually exists for every unitary representation of a Lie group G (see Helgason [1], p. 441, and references given there). Thus it is always possible to use Lie algebraic methods to derive information about (possibly infinite-dimensional) unitary representations of Lie groups. We will not need this result but quote it here to show that the methods used in the rest of this chapter are not as special as they might appear.

Lemma 3.1 If the operators $\mathbf{U}(g)$ form a unitary representation of G, then

$$\langle L_\alpha f, h\rangle = -\langle f, L_\alpha h\rangle \tag{3.47}$$

for all $f, h \in \mathscr{D}$ and all $\alpha \in \mathscr{G}$.

PROOF Since the operators $\mathbf{U}(g)$ are unitary and form a representation of G we have

$$\langle \mathbf{U}(\exp t\alpha)f, h\rangle = \langle f, \mathbf{U}(\exp - t\alpha)h\rangle$$

for all real t. Differentiating both sides of this equation with respect to t and evaluating at $t = 0$ we obtain the lemma.

Lemma 3.1 gives necessary conditions that operators L_α must satisfy to be obtained as the infinitesimal operators of a unitary representation of G.

3-7 Induced Representations of $\mathscr{E}_3$

The above results can now be applied to find unitary irreducible representations of E_3.

Let U be an irreducible unitary representation of E_3 on the Hilbert space $\mathscr{H}$ and let $\mathscr{D}$ be a dense subspace of $\mathscr{H}$ satisfying the properties (3.45). First of all, from (3.39) and (3.41) we find

$$g(\theta, x_1, x_2) = (\exp x_1 \mathscr{J}_1)(\exp x_2 \mathscr{J}_2)(\exp \theta \mathscr{J}_3). \tag{3.48}$$

Thus E_3 is uniquely determined by $\mathscr{J}_1, \mathscr{J}_2, \mathscr{J}_3$. Equations (3.46) then show that the operators $\mathbf{U}(g)$ for all $g \in E_3$ are uniquely determined by the infinitesimal operators L_α.

Second, if $\mathscr{D}' \subset \mathscr{D}$ is invariant under all the infinitesimal operators L_α of an irreducible unitary representation of E_3, then $\mathscr{D}'$ is dense in $\mathscr{D}$. For, if $\mathscr{D}'$ is invariant under the L_α, it is clear from (3.46) that $\bar{\mathscr{D}}'$ (the closure of $\mathscr{D}'$) is invariant under $\mathbf{U}(g)$ for all $g \in E_3$. Since the unitary representation is irreducible we have $\bar{\mathscr{D}}' = \mathscr{H}$. Thus, $\mathscr{D}'$ is dense in both $\mathscr{D}$ and $\mathscr{H}$. We can conclude that $\mathscr{D}$ contains no proper closed subspaces invariant under the L_α.

Define the infinitesimal operators J_k on $\mathscr{D}$ by

$$\mathrm{J}_k f = \frac{d}{dt} \mathbf{U}(\exp t \mathscr{J}_k) f \Big|_{t=0}, \qquad k = 1, 2, 3, \tag{3.49}$$

for all $f \in \mathscr{D}$. These operators satisfy the commutation relations

$$[\mathrm{J}_1, \mathrm{J}_2] = 0, \qquad [\mathrm{J}_3, \mathrm{J}_1] = \mathrm{J}_2, \qquad [\mathrm{J}_3, \mathrm{J}_2] = -\mathrm{J}_1$$

and determine a representation of $\mathscr{E}_3$ on $\mathscr{D}$. Therefore, the operators $\mathrm{J}^\pm = \mp \mathrm{J}_2 + i\mathrm{J}_1$, $\mathrm{J}^3 = i\mathrm{J}_3$ satisfy the relations

$$[\mathrm{J}^+, \mathrm{J}^-] = 0, \qquad [\mathrm{J}^3, \mathrm{J}^+] = \mathrm{J}^+, \qquad [\mathrm{J}^3, \mathrm{J}^-] = -\mathrm{J}^-.$$

Clearly, $\mathrm{J}^\pm$, J^3 induce a representation ρ of the complex Lie algebra $\mathscr{T}_3$ on $\mathscr{D}$. We will investigate which of the irreducible representations $Q(\omega, m_o)$ of $\mathscr{T}_3$ can be obtained in this way on some dense subspace $\mathscr{D}'$ of $\mathscr{D}$, i.e., the conditions under which ρ restricted to $\mathscr{D}'$ is isomorphic to $Q(\omega, m_o)$.

According to Lemma 3.1,

$$\langle \mathrm{J}_k f, h \rangle = -\langle f, \mathrm{J}_k h \rangle, \qquad k = 1, 2, 3,$$

for all $f, h \in \mathscr{D}$. Thus,

$$\langle J^3 f, h\rangle = -i\langle J_3 f, h\rangle = +i\langle f, J_3 h\rangle = \langle f, J^3 h\rangle \tag{3.50}$$

$$\langle J^+ f, h\rangle = \langle(-J_2 + iJ_1) f, h\rangle = \langle f, (J_2 + iJ_1)h\rangle = \langle f, J^- h\rangle. \tag{3.51}$$

Recall that the representation $Q(\omega, m_o)$ of $\mathscr{T}_3$ was determined by the relations

$$J^3 f_m = m f_m, \qquad J^{\pm} f_m = \omega f_{m\pm 1}$$

where $\omega \neq 0$, $m = m_o + k$, and k runs over the integers. Now we assume that the vectors f_m are in $\mathscr{D}$ and use (3.50) and (3.51) to find restrictions on ω and m_o. From (3.50),

$$\bar{m}\langle f_m, f_n\rangle = \langle J^3 f_m, f_n\rangle = \langle f_m, J^3 f_n\rangle = n\langle f_m, f_n\rangle.$$

Thus $(\bar{m} - n)\langle f_m, f_n\rangle = 0$ for all m, n in the spectrum of J^3. If $m \neq n$ this relation proves $\langle f_m, f_n\rangle = 0$. However, if $m = n$ we have $(\bar{m} - m)|f_m|^2 = 0$. Since $f_m \neq 0$, the eigenvalue m must be real. Further, from (3.46) we find

$$\mathbf{U}(\exp \theta \mathscr{J}_3) f_m = \exp(-i\theta J^3) f_m = e^{-im\theta} f_m.$$

Since $\exp(2\pi \mathscr{J}_3)$ is the identity element of E_3, we must have $e^{-2\pi i m} = 1$ for all m in the spectrum of J^3. Thus, m must be an integer and $m_o = 0$.

From (3.51)

$$\bar{\omega}\langle f_{m+1}, f_{m+1}\rangle = \langle J^+ f_m, f_{m+1}\rangle = \langle f_m, J^- f_{m+1}\rangle = \omega\langle f_m, f_m\rangle$$

for all integers m. Hence, $\omega/\bar{\omega} = |f_{m+1}|^2/|f_m|^2 > 0$. This relation can be satisfied only if ω is real and $|f_{m+1}| = |f_m|$. Consequently, all the vectors f_m have the same length and without loss of generality we can assume $|f_m| = 1$ for all integers m. This analysis proves that any irreducible representation of $\mathscr{T}_3$ induced by a unitary representation of E_3 must be of the form $Q(\omega, 0)$ where ω is a nonzero real number. Furthermore, the vectors f_m, m an integer, form an orthonormal basis for $\mathscr{D}$ (hence, for $\mathscr{H}$):

$$\langle f_m, f_n\rangle = \delta_{m,n} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

3-8 The Unitary Representations (ρ) of E_3

Conversely, we will show by explicit computation that, in fact, all of the representations $Q(\omega, 0)$, ω real, induce unitary irreducible representations of E_3. (Since $Q(\omega, 0) \cong Q(-\omega, 0)$, without loss of generality we

can assume $\omega < 0$.) The unitary representations of E_3 can be obtained formally through an examination of the multiplier representation (3.6) of T_3. If $m_o = 0$, (3.6) depends on the parameter τ only in the form e^τ (if $m_o \neq 0$ this is not true):

$$[\mathbf{A}(g)f](z) = \exp\left(\omega bz + \frac{\omega c}{z}\right) f(e^\tau z), \qquad f \in \mathcal{A}_1. \tag{3.6$'$}$$

Thus, the operators (3.6)′ define a single-valued multiplier representation of the multiplicative matrix group T_3' with elements

$$\tilde{g}(\tau, b, c) = \begin{pmatrix} e^{-\tau} & 0 & c \\ 0 & e^\tau & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad b, c, \tau \in \mathbb{C}. \tag{3.52}$$

The matrices (3.52) are obtained from the matrices (3.4) of T_3 by eliminating the first row and last column. T_3' is not simply connected. However, as local Lie groups, T_3' and T_3 are isomorphic.

Comparing (3.52) and (3.42) we can consider E_3 as the real subgroup of T_3' consisting of the matrices (3.52) such that $\operatorname{Re}\tau = 0$ and $b = -\bar{c}$. In fact,

$$g[\theta, r, \varphi] = \tilde{g}\left(-i\theta, -\frac{ire^{-i\varphi}}{2}, -\frac{ire^{+i\varphi}}{2}\right)$$

$$= \begin{pmatrix} e^{+i\theta} & 0 & -\dfrac{ire^{+i\varphi}}{2} \\ 0 & e^{-i\theta} & -\dfrac{ire^{-i\varphi}}{2} \\ 0 & 0 & 1 \end{pmatrix},$$

$$0 \leqslant \theta, \quad \varphi < 2\pi, \quad r \geqslant 0. \tag{3.53}$$

Using this embedding of E_3 in T_3' we can define a multiplier representation of E_3 directly from (3.6)′ by restricting the group elements to E_3:

$$[\mathbf{A}(g')f](z) = \exp\left[-\frac{i\omega}{2}\left(re^{-i\varphi}z + \frac{re^{+i\varphi}}{z}\right)\right] f(e^{-i\theta}z), \qquad g'[\theta, r, \varphi] \in E_3. \tag{3.54}$$

Since $|e^{i\theta}z| = |z|$ for all θ, the representation (3.54) can be restricted to functions defined on the unit circle: $z = e^{i\alpha}$, $0 \leqslant \alpha < 2\pi$. The basis functions $f_m(z) = z^m$ restrict to $f_m(\alpha) = e^{im\alpha}$ for all integers m.

Now we have the ingredients required to define irreducible unitary representations of E_3. Let $\mathcal{H}$ be the complex Hilbert space consisting of all functions $f(\alpha)$, $0 \leqslant \alpha < 2\pi$, mod 2π, such that

$$\int_0^{2\pi} |f(\alpha)|^2 \, d\alpha < \infty.$$

The inner product on $\mathscr{H}$ is

$$\langle f, h\rangle = (2\pi)^{-1}\int_0^{2\pi} \overline{f(\alpha)}\, h(\alpha)\, d\alpha, \qquad f, h \in \mathscr{H}. \tag{3.55}$$

As is well known, the functions

$$f_m(\alpha) = e^{im\alpha}, \qquad m = 0, \pm 1, \pm 2, \ldots,$$

form an orthonormal basis for $\mathscr{H}$,

$$\langle f_m, f_n\rangle = \delta_{m,n}.$$

We define a representation (ρ), $\rho > 0$, of E_3 by unitary operators $\mathbf{U}(g)$ on $\mathscr{H}$ such that

$$[\mathbf{U}(g)f](\alpha) = e^{i\rho r\cos(\alpha-\varphi)} f(\alpha - \theta) \tag{3.56}$$

where $f \in \mathscr{H}$ and $g = g[\theta, r, \varphi] \in E_3$. The expression for $\mathbf{U}(g)$ follows immediately from (3.54) by setting $z = e^{i\alpha}$, $\rho = -\omega$. The operators $\mathbf{U}(g)$ are unitary since

$$\begin{aligned}\langle \mathbf{U}(g)f, \mathbf{U}(g)h\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \overline{f(\alpha-\theta)}\, h(\alpha-\theta)\, d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi} \overline{f(\alpha)}\, h(\alpha)\, d\alpha = \langle f, h\rangle,\end{aligned}$$

for all $f, h \in \mathscr{H}$, $g \in E_3$. We define the **matrix elements** of (ρ) with respect to the basis f_m by

$$\begin{aligned}U_{nm}(g) &= \langle f_n, \mathbf{U}(g) f_m\rangle \\ &= \frac{1}{2\pi}\int_0^{2\pi} \exp[+i\rho r\cos(\alpha-\varphi) - im\theta + i(m-n)\alpha]\, d\alpha \\ &= i^{n-m} e^{i[(m-n)\varphi - m\theta]} J_{n-m}(\rho r), \qquad -\infty < n, \quad m < \infty,\end{aligned} \tag{3.57}$$

where the J_m are Bessel functions of integral order. To justify the last equality note that for $z = ie^{i\alpha}$, Eq. (3.15) becomes

$$e^{ir\cos\alpha} = \sum_{l=-\infty}^{\infty} i^l J_l(r)\, e^{il\alpha}.$$

Thus the exponential functions under the integral sign in (3.57) can be expanded in a series of Bessel functions and integrated term by term to obtain the indicated result. Note the similarity between (3.14) and (3.57).

By construction, the operators $\mathbf{U}(g)$ satisfy the representation property

$$\mathbf{U}(gg') = \mathbf{U}(g)\,\mathbf{U}(g') \tag{3.58}$$

for all $g, g' \in E_3$. However, we can also verify this directly. In terms of the coordinates (θ, x_1, x_2) for g, (3.56) becomes

$$[\mathbf{U}(g)f](\alpha) = \exp[i\rho(x_1 \cos\alpha + x_2 \sin\alpha)]f(\alpha - \theta).$$

Thus, if the group elements g and g' have coordinates (θ, x_1, x_2), (θ', x_1', x_2'), respectively, we find

$$\begin{aligned}
\mathbf{U}(g)[\mathbf{U}(g')f](\alpha) &= \exp[i\rho(x_1 \cos\alpha + x_2 \sin\alpha)][\mathbf{U}(g')f](\alpha - \theta) \\
&= \exp\{i\rho[(x_1 \cos\alpha + x_2 \sin\alpha) + (x_1' \cos(\alpha - \theta) \\
&\quad + x_2' \sin(\alpha - \theta))]\} f(\alpha - \theta - \theta') \\
&= \exp\{i\rho[(x_1' \cos\theta - x_2' \sin\theta + x_1) \cos\alpha \\
&\quad + (x_1' \sin\theta + x_2' \cos\theta + x_2) \sin\alpha]\} f(\alpha - \theta - \theta') \\
&= [\mathbf{U}(gg')f](\alpha) \qquad \text{for all} \quad f \in \mathscr{H},
\end{aligned}$$

since the coordinates of gg' are

$$(\theta + \theta', x_1' \cos\theta - x_2' \sin\theta + x_1, x_1' \sin\theta + x_2' \cos\theta + x_2).$$

This proves that (ρ) is indeed a unitary representation of E_3. The methods introduced in Section 3-7 could also be used to show that (ρ) is irreducible. However, we will give a direct proof of this fact.

Lemma 3.2 The representation (ρ), $\rho > 0$, is irreducible.

PROOF We assume (ρ) is reducible and obtain a contradiction. Thus, suppose there exists a proper closed subspace $\mathscr{S}$ of $\mathscr{H}$ such that $\mathbf{U}(g)f \in \mathscr{S}$ for all $g \in E_3$, $f \in \mathscr{S}$. Let $\mathbf{P}$ be the self-adjoint projection operator on $\mathscr{S}$. That is, $\mathbf{P}$ is the operator on $\mathscr{H}$ uniquely defined by the conditions (a) $\mathbf{P}f = f$ for all $f \in \mathscr{S}$; and (b) $\mathbf{P}h = 0$ for all $h \in \mathscr{S}^\perp$ where

$$\mathscr{S}^\perp = \{h \in \mathscr{H} : \langle h, f\rangle = 0 \quad \text{for all} \quad f \in \mathscr{S}\}.$$

From elementary considerations in functional analysis it follows that (i) $\langle \mathbf{P}h', h''\rangle = \langle h', \mathbf{P}h''\rangle$ for all $h', h'' \in \mathscr{H}$, i.e., $\mathbf{P}$ is self-adjoint; (ii) $\mathbf{P}^2 = \mathbf{P}$; and (iii) $\mathbf{U}(g)\mathbf{P} = \mathbf{P}\mathbf{U}(g)$ for all $g \in E_3$ (see Naimark [1], Chapters 4, 6). Furthermore $\mathbf{P} \neq \mathbf{0}, \mathbf{I}$, since $\mathscr{S}$ is a proper subspace of $\mathscr{H}$.

Let g_θ be the element of E_3 with coordinates $(\theta, 0, 0)$. Then $\mathbf{U}(g_\theta) f_m = e^{-im\theta} f_m$ where f_m is a basis vector for $\mathscr{H}$. From (ii) we obtain $\mathbf{U}(g_\theta)\mathbf{P}f_m =$

$e^{-im\theta}\mathbf{P}f_m$, for all integers m. Since the vectors f_m form a basis for $\mathscr{H}$ we must have $\mathbf{P}f_m = \alpha_m f_m$, α_m a constant. The property $\mathbf{P}^2 = \mathbf{P}$ implies $\alpha_m^2 = \alpha_m$, hence $\alpha_m = 0$ or $\alpha_m = 1$ for each integer m. By hypothesis there exist nonnegative integers m', m'' such that $\mathbf{P}f_{m'} = 0$, $\mathbf{P}f_{m''} = f_{m''}$. Thus, $U_{m'm''}(g) = \langle f_{m'}, \mathbf{U}(g)\mathbf{P}f_{m''}\rangle = \langle f_{m'}, \mathbf{P}\mathbf{U}(g)f_{m''}\rangle = \langle \mathbf{P}f_{m'}, \mathbf{U}(g)f_{m''}\rangle = 0$ for all $g \in E_3$. However, from (3.57) we see that no matrix element $U_{nm}(g)$ of (ρ) is identically zero. This contradiction proves the lemma.

3-9 The Matrix Elements of (ρ)

We can obtain information about Bessel functions of integral order from expression (3.57) for the matrix elements of (ρ). Since the operators $\mathbf{U}(g)$ are unitary and form a representation of E_3, the matrix elements satisfy

$$U_{nm}(g^{-1}) = \overline{U_{mn}(g)},$$

or

$$J_{-n}(r) = (-1)^n J_n(r). \tag{3.59}$$

Moreover, $U_{nm}(\mathbf{e}) = \delta_{n,m}$ where $\mathbf{e}$ is the identity element of E_3. In terms of Bessel functions this implies

$$J_n(0) = 0 \quad \text{if} \quad n \neq 0, \qquad J_0(0) = 1. \tag{3.60}$$

Using the Schwarz inequality, we have

$$|U_{nm}(g)| = |\langle f_n, \mathbf{U}(g)f_m\rangle| \leqslant |f_n| \cdot |\mathbf{U}(g)f_m| = |f_n| \cdot |f_m| = 1$$

so $|J_n(r)| \leqslant 1$.

Since $\mathbf{U}(g'g'') = \mathbf{U}(g')\,\mathbf{U}(g'')$ for all $g', g'' \in E_3$ the matrix elements satisfy the addition theorem

$$U_{nm}(g'g'') = \sum_{k=-\infty}^{\infty} U_{nk}(g')\,U_{km}(g''). \tag{3.61}$$

Substituting expressions (3.57) for the matrix elements and simplifying we obtain

$$e^{-im\varphi}J_m(r) = \sum_{k=-\infty}^{\infty} e^{-i[k\varphi''+(m-k)\varphi']}\,J_{m-k}(r')\,J_k(r'') \tag{3.62}$$

where $re^{i\varphi} = r'e^{i\varphi'} + r''e^{i\varphi''}$. This is Graf's addition theorem, a special case of (3.32).

If $r' = r'' = x$, $\varphi' = 0$, and $\varphi'' = \pi$, then $r = 0$, $\varphi = 0$, and

$$\sum_{k=-\infty}^{\infty} J_{k+n}(x)\, J_k(x) = \delta_{n,0}$$

where we have used (3.59) and (3.60).

If $\varphi' = \varphi'' = 0$ then $\varphi = 0$, $r = r' + r''$, and (3.62) becomes

$$J_m(r' + r'') = \sum_{k=-\infty}^{\infty} J_{m-k}(r')\, J_k(r'').$$

Finally, we can derive a useful integral formula for the product of two Bessel functions from (3.62) by setting $\varphi' = 0$ and multiplying by $e^{in\varphi''}$. Integrating both sides of the resulting equation with respect to φ'' we obtain

$$J_{m-n}(r')\, J_n(r'') = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n\varphi'' - m\varphi)}\, J_m(r)\, d\varphi'' \tag{3.63}$$

where $re^{i\varphi} = r' + r''e^{i\varphi''}$.

3-10 The Infinitesimal Operators of (ρ)

The infinitesimal operators of the representation (ρ) defined by (3.49) are easily computed to be

$$\mathrm{J}_1 = i\rho \cos\alpha, \qquad \mathrm{J}_2 = i\rho \sin\alpha, \qquad \mathrm{J}_3 = -\frac{\partial}{\partial\alpha}. \tag{3.64}$$

These operators are all defined on any dense subspace $\mathscr{D}$ of $\mathscr{H}$ satisfying properties (3.46) and they leave $\mathscr{D}$ invariant. Forming the operators J^+, J^-, J^3 in the usual manner, we have

$$\mathrm{J}^+ = \omega e^{i\alpha}, \qquad \mathrm{J}^- = \omega e^{-i\alpha}, \qquad \mathrm{J}^3 = -i\frac{\partial}{\partial\alpha}, \qquad \omega = -\rho.$$

The action of these operators on the analytic basis vectors $f_m(\alpha) = e^{im\alpha}$ is given by

$$\mathrm{J}^+ f_m = \omega f_{m+1}, \qquad \mathrm{J}^- f_m = \omega f_{m-1}, \qquad \mathrm{J}^3 f_m = m f_m$$

in agreement with Eqs. (3.1) for the representation $Q(\omega, 0)$ of $\mathscr{T}_3$. Thus, each representation $Q(\omega, 0)$, $\omega < 0$, induces an irreducible unitary representation $(-\omega)$ of E_3.

As in Section 3-2 we could derive recursion relations and differential equations for the matrix elements of (ρ). However, we shall omit this as the results are merely special cases of the corresponding results in Section 3-2. Similarly the addition theorem (3.62) is a special case of the addition theorem (3.16).

From the general theory of Lie groups it can be shown that up to unitary equivalence all of the irreducible unitary representations of E_3 are of the form (ρ), $\rho > 0$, except for the trivial 1-dimensional representations χ_n, n an integer, defined by

$$\chi_n[g(\theta, x_1, x_2)] = e^{in\theta}$$

(see Bingen [1], Vilenkin [1, 3]). Thus, the Lie algebraic methods presented in this chapter suffice to find all the irreducible unitary representations of E_3. The reason for this is essentially topological: The 1-parameter subgroup $C = \{\exp \theta \mathscr{J}_3\}$ of E_3 is compact. Because C is compact, the Peter–Weyl theorem guarantees that any unitary representation U of E_3 when restricted to C can be decomposed into a direct sum of 1-dimensional irreducible representations of C. Since the 1-dimensional representations of C are of the form $\mu_n(\exp \theta \mathscr{J}_3) = e^{in\theta}$, n an integer, it is always possible to find a basis for the representation space of U consisting of eigenvectors of the operator $\mathrm{J}_3 = -i\mathrm{J}^3$. If U is irreducible it is not difficult to show that each eigenvalue has multiplicity one. Thus, the hypotheses (2.19) of Section 2-6 are satisfied and our methods succeed in determining all unitary irreducible representations of E_3.

Another real Lie group whose Lie algebra is a real form of $\mathscr{T}_3$ is T_3^R. The elements g of T_3^R are those matrices of T_3 which are real:

$$g = \begin{pmatrix} 1 & 0 & 0 & \tau \\ 0 & e^{-\tau} & 0 & c \\ 0 & 0 & e^{\tau} & b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad b, c, \tau \in R.$$

In contrast to E_3, the group T_3^R has no compact 1-parameter subgroups. In this case the Peter–Weyl theorem will not help and, as the reader can verify for himself from Lemma 2.1, none of the representations $Q(\omega, m_o)$ of $\mathscr{T}_3$ induce unitary representations of T_3^R. The importance of T_3^R in special function theory is related to the study of integral transforms and is beyond the scope of this book (see Vilenkin [3]).