CHAPTER **2**

Representations and Realizations of Lie Algebras

In Chapter 2 we establish a fundamental relationship between Lie groups and certain special functions: Special functions appear as basis vectors and matrix elements corresponding to local multiplier representations of Lie groups. The correspondence is sketched in Section 2-2, then illustrated by the familiar example of angular momentum operators and spherical harmonics.

The 4-dimensional Lie algebras $\mathscr{G}(a, b)$ are introduced in Section 2-5. Later, it will be shown that the representation theory of $\mathscr{G}(a, b)$ corresponds to a study of special functions of hypergeometric type. As preliminary material toward the demonstration of this fact we classify the abstract irreducible representations of $\mathscr{G}(a, b)$ in Section 2-6, and construct realizations of $\mathscr{G}(a, b)$ by means of generalized Lie derivatives in one and two complex variables in Sections 2-7 and 2-8.

2-1 Representations of Lie Algebras

In this section the concept of a **representation** of a Lie algebra on an abstract vector space will be introduced. It is assumed that the reader is familiar with the basic definitions and results of the theory of vector spaces (Halmos [1], Dunford and Schwartz [1]). However, since most elementary textbooks on linear algebra are concerned primarily with **finite**-dimensional vector spaces it may be useful to review some of the fundamental definitions in a wide enough context to apply to **infinite**-dimensional spaces. Infinite-dimensional abstract vector spaces will appear frequently in this book.

Let V be a vector space over the field F. (Here, F will always be either the real numbers R, or the complex numbers, $\mathcal{C}$.) We denote the additive identity of V by θ and the additive identity of F by 0. A subset S of V is said to be **linearly independent** over F if for every finite subset $v_1, v_2, \ldots, v_n$ of distinct elements of S and every sequence $(a_1, a_2, \ldots, a_n)$ of elements of F the equality $\sum_{i=1}^n a_i v_i = \theta$ implies $a_1 = a_2 = \cdots = a_n = 0$. A nonvoid linearly independent subset B is called a **basis** of V if it is maximal, i.e., if B is not a proper subset of some linearly independent set in V. It can be shown using Zorn's lemma that every vector space with at least two elements contains a basis (Dunford and Schwartz [1], Chapter 1). Furthermore, if B is a basis for V then any $v \in V$ can be written uniquely as a finite linear combination of elements of B:

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n, \qquad a_1, \ldots, a_n \in F, \quad v_1, \ldots, v_n \in B.$$

The cardinality of a basis depends only on V and is called the **dimension** of V. If B contains m elements, V is ***m*-dimensional**. If B contains an infinite number of elements, V is **infinite-dimensional**. (Similarly, if $\mathcal{G}$ is a Lie algebra over F a **basis** of $\mathcal{G}$ is a subset $\mathcal{S}$ which is a basis of $\mathcal{G}$ considered as a vector space. The **dimension** of $\mathcal{G}$ is the cardinality of a basis.)

A **linear operator** on V is a mapping $T\colon V \to V$ such that

$$T(a_1 v_1 + a_2 v_2) = a_1 T(v_1) + a_2 T(v_2) \in V, \qquad a_1, a_2 \in F, \quad v_1, v_2 \in V.$$

The **product** $T_1 T_2$ of two linear operators T_1 and T_2 on V is the linear operator defined by $(T_1 T_2)(v) = T_1(T_2(v))$ for $v \in V$. The **sum** $T_1 + T_2$ is the linear operator defined by $(T_1 + T_2)(v) = T_1(v) + T_2(v)$. If $a \in F$ and T is a linear operator we define the **scalar multiple** aT of T by $(aT)(v) = aT(v)$ for all $v \in V$. The set of all linear operators on V with the operations of product, sum, and scalar multiple forms an algebra. Moreover, if we define the **commutator** $[T_1, T_2]$ of the linear operators T_1, T_2 by $[T_1, T_2] = T_1 T_2 - T_2 T_1$, the operators on V together with the commutator $[.,.]$ form a Lie algebra $\mathcal{L}(V)$.

Let $\mathcal{G}$ be a Lie algebra over F.

Definition A **representation** of $\mathcal{G}$ on V is a homomorphism $\rho: \mathcal{G} \to \mathcal{L}(V)$. That is, ρ satisfies the conditions

(1) $\rho(\alpha) \in \mathcal{L}(V)$ for all $\alpha \in \mathcal{G}$,
(2) $\rho([\alpha, \beta]) = [\rho(\alpha), \rho(\beta)]$,
(3) $\rho(a\alpha + b\beta) = a\rho(\alpha) + b\rho(\beta)$, $a, b \in F$, $\alpha, \beta \in \mathcal{G}$.

A subspace W of V is said to be **invariant** under ρ if $\rho(\alpha)w \in W$ for all $\alpha \in \mathcal{G}$, $w \in W$. A representation ρ of $\mathcal{G}$ on V is **irreducible** if there is no proper subspace W of V which is invariant under ρ.

Two vector spaces V, V' over F are **isomorphic** if there exists a mapping μ of V onto V' such that

(1) $\mu(a_1 v_1 + a_2 v_2) = a_1\mu(v_1) + a_2\mu(v_2)$ for all $a_1, a_2 \in F$, $v_1, v_2 \in V$.
(2) If $\mu(v) = \theta'$ then $v = \theta$.
(3) For every $v' \in V'$ there exists a $v \in V$ such that $\mu(v) = v'$.

Let ρ, ρ' be representations of $\mathcal{G}$ on V, V', respectively. ρ and ρ' are said to be **isomorphic** if there exists an isomorphism μ of V and V' such that $\mu(\rho(\alpha)v) = \rho'(\alpha)\mu(v)$ for all $\alpha \in \mathcal{G}$, $v \in V$.

2-2 Realizations of Representations

Let G be a local Lie group with Lie algebra $L(G)$ and suppose ρ is a representation of $L(G)$ on the abstract vector space V. In analogy with the theory of local transformation groups presented in Chapter 1, it would seem natural to construct a mapping $\hat{\rho}$ of G into $\mathcal{L}(V)$ as follows:

$$\hat{\rho}(\exp t\alpha)v = e^{t\rho(\alpha)}v = \sum_{l=0}^{\infty} \frac{t^l \rho(\alpha)^l}{l!} v \tag{2.1}$$

where $\alpha \in L(G)$, $v \in V$, and $t \in \mathcal{C}$. Unfortunately, the right-hand side of (2.1) may not be defined as an element of V since it involves an infinite linear combination of vectors. Thus, expression (2.1) may be meaningless and the analogy between Lie algebras and local Lie groups breaks down.

However, suppose ρ satisfies

CONDITION (A) V can be realized as a vector space whose elements are functions analytic in a neighborhood of some point $\mathbf{z}^o \in \mathcal{C}^m$ and such that the operators $\rho(\alpha)$, $\alpha \in L(G)$ are differential operators analytic at $\mathbf{z}^o$.

To be more precise, let $\mathcal{O}_{\mathbf{z}^o}$ be the set of all functions analytic in some neighborhood of $\mathbf{z}^o$ (the **germs** of functions at $\mathbf{z}^o$, see Gunning and Rossi [1], Chapter 2). $\mathcal{O}_{\mathbf{z}^o}$ can be given the structure of a complex vector space in the usual manner. Assume the existence of an isomorphism μ of the abstract vector space V onto a subspace $\mathscr{V}$ of $\mathcal{O}_{\mathbf{z}^o}$. Then, the operators $\rho^\mu(\alpha)$ on $\mathscr{V}$ induced by the operators $\rho(\alpha)$, $\alpha \in L(G)$ on V ($\rho^\mu(\alpha)\mu(v) = \mu[\rho(\alpha)v]$, $v \in V$) obviously define a representation of $L(G)$. Second, assume there exists a representation of $L(G)$ by generalized Lie derivatives D_α on $\mathcal{O}_{\mathbf{z}^o}$ such that $D_\alpha = \rho^\mu(\alpha)$ on $\mathscr{V}$ for all $\alpha \in L(G)$. If these two hypotheses are satisfied the vector space V can be considered as a subspace of $\mathcal{O}_{\mathbf{z}^o}$ and the operators $\rho(\alpha)$ can be identified with the generalized Lie derivatives D_α. Under these circumstances Theorem 1.10 (or a slight modification of it; see Section 8-1) states that the representation ρ of $L(G)$ induces a multiplier representation T^ν of G on $\mathcal{O}_{\mathbf{z}^o}$. In particular, Eq. (2.1) is now well defined since the right-hand side of the equation is an infinite sum of analytic functions which converges to an element of $\mathcal{O}_{\mathbf{z}^o}$ for $|t|$ sufficiently small. (Here, the topology on $\mathcal{O}_{\mathbf{z}^o}$ is the usual one of uniform convergence on compact sets of $\mathcal{C}^m$.) According to Theorem 1.10, $\hat{\rho}(\exp t\alpha)v = \mathbf{T}^\nu(\exp t\alpha)v$ for all $\alpha \in L(G)$, $v \in \mathscr{V}$, and $|t|$ sufficiently small.

As noted above, $\mathscr{V}$ is invariant under the operators D_α but not necessarily invariant under the operators $\mathbf{T}^\nu(g)$, $g \in G$. To remedy this we extend $\mathscr{V} \subset \mathcal{O}_{\mathbf{z}^o}$ to a larger subspace $\bar{\mathscr{V}}$ which is invariant under T^ν. Namely, we define $\bar{\mathscr{V}}$ to be the intersection of all subspaces $\mathscr{W}$ of $\mathcal{O}_{\mathbf{z}^o}$ which obey the conditions: $\mathscr{W} \supset \mathscr{V}$ and $\mathbf{T}^\nu(g)w \in \mathscr{W}$ for all $g \in G$, $w \in \mathscr{W}$ such that $\mathbf{T}^\nu(g)w$ is defined. Clearly, $\bar{\mathscr{V}}$ is the smallest subspace of $\mathcal{O}_{\mathbf{z}^o}$ which contains $\mathscr{V}$ and which is invariant under T^ν. The elements of $\bar{\mathscr{V}}$ are just the finite sums of elements of the form $\mathbf{T}^\nu(g)v$, where $g \in G$, $v \in \mathscr{V}$.

Consequently, if condition (A) is satisfied a representation ρ of $L(G)$ will induce a multiplier representation T^ν of G which leaves $\bar{\mathscr{V}}$ invariant. In general, T^ν depends critically on the dimension m of $\mathcal{C}^m$ and on the choice of $\mathscr{V}$. To preserve the one-to-one correspondence between local Lie groups and Lie algebras, we need to find conditions which guarantee that the action of T^ν on $\bar{\mathscr{V}}$ is in some way uniquely determined by ρ.

If $v_1, \ldots, v_k, \ldots$ is a basis for V, the functions $\mu(v_1)$, $\mu(v_2), \ldots, \mu(v_k), \ldots$ form a basis for $\mathscr{V}$. We say that the $\{\mu(v_k)\}$ form an **analytic basis** for $\bar{\mathscr{V}}$ if every element u of $\bar{\mathscr{V}}$ can be expressed uniquely as a countable linear combination of the basis functions $\mu(v_k)$, uniformly convergent in some neighborhood of $\mathbf{z}^o$. Furthermore, it is required that the coefficients in this expansion be bounded linear functionals of the argument u (in the topology of uniform convergence on compact sets), i.e., if $u = \sum_k c_k(u)\mu(v_k)$ then $c_k(u) \to 0$ as $u \to 0$.

From Eq. (2.1) we obtain

$$\mathbf{T}^{\nu}(\exp t\alpha)\,\mu(v_k) = \sum_{l=0}^{\infty} \frac{t^l D_\alpha^l}{l!}\,\mu(v_k) = \sum_{l=0}^{\infty} \frac{t^l}{l!}\,\mu(\rho(\alpha)^l v_k),$$
$$\alpha \in L(G), \quad k = 1, 2, \ldots, \tag{2.2}$$

for $|\,t\,|$ sufficiently small. If $\{\mu(v_k)\}$ is an analytic basis the right-hand side of (2.2) converges to an element of $\mathscr{V}$ which is a (possibly infinite) linear combination of the functions $\mu(v_k)$. In terms of this basis the **matrix elements** $T_{lk}(\exp t\alpha)$ are uniquely defined by

$$\mathbf{T}^{\nu}(\exp t\alpha)\,\mu(v_k) = \sum_{l=1}^{\infty} T_{lk}(\exp t\alpha)\,\mu(v_l), \qquad k = 1, 2, \ldots\,. \tag{2.3}$$

In fact, Eqs. (2.2) show that the matrix elements are uniquely determined by ρ for g in a sufficiently small neighborhood of $\mathbf{e} \in G$ and are independent of the choice of $\mathscr{C}^m$ and $\mathscr{V}$.

Thus, restricting ourselves to vector spaces $\mathscr{V}$ of analytic functions which satisfy

CONDITION (B) $\mu(v_1), \mu(v_2), \ldots, \mu(v_k), \ldots,$ is an analytic basis for $\mathscr{V}$,

we find that the matrix elements of the operators $\mathbf{T}^{\nu}(g)$ depend only on ρ and the chosen basis of V; not on $\mathscr{V}$.

Let $G, \rho, V, \{v_k\}, \mu, \mathscr{V}$ be given such that conditions (A) and (B) are satisfied. Clearly, $\mu(v_1), \mu(v_2), \ldots,$ is an analytic basis for $\mathscr{V}$. Using this analytic basis and (2.3) we can easily derive the following **addition theorem** for the matrix elements:

$$T_{lk}(g_1 g_2) = \sum_{j=1}^{\infty} T_{lj}(g_1)\,T_{jk}(g_2), \qquad l, k = 1, 2, \ldots, \tag{2.4}$$

defined for g_1, g_2 in a suitably small neighborhood of $\mathbf{e}$. The relation

$$\mathbf{T}^{\nu}(g)\,\mu(v_k) = \sum_{l=1}^{\infty} T_{lk}(g)\,\mu(v_l), \qquad g \in G, \tag{2.3$'$}$$

can be regarded as a **generating function** for the matrix elements $T_{lk}(g)$ and the basis functions $\mu(v_l)$. Finally, the equations

$$D_\alpha \mu(v_k) = \mu(\rho(\alpha)\,v_k), \qquad \alpha \in L(G), \quad k = 1, 2, \ldots, \tag{2.5}$$

can be interpreted as **differential recursion relations** for the functions $\mu(v_k)$.

The connection between these results and special function theory is now apparent. The matrix elements $T_{lk}(g)$ are functions on the local group G and, if we make the proper choices for G and a basis of V, they will turn out to be familiar special functions. Moreover, the analytic functions $\mu(v_k)$ will often be expressible as special functions. In this case Eqs. (2.4), (2.3)$'$, and (2.5) constitute addition theorems, generating functions, and recursion relations for the special functions $T_{lk}(g)$ and $\mu(v_k)$.

I claim that a significant portion of special function theory is contained in these three sets of equations. Much of the remainder of this book will be devoted to the documentation of this claim by means of explicit computations. (Recall Section 1-4 where it was shown that the Jacobi polynomials appear as matrix elements of local multiplier representations of $SL(2)$.)

To systematically study the special functions related to a given Lie algebra $L(G)$ we could proceed as follows:

(1) Classify "all" representations ρ of $L(G)$.

This is primarily an algebraic problem. In order to obtain practical results it will ordinarily be necessary to study only those representations which possess certain convenient properties. In particular, we shall classify only the irreducible representations of $L(G)$.

(2) Classify "all" realizations of $L(G)$ by generalized Lie derivatives.

This is a problem in classical Lie theory which will be studied in Chapter 8. The basic difficulty here is how to decide when two realizations of $L(G)$ by generalized Lie derivatives are "essentially" the same.

(3) For each representation ρ of $L(G)$ and each realization of $L(G)$ by generalized Lie derivatives D_α, find a realization $\mathscr{V}$ of V such that conditions (A) and (B) are satisfied.

(4) Choose a "suitable" basis in V and compute the matrix elements $T_{lk}(g)$ and basis functions $\mu(v_k)$. For the low-dimensional Lie algebras considered in this book it will ordinarily be clear which basis in V should be chosen.

The above description of a relation between Lie algebras and special functions is general and heuristic. To show that this procedure leads to practical methods for uncovering the group structure of special functions we will henceforth be very specific. In particular, we will examine in detail the special functions associated with the Lie algebras $L(SL(2))$, $L(G(0, 1))$, and $L(T_3)$. The special functions will turn out to be hypergeometric functions, confluent hypergeometric functions, and Bessel functions. These familiar functions will occur both as matrix elements and as basis vectors for irreducible representations of the above Lie algebras.

2-3 Representations of $L(O_3)$

Perhaps the best known example of the relation between Lie groups and special functions presented in Section 2-2 is the connection between the rotation group and spherical harmonics. Although this example is treated in most books on quantum mechanics, it may be instructive to examine it again to see how it fits into the general framework of the last section. It is a remarkable fact that essentially the same methods which are used to derive the fundamental properties of spherical harmonics from the representation theory of the rotation group, suffice to derive the fundamental properties of hypergeometric, confluent hypergeometric and Bessel functions.

The rotation group O_3 in 3-dimensional space is the group of real 3×3 matrices A such that $AA^t = I$ and $\det A = +1$ (Hamermesh [1]). Here A^t is the transpose of A and I is the 3×3 identity matrix. O_3 is a real 3-parameter Lie group. The real Lie algebra $L(O_3)$ has a basis $\mathscr{J}_1, \mathscr{J}_2, \mathscr{J}_3$ with commutation relations

$$[\mathscr{J}_1, \mathscr{J}_2] = \mathscr{J}_3, \qquad [\mathscr{J}_3, \mathscr{J}_1] = \mathscr{J}_2, \qquad [\mathscr{J}_2, \mathscr{J}_3] = \mathscr{J}_1. \tag{2.6}$$

Let ρ be a finite-dimensional irreducible representation of $L(O_3)$ on the complex vector space V and define the operators $\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3$ on V by $\rho(\mathscr{J}_k) = \mathrm{J}_k$, $k = 1, 2, 3$. These operators generate a Lie algebra which is a homomorphic image of $L(O_3)$. However, for many purposes it is more convenient to use the operators $\mathrm{J}^+, \mathrm{J}^-, \mathrm{J}^3$:

$$\mathrm{J}^3 = i\mathrm{J}_3, \qquad \mathrm{J}^+ = -\mathrm{J}_2 + i\mathrm{J}_1, \qquad \mathrm{J}^- = \mathrm{J}_2 + i\mathrm{J}_1 \qquad (i = \sqrt{-1}). \tag{2.7}$$

The commutation relations become

$$[\mathrm{J}^3, \mathrm{J}^+] = \mathrm{J}^+, \qquad [\mathrm{J}^3, \mathrm{J}^-] = -\mathrm{J}^-, \qquad [\mathrm{J}^+, \mathrm{J}^-] = 2\mathrm{J}^3 \tag{2.8}$$

where now $[A, B] = AB - BA$. Note that (2.8) is formally identical to Eq. (1.19) for the commutation relations of the generators of the complex Lie algebra $sl(2)$.

To determine all finite-dimensional irreducible representations of $L(O_3)$ it is sufficient to classify (up to isomorphism) all nonzero complex vector spaces V and operators $\mathrm{J}^+, \mathrm{J}^-, \mathrm{J}^3$ on V satisfying (2.8), such that no proper subspace of V is invariant under $\mathrm{J}^+, \mathrm{J}^-, \mathrm{J}^3$.

Let V be one such vector space and let $h_q \in V$ be a nonzero eigenvector of J^3 with eigenvalue q:

$$\mathrm{J}^3 h_q = q h_q .$$

The relation $[J^3, J^+]\, h_q = J^+h_q$ implies $J^3(J^+h_q) = (q+1)\, J^+h_q$. Hence, either $J^+h_q = 0$ or J^+h_q is an eigenvector of J^3 with eigenvalue $q+1$. Similarly it is easy to show that either $J^-h_q = 0$ or J^-h_q is an eigenvector of J^3 with eigenvalue $q-1$. By repeating the above argument we find

$$J^3(J^+)^k h_q = (q+k)(J^+)^k h_q\,, \qquad J^3(J^-)^k h_q = (q-k)(J^-)^k h_q$$

for all positive integers k. Since V is finite-dimensional there must exist an integer $r \geqslant 0$ satisfying $(J^+)^r h_q \neq 0$ and $(J^+)^{r+1} h_q = 0$. Set $(J^+)^r h_q = f_l$ where $l = q + r$, so that $J^3 f_l = l f_l$. By a similar argument there is an integer $s \geqslant 0$ such that $(J^-)^s f_l \neq 0$ and $(J^-)^{s+1} f_l = 0$. We will show that the vectors f_k, $k = l, l-1,\ldots, l-s$, defined by $f_{l-j} = (J^-)^j f_l$, $j = 0,\ldots, s$, form a basis for V. One way to demonstrate this is by means of the operator $C_{1,0} = J^+J^- + J^3J^3 - J^3$ (the Casimir operator). As can easily be verified from the commutation relations (2.8), $C_{1,0}$ commutes with J^+, J^-, and J^3:

$$[C_{1,0}\,, J^+] = [C_{1,0}\,, J^-] = [C_{1,0}\,, J^3] = 0. \tag{2.9}$$

Moreover,

$$\begin{aligned} C_{1,0} f_l &= J^+J^- f_l + J^3J^3 f_l - J^3 f_l \\ &= (J^-J^+ + 2J^3) f_l + (l^2 - l) f_l = J^-J^+ f_l + l(l+1) f_l\,; \end{aligned}$$

so $C_{1,0} f_l = l(l+1) f_l$, because $J^+ f_l = 0$. Since $C_{1,0}$ commutes with J^- we have

$$C_{1,0} f_{l-j} = C_{1,0}(J^-)^j f_l = (J^-)^j\, C_{1,0} f_l = l(l+1)(J^-)^j f_l = l(l+1) f_{l-j}$$

for $j = 1, 2,\ldots, s$. Thus, $C_{1,0} f_k = l(l+1) f_k$ for $k = l, l-1,\ldots, l-s$. On the other hand we can evaluate $C_{1,0} f_{l-s}$ directly:

$$C_{1,0} f_{l-s} = (J^+J^- + J^3J^3 - J^3) f_{l-s} = (l-s)(l-s-1) f_{l-s}\,,$$

since $J^- f_{l-s} = 0$. Comparison of these two results yields $l(l+1) = (l-s)(l-s-1)$ or $s = 2l$. Since s is a nonnegative integer, l is either an integer or half an integer. Another direct computation proves

$$\begin{aligned} C_{1,0} f_{l-j} &= J^+(J^-)^{j+1} f_l + J^3J^3 f_{l-j} - J^3 f_{l-j} \\ &= J^+ f_{l-j-1} + (l-j)(l-j-1) f_{l-j} \end{aligned}$$

for $j = 1, 2,\ldots, s-1$. From $C_{1,0} f_{l-j} = l(l+1) f_{l-j}$ we obtain $J^+ f_k = (l-k)(l+k+1) f_{k+1}$ for $k = l-1, l-2,\ldots, -l$. These results show that the $(2l+1)$-dimensional subspace of V generated by the vectors f_k, $k = l, l-1,\ldots, -l$, is invariant under J^+, J^-, and J^3.

Moreover, this subspace must coincide with V itself because V is irreducible under ρ. We know how the operators J^+, J^-, J^3 act on the basis vectors f_k of V, so ρ is completely determined:

$$J^3 f_k = k f_k , \qquad J^- f_k = f_{k-1} , \qquad J^+ f_k = (l - k)(l + k + 1) f_{k+1} ,$$
$$C_{1,0} f_k = l(l + 1) f_k , \qquad k = l, l - 1, \ldots, -l. \tag{2.10}$$

(On the right-hand sides of Eqs. (2.10) we make the convention: $f_k = 0$ if k is not in the spectrum of J^3.) Conversely, for any nonnegative integer $2l$ it is easy to verify that the operators J^+, J^-, J^3 on V defined by (2.10) satisfy the commutation relations (2.8) and thus determine an irreducible representation $D(2l)$ of O_3.

The irreducible representation $D(2l)$ is uniquely determined by and uniquely determines the spectrum of the operator J^3 in this representation, i.e., the eigenvalues $\{+l, l - 1, \ldots, -l\}$. However, the basis vectors $\{f_k\}$ are not uniquely determined by $D(2l)$. In particular, if $\{\gamma_k\}$, $k = l, \ldots, -l$, is a set of nonzero complex constants the vectors $f_k' = \gamma_k f_k$ will also form a basis of V consisting of eigenvectors of J^3. A most convenient basis for V is obtained by choosing the constants γ_k such that $\gamma_{k+1}/\gamma_k = [(l + k + 1)(l - k)]^{1/2}$, $k = l - 1, l - 2, \ldots, -l$. For the new basis vectors $\{f_k'\}$, relations (2.10) become

$$J^3 f_k' = k f_k' , \qquad J^- f_k' = [(l + k)(l - k + 1)]^{1/2} f_{k-1}' ,$$
$$J^+ f_k' = [(l - k)(l + k + 1)]^{1/2} f_{k+1}' , \qquad C_{1,0} f_k' = l(l + 1) f_k' . \tag{2.11}$$

Relations (2.11) are especially convenient for the study of unitary representations of O_3 (see Section 5-16).

From the above discussion we can conclude that the only finite-dimensional irreducible representations of $L(O_3)$ are the representations $D(2l)$, $2l$ a nonnegative integer. The representation space corresponding to $D(2l)$ has dimension $2l + 1$ and contains a basis $f_l', f_{l-1}', \ldots, f_{-l}'$ such that the action of J^+, J^-, J^3 on the basis is given by (2.11).

2-4 The Angular Momentum Operators

In the study of the quantum theory of angular momentum the differential operators

$$J_1 = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} , \qquad J_2 = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} ,$$
$$J_3 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \tag{2.12}$$

occur, where x_1, x_2, x_3 are real cartesian coordinates (Landau and Lifshitz [1], Chapter 4). These operators satisfy the commutation relations (2.6) and thus generate a Lie algebra isomorphic to $L(O_3)$. In terms of the spherical coordinates r, θ, φ defined by

$$x_1 = r\cos\theta\cos\varphi, \qquad x_2 = r\cos\theta\sin\varphi, \qquad x_3 = r\sin\theta,$$
$$r \geqslant 0, \qquad 0 \leqslant \theta \leqslant \pi, \qquad 0 \leqslant \varphi < 2\pi,$$

the operators J^+, J^-, J^3 take the form

$$J^\pm = e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right), \qquad J^3 = -i\frac{\partial}{\partial\varphi}. \tag{2.13}$$

To relate the above results to special function theory we look for a realization of the representation $D(2l)$ such that the basis space $\mathscr{V}$ is a space of analytic functions of the real variables θ, φ and the operators J^+, J^-, J^3 are given by (2.13). Thus, we try to find the functions $Y_l^k(\theta, \varphi)$ such that

$$\begin{gathered} J^3Y_l^k = kY_l^k, \qquad J^-Y_l^k = [(l+k)(l-k+1)]^{1/2}Y_l^{k-1}, \\ J^+Y_l^k = [(l-k)(l+k+1)]^{1/2}Y_l^{k+1}, \qquad C_{1,0}Y_l^k = l(l+1)Y_l^k, \\ k = l, l-1, \ldots, -l. \end{gathered} \tag{2.14}$$

From $J^3 = -i\,\partial/\partial\varphi$ there follows the relation

$$-i\frac{\partial Y_l^k}{\partial\varphi}(\theta, \varphi) = kY_l^k(\theta, \varphi),$$

so Y_l^k can be written in the form $Y_l^k(\theta, \varphi) = P_l^k(\theta)e^{ik\varphi}$. The problem of finding functions Y_l^k satisfying (2.12) thus reduces to the problem of determining the functions P_l^k. Furthermore, the requirement $J^+Y_l^l = 0$ leads to the differential equation

$$\frac{dP_l^l}{d\theta} - l\cot\theta\, P_l^l = 0$$

with solution

$$P_l^l(\theta) = c_l\sin^l\theta = c_l(1-\cos^2\theta)^{l/2}$$

where c_l is an arbitrary nonzero constant. The functions P_l^k can now be defined recursively from P_l^l by the condition

$$-\frac{dP_l^{k+1}}{d\theta} - (k+1)\cot\theta\, P_l^{k+1} = [(l+k+1)(l-k)]^{1/2}P_l^k.$$

By induction, we obtain the explicit expression

$$P_l^k(\theta) = c_l \left[\frac{(l+k)!}{(2l)!\,(l-k)!}\right]^{1/2} (1-\cos^2\theta)^{-k/2} \frac{d^{l-k}}{d(\cos\theta)^{l-k}} (1-\cos^2\theta)^l,$$
$$k = l, l-1, \ldots, -l. \tag{2.15}$$

The requirement $J^- Y_l^{-l} \equiv 0$ leads to the condition

$$-\frac{dP_l^{-l}}{d\theta} - l\cot\theta\, P_l^{-l} = \frac{c_l}{(2l)!}(1-\cos^2\theta)^{-(l+1)/2}\frac{d^{2l+1}}{d(\cos\theta)^{2l+1}}(1-\cos^2\theta)^l \equiv 0.$$

This condition is satisfied only if l is an integer; if l is not an integer the constructions fails.

In case l is an integer the functions $f_k' = Y_l^k$, $k = l, l-1, \ldots, -l$, form a basis for the representation $D(2l)$ of $L(O_3)$. To see this, note that we have mimicked the abstract construction of the representation $D(2l)$ given in Section 2-3. Thus, we have found a function $f_l' = Y_l^l$ such that $J^+ f_l' = 0$ and have determined the functions f_k' defined by

$$f_k' = Y_l^k = \left[\frac{(l+k)!}{(2l)!\,(l-k)!}\right]^{1/2} (J^-)^{l-k} Y_l^l, \qquad k = l, l-1, \ldots, -l.$$

Since $J^- Y_l^{-l} = J^- f_{-l}' = 0$, the argument in the previous section proves that the basis functions satisfy Eqs. (2.11). That is,

$$\begin{aligned}
-\frac{d}{d\theta}P_l^k - k\cot\theta\, P_l^k &= [(l+k)(l-k+1)]^{1/2} P_l^{k-1} \\
\frac{d}{d\theta}P_l^k - k\cot\theta\, P_l^k &= [(l-k)(l+k+1)]^{1/2} P_l^{k+1} \qquad (2.16)\\
\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP_l^k}{d\theta}\right) + \left[l(l+1) - \frac{k^2}{\sin^2\theta}\right] P_l^k &= 0, \\
k &= l, l-1, \ldots, -l,
\end{aligned}$$

where the last equation is obtained by writing $C_{1,0} f_k' = l(l+1) f_k'$ in terms of the differential operators (2.13). The arbitrary constant c_l in Eq. (2.15) is usually fixed by the requirement

$$\int_0^{2\pi}\int_0^{\pi} |\, Y_l^l(\theta,\varphi)|^2 \sin\theta\, d\theta\, d\varphi = 1.$$

Evaluating this integral we find

$$c_l = (-1)^l \left[\frac{(2l+1)!}{4\pi(l!)^2}\right]^{1/2}$$

where the phase factor $(-1)^l$ is introduced to conform to convention.

The functions $Y_l^k(\theta, \varphi) = P_l^k(\theta)e^{ik\varphi}$ are known as spherical harmonics and Eqs. (2.16) give fundamental recursion relations and a differential equation for these functions (Erdélyi *et al.* [1], Vol. II). Thus, if l is a nonnegative integer the spherical harmonics Y_l^k form a basis for the representation $D(2l)$. Although we have failed to find a realization of those representations for which l is not an integer, in Section 5-14 we will find such realizations by choosing homomorphisms of $L(O_3)$ by generalized Lie derivatives different from (2.12).

The representations of $L(O_3)$ realized here can be extended to local multiplier representations of O_3. The matrix elements can be computed and the results yield generating functions and addition theorems for the spherical harmonics. These results are well known (Gel'fand *et al.* [1]) and will be included in Section 5-16.

The methods used to relate spherical harmonics to representations of $L(O_3)$ are applicable to a wide variety of special functions. In the next section a family of Lie algebras $\mathcal{G}(a, b)$ will be introduced and the irreducible representations of these Lie algebras will be classified, subject to suitable restrictions. We will find realizations of the irreducible representations in terms of generalized Lie derivatives acting on spaces of analytic functions. For each such realization there will exist a natural basis of special functions. The special functions so obtained are the hypergeometric, confluent hypergeometric, and Bessel functions. This relation between Lie algebras and special functions provides insight into special function theory.

2-5 The Lie Algebras $\mathcal{G}(a, b)$

For any pair of complex numbers (a, b) define the 4-dimensional complex Lie algebra $\mathcal{G}(a, b)$ with basis $\mathcal{J}^+$, $\mathcal{J}^-$, $\mathcal{J}^3$, $\mathcal{E}$ by

$$\begin{gathered}[\mathcal{J}^+, \mathcal{J}^-] = 2a^2\mathcal{J}^3 - b\mathcal{E}, \qquad [\mathcal{J}^3, \mathcal{J}^+] = \mathcal{J}^+, \qquad [\mathcal{J}^3, \mathcal{J}^-] = -\mathcal{J}^- \\ [\mathcal{J}^+, \mathcal{E}] = [\mathcal{J}^-, \mathcal{E}] = [\mathcal{J}^3, \mathcal{E}] = \mathcal{O},\end{gathered} \tag{2.17}$$

where [.,.] is the commutator bracket and $\mathcal{O}$ is the additive identity element. (It is easy to verify that relations (2.17) do in fact define a Lie algebra.) For special choices of the parameters a, b, $\mathcal{G}(a, b)$ essentially coincides with one of the Lie algebras introduced in Section 1-2. In particular we have the following isomorphisms:

$$\mathcal{G}(1, 0) \cong sl(2) \oplus (\mathcal{E}), \qquad \mathcal{G}(0, 1) \cong L[G(0, 1)], \qquad \mathcal{G}(0, 0) \cong L(T_3) \oplus (\mathcal{E})$$

where $(\mathcal{E})$ is the 1-dimensional Lie algebra generated by $\mathcal{E}$. Recall that a subset $\mathcal{G}'$ of a Lie algebra $\mathcal{G}$ is a **subalgebra** of $\mathcal{G}$ if $\mathcal{G}'$ is itself a Lie algebra under the scalar multiplication, addition, and commutator bracket of $\mathcal{G}$. If $\mathcal{G}$ is a Lie algebra and $\mathcal{G}_1$, $\mathcal{G}_2$ are subalgebras of $\mathcal{G}$, then $\mathcal{G}$ is the **direct sum** of $\mathcal{G}_1$ and $\mathcal{G}_2$, $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$, if considered as a vector space $\mathcal{G}$ is the direct sum of its subspaces $\mathcal{G}_1$, $\mathcal{G}_2$, and further, $[\alpha_1, \alpha_2] = 0$ for all $\alpha_1 \in \mathcal{G}_1$, $\alpha_2 \in \mathcal{G}_2$.

Lemma 2.1

$$\mathcal{G}(a, b) \cong \begin{cases} \mathcal{G}(1, 0) & \text{if } \; a \neq 0, \\ \mathcal{G}(0, 1) & \text{if } \; a = 0, \quad b \neq 0, \\ \mathcal{G}(0, 0) & \text{if } \; a = b = 0. \end{cases}$$

PROOF If $a \neq 0$, set $\mathcal{J}'^{+} = a^{-1}\mathcal{J}^{+}$, $\mathcal{J}'^{-} = a^{-1}\mathcal{J}^{-}$, $\mathcal{J}'^{3} = \mathcal{J}^{3} - (ba^{-2}/2)\,\mathcal{E}$, $\mathcal{E}' = \mathcal{E}$. In terms of the primed basis elements the commutation relations for $\mathcal{G}(a, b)$ given by (2.17) become identical with those for $\mathcal{G}(1, 0)$. If $a = 0$, $b \neq 0$, set $\mathcal{E}' = b\mathcal{E}$. In terms of the basis $\mathcal{J}^{+}$, $\mathcal{J}^{-}$, $\mathcal{J}^{3}$, $\mathcal{E}'$, the isomorphism between $\mathcal{G}(a, b)$ and $\mathcal{G}(0, 1)$ is evident.

This lemma shows that there are only three distinct Lie algebras of the form $\mathcal{G}(a, b)$, up to isomorphism. However, it is often useful to consider the entire family $\{\mathcal{G}(a, b),\, a, b \in \mathcal{C}\}$ of such Lie algebras because Eqs. (2.17) determine relationships between the nonisomorphic algebras $\mathcal{G}(1, 0)$, $\mathcal{G}(0, 1)$, and $\mathcal{G}(0, 0)$. To spell out these relationships we introduce the notion of contraction of Lie algebras.

Let $\mathcal{G}$ be an n-dimensional complex Lie algebra with basis $\{\alpha_i\}$, $i = 1,\ldots, n$. The structure constants C_{ij}^{k} of $\mathcal{G}$ relative to this basis are given by

$$[\alpha_i, \alpha_j] = \sum_{k=1}^{n} C_{ij}^{k}\alpha_k, \qquad i, j = 1,\ldots, n.$$

If $P = (P_i^l)$ is a nonsingular $n \times n$ complex matrix we can define a new basis $\{\alpha_i'\}$ for $\mathcal{G}$ by $\alpha_i' = \sum_{l=1}^{n} P_i^l \alpha_l$, $i = 1,\ldots, n$. In terms of the new basis the structure constants become

$$C_{ij}'^{k} = \sum_{l,h,r=1}^{n} P_i^l P_j^h C_{lh}^{r} (P^{-1})_r^k, \qquad i, j, k = 1,\ldots, n.$$

We now introduce a one-parameter family $P(t)$ of nonsingular matrices defined for all values of $t > 0$ in such a way that the matrix elements $P_i^l(t)$ are continuous in t. Suppose this family of matrices has the property that the limits

$$C_{ij}^{ok} = \lim_{t \to 0} \sum_{l,h,r=1}^{n} P_i^l(t)\, P_j^h(t)\, C_{lh}^{r} (P^{-1}(t))_r^k, \qquad i, j, k = 1,\ldots, n,$$

exist. It is easy to show that the $\{C_{ij}^{ok}\}$ are the structure constants of an n-dimensional Lie algebra (Saletan [1], Sharp [1]). If $\lim_{t\to 0} P(t) = P^o$ where P^o is a nonsingular matrix then the constants C_{ij}^{ok} are structure constants for $\mathscr{G}$ relative to the basis

$$\alpha_i^o = \sum_{l=1}^{n} (P^o)_i^l \alpha_l .$$

However, if $\lim_{t\to 0} P(t)$ is a singular matrix or if the limit does not exist, the constants C_{ij}^{ok} may be structure constants for an n-dimensional Lie algebra $\mathscr{G}'$ which is not isomorphic to $\mathscr{G}$. In such a case we say that $\mathscr{G}'$ is a **contraction** of $\mathscr{G}$ (Wigner and Inönu [1], Saletan [1], Sharp [1]).

According to Lemma 2.1 we can always find a basis for the Lie algebra $\mathscr{G}(1, 0)$ in the form (2.17) where $a \neq 0$ and b is arbitrary. If $a \to 0$ in (2.17) we get in the limit either $\mathscr{G}(0, 1)$ or $\mathscr{G}(0, 0)$ depending on whether or not $b = 0$. This shows that $\mathscr{G}(0, 1)$ and $\mathscr{G}(0, 0)$ are contractions of $\mathscr{G}(1, 0)$. Similarly, $\mathscr{G}(0, 0)$ is a contraction of $\mathscr{G}(0, 1)$. This relationship between the Lie algebras $\mathscr{G}(a, b)$ will turn out to be of fundamental importance in special function theory.

2-6 Representations of $\mathscr{G}(a, b)$

Let ρ be a representation of $\mathscr{G}(a, b)$ on the complex vector space V and set

$$\mathrm{J}^+ = \rho(\mathscr{J}^+), \qquad \mathrm{J}^- = \rho(\mathscr{J}^-), \qquad \mathrm{J}^3 = \rho(\mathscr{J}^3), \qquad \mathrm{E} = \rho(\mathscr{E}).$$

These linear operators obey the commutation relations

$$\begin{gathered}[\mathrm{J}^+, \mathrm{J}^-] = 2a^2\mathrm{J}^3 - b\mathrm{E}, \qquad [\mathrm{J}^3, \mathrm{J}^+] = \mathrm{J}^+, \qquad [\mathrm{J}^3, \mathrm{J}^-] = -\mathrm{J}^-, \\ [\mathrm{J}^+, \mathrm{E}] = [\mathrm{J}^-, \mathrm{E}] = [\mathrm{J}^3, \mathrm{E}] = \mathrm{O},\end{gathered} \tag{2.18}$$

where now $[A, B] = AB - BA$ for linear operators A and B on V.

Define the **spectrum** S of J^3 to be the set of eigenvalues of J^3. The **multiplicity** of the eigenvalue $\lambda \in S$ is the dimension of the eigenspace V^λ,

$$V^\lambda = \{v \in V\colon \mathrm{J}^3 v = \lambda v\}.$$

In analogy with the example presented in Section 2-3 we will analyze the irreducible representations of $\mathscr{G}(a, b)$ and for each such representation find a basis for V consisting of eigenvectors of J^3. To be precise, we will classify all representations ρ of $\mathscr{G}(a, b)$ satisfying the conditions:

(i) ρ is irreducible

(ii) Each eigenvalue of J^3 has multiplicity equal to one. There is a countable basis for V consisting of eigenvectors of J^3. (2.19)

These conditions are satisfied by the representations of $L(O_3)$ classified in Section 2-3. Moreover, they enable us to construct representations of $\mathscr{G}(a, b)$ by mimicking the construction of representations of $L(O_3)$. The basic justification for the requirements (2.19) is that they quickly lead to connections between $\mathscr{G}(a, b)$ and certain special functions. This claim will be thoroughly documented in the chapters to follow. Also, $\mathscr{J}^3$ and $\mathscr{E}$ generate a Cartan subalgebra of $\mathscr{G}(a, b)$ (Jacobson [1]), so in analogy with the theory of finite-dimensional representations of semi-simple Lie algebras it is natural to try to find basis vectors for V which are simultaneous eigenvectors of the operators corresponding to the elements of this Cartan subalgebra. (We will show that if ρ satisfies (2.19) then $E = \rho(\mathscr{E})$ is a multiple of the identity operator and every non-zero vector in V is an eigenvector of E.)

Condition (ii) can be written in forms which are apparently much weaker, though actually equivalent. In particular we could replace (ii) by

(ii)′ J^3 has an eigenvalue of finite multiplicity.

It is not difficult to show that conditions (i) and (ii)′ imply (ii). The necessary ingredients for the proof will be developed in the process of classifying the representations satisfying (i) and (ii), but the details will be left to the reader. We use condition (ii) for the sake of convenience.

To classify the representations of $\mathscr{G}(a, b)$ for arbitrary $a, b \in \mathscr{C}$ it is enough to consider the three cases: $\mathscr{G}(1, 0)$, $\mathscr{G}(0, 1)$, $\mathscr{G}(0, 0)$. However, for the time being it will be convenient to treat these three cases simultaneously by studying $\mathscr{G}(a, b)$ without being specific as to the values of a and b. We will consider, therefore, a representation ρ of $\mathscr{G}(a, b)$ on the complex vector space V such that (2.19) is satisfied. Our objective will be the enumeration of all possibilities for ρ. To carry out this enumeration the following remarks will be helpful:

(*A*) Define the operator $C_{a,b}$ on V by

$$C_{a,b} = J^+J^- + a^2J^3J^3 - a^2J^3 - bJ^3E. \tag{2.20}$$

It is easy to check that $C_{a,b}$ commutes with every operator $\rho(\alpha)$, $\alpha \in \mathcal{G}(a, b)$. Thus,

$$[C_{a,b}, J^+] = [C_{a,b}, J^-] = [C_{a,b}, J^3] = [C_{a,b}, E] = O.$$

We will show that $C_{a,b} = \lambda I$ where I is the identity operator and λ is a constant depending on ρ.

(*B*) If S is the spectrum of J^3, condition (ii) guarantees that S is countable and that there exists a basis for V consisting of vectors f_m, $J^3 f_m = m f_m$, defined for every $m \in S$. Suppose $q \in S$. Then the equation $[J^3, J^+] f_q = J^+ f_q$ leads to the result $J^3(J^+ f_q) = (q + 1) J^+ f_q$; so either $J^+ f_q = \xi_{q+1} f_{q+1}$ where ξ_{q+1} is a nonzero constant and $q + 1 \in S$, or $J^+ f_q = 0$. Similarly, the equation $[J^3, J^-] f_q = -J^- f_q$ implies either $J^- f_q = \eta_q f_{q-1}$ where η_q is a nonzero constant and $q - 1 \in S$, or $J^- f_q = 0$. The equation $[E, J^3] f_q = 0$ implies $E f_q = \mu_q f_q$ for some constant μ_q and $[C_{a,b}, J^3] f_q = 0$ implies $C_{a,b} f_q = \lambda_q f_q$ for some constant λ_q.

(*C*) Since ρ is irreducible the results of (*B*) show that the spectrum S must be connected, i.e., if $q \in S$ then S is of the form

$$S = \{q + n \colon n \text{ an integer such that } n_1 < n < n_2\}$$

where n_1 and n_2 are integers. We do not exclude the possibilities $n_1 = -\infty$ or $n_2 = +\infty$. In addition, if $m, m + 1 \in S$ then $\xi_{m+1}, \eta_{m+1} \neq 0$, since otherwise the irreducibility of ρ would be violated.

(*D*) Suppose $m, m + 1 \in S$. Then the equation $[E, J^+] f_m = 0$ leads to $\xi_{m+1}(\mu_{m+1} - \mu_m) = 0$, which implies $\mu_m = \mu_{m+1}$. Thus, $\mu_m = \mu =$ constant for all $m \in S$ and $E = \mu I$ is a multiple of the identity operator on V. A similar argument proves $\lambda_m = \lambda$ for all $m \in S$ so that $C_{a,b} = \lambda I$. The equation $C_{a,b} f_m = \lambda f_m$ leads to the relations

$$\xi_m \eta_m + a^2 m^2 - a^2 m - b m \mu = \lambda \qquad \text{for all} \quad m \in S,$$

obtained from the definition (2.20) of $C_{a,b}$. (If $m - 1 \notin S$ then $\eta_m = 0$.) These relations can be rewritten in the form

$$\xi_m \eta_m = \lambda - a^2 m(m - 1) + b\mu m. \tag{2.21}$$

Moreover, since $[J^+, J^-] = 2a^2 J^3 - bE$ the operator $C_{a,b}$ is also given by

$$C_{a,b} = J^- J^+ + a^2 J^3 J^3 + a^2 J^3 - b J^3 E - bE$$

From this expression we obtain the relations

$$\xi_{m+1} \eta_{m+1} = \lambda - a^2 m(m + 1) + b\mu(m + 1) \tag{2.22}$$

defined for all $m \in S$, where $\xi_{m+1} = 0$ if $m + 1 \notin S$. Thus, Eq. (2.21) is valid for all m such that $m \in S$ or $m - 1 \in S$.

(*E*) If $\{\gamma_m, m \in S\}$ is a set of nonzero constants, we can introduce a new basis $\{f'_m\}$ for V by means of the definition $f'_m = \gamma_m f_m$, all $m \in S$. In terms of the new basis:

$$\begin{gathered} \mathrm{J}^+ f'_m = \xi'_{m+1} f'_{m+1}, \qquad \mathrm{J}^- f'_m = \eta'_m f'_{m-1}, \\ \mathrm{J}^3 f'_m = m f'_m, \qquad \mathrm{E} f'_m = \mu f'_m, \\ \mathrm{C}_{a,b} f'_m = \lambda f'_m, \qquad m \in S, \end{gathered} \tag{2.23}$$

where

$$\xi'_m = \frac{\gamma_{m-1}}{\gamma_m} \xi_m, \qquad \eta'_m = \frac{\gamma_m}{\gamma_{m-1}} \eta_m.$$

(We assume $\xi_{m+1} = 0$ if $m + 1 \notin S$ and $\eta_m = 0$ if $m - 1 \notin S$.) Again the constants ξ'_m, η'_m must satisfy the condition

$$\xi'_m \eta'_m = \lambda - a^2 m(m - 1) + b\mu m.$$

It follows from these considerations that for all $m \in S$ such that $m - 1 \in S$ we can choose the nonzero constants η_m arbitrarily and define the constants ξ_m by (2.21). (If $m - 1 \notin S$ then $\eta_m = 0$ and ξ_m is not defined.) We can always find a basis $\{f_m\}$ of V such that η_m, ξ_m satisfy (2.23) (without the primes).

(*F*) The representation ρ of $\mathcal{G}(a, b)$ is uniquely determined by the constants λ, μ and the spectrum S of J^3. The nonzero constants ξ_m, η_m are not unique and may be chosen arbitrarily, subject only to conditions (2.21).

This is as far as we can go without making any assumptions as to the values of a and b. At this point we use the fact that we need consider only the Lie algebras $\mathcal{G}(1, 0)$, $\mathcal{G}(0, 1)$, and $\mathcal{G}(0, 0)$.

Theorem 2.1 Every representation of $\mathcal{G}(0, 0)$ which satisfies (2.19) and for which $\mathrm{J}^+\mathrm{J}^- \not\equiv 0$ on V is isomorphic to a representation $Q^\mu(\omega, m_0)$ defined for μ, ω, $m_0 \in \mathcal{C}$ such that $\omega \neq 0$ and $0 \leqslant \mathrm{Re}\, m_0 < 1$. $S = \{m_0 + n\colon n \text{ an integer}\}$. For each representation $Q^\mu(\omega, m_0)$ there is a basis for V consisting of vectors f_m, $m \in S$, such that

$$\begin{gathered} \mathrm{J}^3 f_m = m f_m, \qquad \mathrm{E} f_m = \mu f_m, \\ \mathrm{J}^+ f_m = \omega f_{m+1}, \qquad \mathrm{J}^- f_m = \omega f_{m-1}, \\ \mathrm{C}_{0,0} f_m = (\mathrm{J}^+\mathrm{J}^-) f_m = \omega^2 f_m. \end{gathered} \tag{2.24}$$

The representations $Q^\mu(\omega, m_0)$ and $Q^\mu(-\omega, m_0)$ are isomorphic.

PROOF Set $a = b = 0$ in Eq. (2.22). If ρ is a representation of $\mathcal{G}(0,0)$ satisfying (2.19) and such that $J^+J^- \not\equiv 0$ on V, we have $\xi_m\eta_m = \lambda \neq 0$ for all m such that $m \in S$ or $m - 1 \in S$. Thus, there exists no element m_1 in S for which $J^+f_{m_1} = 0$ for a nonzero $f_{m_1} \in V^{m_1}$ and no element m_2 such that $J^-f_{m_2} = 0$ for a nonzero $f_{m_2} \in V^{m_2}$. If $m_o \in S$ then S must be of the form $\{m_o + n: n \text{ an integer}\}$. Without loss of generality we can assume m_o is the element of S with smallest positive real part, i.e., $0 \leqslant \operatorname{Re} m_o < 1$. Since $\xi_m\eta_m = \lambda \neq 0$ we can also assume that $\xi_m = \eta_m = \omega$ for all $m \in S$. Then $\lambda = \omega^2$ and ω is determined only up to sign. This proves that every representation ρ satisfying the hypotheses of the theorem can be cast into the form (2.24). Conversely, it is easy to show that the representations $Q^\mu(\omega, m_o)$ satisfy the hypotheses of the theorem. Q.E.D.

The irreducible representations of $\mathcal{G}(0,0)$ for which $J^+J^- \equiv 0$ are easily seen to be 1-dimensional. They are of little interest for our purposes so we will not bother to classify them.

Theorem 2.2 Every representation of $\mathcal{G}(0,1)$ satisfying (2.19) and for which $E \not\equiv 0$ is isomorphic to a representation in the following list:

(i) The representations $R(\omega, m_o, \mu)$ defined for all $\omega, m_o, \mu \in \mathcal{C}$ such that $\mu \neq 0$, $0 \leqslant \operatorname{Re} m_o < 1$, and $\omega + m_o$ is not an integer. $S = \{m_o + n: n \text{ an integer}\}$.

(ii) The representations $\uparrow_{\omega,\mu}$ defined for all $\omega, \mu \in \mathcal{C}$ such that $\mu \neq 0$. $S = \{-\omega + n: n \text{ a nonnegative integer}\}$.

For each of the cases (i) and (ii) there is a basis of V consisting of vectors f_m defined for each $m \in S$ such that

$$\begin{gathered} J^3 f_m = m f_m, \qquad E f_m = \mu f_m, \\ J^+ f_m = \mu f_{m+1}, \qquad J^- f_m = (m + \omega) f_{m-1}, \\ C_{0,1} f_m = (J^+J^- - EJ^3) f_m = \mu\omega f_m. \end{gathered} \tag{2.25}$$

(On the right-hand side of these equations we assume $f_m = 0$ if $m \notin S$.)

(iii) The representations $\downarrow_{\omega,\mu}$ defined for all $\omega, \mu \in \mathcal{C}$ such that $\mu \neq 0$. $S = \{-\omega - 1 - n: n \text{ a nonnegative integer}\}$. For each of the representations there is a basis of V consisting of vectors f_m defined for each $m \in S$ such that

$$\begin{gathered} J^3 f_m = m f_m, \qquad E f_m = -\mu f_m, \\ J^+ f_m = -(m + \omega + 1) f_{m+1}, \qquad J^- f_m = \mu f_{m-1}, \\ C_{0,1} f_m = (J^+J^- - EJ^3) f_m = -\mu\omega f_m. \end{gathered} \tag{2.26}$$

PROOF Let ρ be a representation of $\mathscr{G}(0, 1)$ satisfying the hypotheses of the theorem and let S be the spectrum of ρ. Then Eq. (2.21) becomes $\xi_m\eta_m = \mu(\omega + m)$ for all m such that $m \in S$ or $m - 1 \in S$. (We have set $a = 0$, $b = 1$, and $\lambda = \mu\omega$ in (2.21).) Let $m_o \in S$ and suppose $\omega + m_o$ is not an integer. Then, $S = \{m_o + n: n \text{ an integer}\}$. Without loss of generality m_o can be taken to be the element of S with smallest positive real part, $0 \leqslant \operatorname{Re} m_o < 1$. According to the remarks in (E) there exists a basis for V such that $\xi_m = \mu$, $\eta_m = \omega + m$ for all $m \in S$. Thus, ρ is isomorphic to the representation $R(\omega, m_o, \mu)$ listed in (i).

If there is an element m_1 in S such that $\omega + m_1$ is a nonnegative integer, the equation $\xi_m\eta_m = \mu(\omega + m)$ implies $-\omega \in S$ and there exists a nonzero vector $f_{-\omega} \in V$ such that $J^3 f_{-\omega} = -\omega f_{-\omega}$ and $J^- f_{-\omega} = 0$. In this case $S = \{-\omega + n: n \text{ a nonnegative integer}\}$. There exists a basis of V such that $\xi_m = \mu$, $\eta_m = \omega + m$ for all $m \in S$. Thus ρ is isomorphic to the representation $\uparrow_{\omega,\mu}$ listed in (ii). We say that this representation is **bounded below**.

If there is an element m_2 in S such that $\omega + m_2$ is a negative integer we find $-\omega - 1 \in S$ and $-\omega \notin S$. Thus, $S = \{-\omega - 1 - n: n \text{ a nonnegative integer}\}$. There exists a basis of V such that $\xi_m = \omega + m$, $\eta_m = \mu$ for all $m \in S$. In this case ρ is isomorphic to the representation $\downarrow_{\omega,\mu}$ listed in (iii). This representation is **bounded above**.

Conversely, it is easy to verify that the representations (i)–(iii) actually satisfy conditions (2.19). Q.E.D.

The irreducible representations of $\mathscr{G}(0, 1)$ for which $E = 0$ can be considered as representations of $\mathscr{T}_3$; hence they are just the representations $Q^0(\omega, m_o)$ classified in Theorem 2.1.

Theorem 2.3 Every representation ρ of $\mathscr{G}(1, 0)$ satisfying conditions (2.19) is isomorphic to a representation in the following list:

(i) The representations $D^\mu(u, m_o)$ defined for all complex μ, u, m_o such that $m_o + u$, $m_o - u$ are not integers and $0 \leqslant \operatorname{Re} m_o < 1$. $S = \{m_o + n: n \text{ an integer}\}$. $D^\mu(u, m_o)$ and $D^\mu(-u - 1, m_o)$ are isomorphic.

(ii) The representations $\uparrow_u^\mu$, $\mu, u \in \mathscr{C}$, where $2u$ is not a nonnegative integer. $S = \{-u + n: n \text{ a nonnegative integer}\}$.

(iii) The representations $\downarrow_u^\mu$, $\mu, u \in \mathscr{C}$, where $2u$ is not a nonnegative integer. $S = \{u - n: n \text{ a nonnegative integer}\}$.

(iv) The representations $D^\mu(2u)$ where $2u$ is a nonnegative integer. $S = \{u, u - 1, \ldots, -u + 1, -u\}$.

For each of these representations there is a basis of V consisting of vectors f_m, defined for each $m \in S$ such that

$$\begin{gathered} \mathrm{J}^3 f_m = m f_m, \qquad \mathrm{J}^+ f_m = (m-u) f_{m+1}, \\ \mathrm{J}^- f_m = -(m+u) f_{m-1}, \qquad \mathrm{E} f_m = \mu f_m, \\ \mathrm{C}_{1,0} f_m = (\mathrm{J}^+\mathrm{J}^- + \mathrm{J}^3\mathrm{J}^3 - \mathrm{J}^3) f_m = u(u+1) f_m. \end{gathered} \tag{2.27}$$

(We make the convention on the right-hand side of Eqs. (2.27) that $f_m = 0$ if $m \notin S$.)

PROOF Let ρ be a representation of $\mathscr{G}(1, 0)$ satisfying the hypotheses of the theorem. From (2.21) we obtain the equations $\xi_m \eta_m = u(u+1) - m(m-1) = -(m+u)(m-u-1)$ valid for all m such that $m \in S$ or $m - 1 \in S$. Here, we have set $\lambda = u(u+1)$. Suppose there is an element m_o in S such that neither $m_o + u$ nor $m_o - u$ is an integer. In this case the product $\xi_m \eta_m$ can never be zero for any element m or $m - 1$ in S. Thus, $S = \{m_o + n\colon n \text{ an integer}\}$. Without loss of generality we can assume that m_o is the element in S with smallest nonnegative real part: $0 \leqslant \operatorname{Re} m_o < 1$. According to remark ($E$) there exists a basis of V such that $\xi_m = m - u - 1$, $\eta_m = -m - u$. Thus, ρ is isomorphic to the representation $D^\mu(u, m_o)$ listed in (i). $D^\mu(u, m_o) \simeq D^\mu(-u-1, m_o)$ since both $u = u_o$ and $u = -u_o - 1$ lead to the same value of λ.

If the spectrum of ρ takes the form $S = \{m_1 + n$, n a nonnegative integer$\}$, the formula $\xi_m \eta_m = \lambda - m(m-1)$, $m \in S$, implies $\lambda = m_1(m_1 - 1)$. Setting $m_1 = -u$ we obtain $\lambda = u(u+1)$ and $S = \{-u + n\colon n$ a nonnegative integer$\}$. As before we can find a basis of V such that $\xi_m = m - u - 1$, $\eta_m = -m - u$. Here, ρ is isomorphic to the representation $\uparrow_u^\mu$. The $\uparrow_u^\mu$ is **bounded below**.

If the spectrum of ρ takes the form $S = \{m_2 - n\colon n$ a nonnegative integer$\}$, the equations $\xi_m \eta_m = \lambda - m(m-1)$ for $m \in S$ or $m - 1 \in S$, imply $\lambda = (m_2 + 1)m_2$, i.e., $\xi_{m_2+1} = 0$. Setting $m_2 = u$ we have $\lambda = u(u+1)$ and $S = \{u - n\colon n$ a nonnegative integer$\}$. Choosing $\xi_m = m - u - 1$, $\eta_m = -m - u$ we see that ρ is isomorphic to $\downarrow_u^\mu$ (**bounded above**).

If S contains finite elements m_3, m_4 such that $S = \{m_3, m_3 + 1, \ldots, m_3 + k, \ldots, m_4\}$ the equations $\xi_m \eta_m = \lambda - m(m+1)$ imply $\lambda = m_3(m_3 - 1) = (m_4 + 1)m_4$ (since $\xi_{m_4+1} = \eta_{m_3} = 0$). Set $m_4 = u$ and $m_3 = u - n$ where n is a nonnegative integer. Thus, $u(u+1) = (u-n)(u-n-1)$. The only possible solution of this equation is $n = 2u$. We conclude that $S = \{-u, -u+1, \ldots, +u\}$ and ρ is isomorphic to the representation $D^\mu(2u)$.

Since the possibilities for S have been exhausted, the theorem lists all representations ρ satisfying (2.19). Q.E.D.

2-7 Realizations of $\mathscr{G}(a, b)$ in Two Variables

In accordance with our general program we will try to find realizations of the irreducible representations ρ of $\mathscr{G}(a, b)$ listed in Section 2-6, such that V becomes a vector space of analytic functions and the operators $\rho(\alpha)$, $\alpha \in \mathscr{G}(a, b)$, form a Lie algebra of analytic differential operators acting on V. As a first step we will determine some possible candidates $\rho(\alpha)$.

Suppose, first, that the $\rho(\alpha)$ are differential operators acting on a space of analytic functions of two complex variables, x and y. For the moment we will not be concerned with the precise domains of the functions in this space.

Define the differential operators J^+, J^-, J^3, E by

$$\rho(\mathscr{J}^+) = \mathrm{J}^+, \qquad \rho(\mathscr{J}^-) = \mathrm{J}^-, \qquad \rho(\mathscr{J}^3) = \mathrm{J}^3, \qquad \rho(\mathscr{E}) = \mathrm{E}.$$

Since ρ is a representation of $\mathscr{G}(a, b)$ these operators must satisfy the usual commutation relations

$$[\mathrm{J}^+, \mathrm{J}^-] = 2a^2\mathrm{J}^3 - b\mathrm{E}, \qquad [\mathrm{J}^3, \mathrm{J}^\pm] = \pm\, \mathrm{J}^\pm, \qquad [\mathrm{J}^\pm, \mathrm{E}] = [\mathrm{J}^3, \mathrm{E}] = 0. \quad (2.28)$$

The number of possible solutions of Eqs. (2.28) is tremendous. To obtain useful results it is convenient to make more restrictive assumptions as to the form of the operators $\rho(\alpha)$. Thus, we assume that these operators take the form

$$\mathrm{J}^3 = \frac{\partial}{\partial y}, \qquad \mathrm{J}^\pm = e^{\pm y}\left(\pm\frac{\partial}{\partial x} - k(x)\frac{\partial}{\partial y} + j(x)\right), \qquad \mathrm{E} = \mu \quad (2.29)$$

where μ is a complex constant and k, j are functions of x to be determined. The operators (2.29) are natural generalizations of the angular momentum operators (2.13) related to O_3. However, we will not give a detailed justification for this choice of differential operators now. In Chapter 8 a theory of generalized Lie derivatives will be developed which will enable us to classify **all** solutions of (2.28) by differential operators in two variables. It will follow from the classification that nothing of importance for special function theory is lost through the restrictions (2.29).

The operators (2.29) automatically satisfy all of the commutation relations (2.28) except $[\mathrm{J}^+, \mathrm{J}^-] = 2a^2\mathrm{J}^3 - b\mathrm{E}$. An easy computation shows that this last relation will be satisfied provided k and j are solutions of the differential equations

$$\frac{dk}{dx}(x) + k(x)^2 = -a^2, \qquad \frac{dj}{dx}(x) + k(x)\,j(x) = -\frac{b\mu}{2}. \quad (2.30)$$

Thus, to find all operators of the form (2.29) it is sufficient to find all solutions of Eqs. (2.30).

$\mathcal{G}(1, 0)$ If $a = 1$, $b = 0$, Eqs. (2.30) have the solutions *type A*

$$k(x) = \cot(x + p), \qquad j(x) = \frac{q}{\sin(x + p)}$$

or *type B*

$$k(x) = i, \qquad j(x) = qe^{-ix}$$

where p, q are complex constants and $i = \sqrt{-1}$. The type designation is based on the classification of factorization types due to Infeld and Hull [1].

$\mathcal{G}(0, 1)$ If $a = 0$, $b = 1$, the solutions are *type C'*

$$k(x) = \frac{1}{x + p}, \qquad j(x) = -\frac{\mu(x + p)}{4} + \frac{q}{x + p}$$

or *type D'*

$$k(x) \equiv 0, \qquad j(x) = -\frac{\mu x}{2} + q$$

where p and q are complex constants.

$\mathcal{G}(0, 0)$ If $a = b = 0$, the solutions are *type C''*

$$k(x) = \frac{1}{x + p}, \qquad j(x) = \frac{q}{x + p}$$

or *type D''*

$$k(x) \equiv 0, \qquad j(x) = q$$

where p and q are complex constants.

Suppose we are able to realize an irreducible representation ρ of $\mathcal{G}(a, b)$ classified in Section 2-6 in such a way that the operators $\rho(\alpha)$ are differential operators of one of the types listed above and the basis space V is a space of analytic functions of x and y. The basis functions $f_m(x, y)$, $m \in S$, satisfy the equations

$$\mathrm{J}^3 f_m(x, y) = \frac{\partial}{\partial y} f_m(x, y) = m f_m(x, y).$$

Thus, $f_m(x, y) = g_m(x)e^{my}$ for all $m \in S$, where $g_m(x)$ is an analytic function of x. Since ρ is irreducible, we have

$$\mathrm{C}_{a,b} f_m = \lambda f_m, \qquad m \in S, \tag{2.31}$$

where the constant λ is uniquely determined by ρ. However, in terms of our realization $C_{a,b}$ is a second order partial differential operator and the relations (2.31) are a system of second order partial differential equations for the basis functions $f_m(x, y)$. Since $f_m(x, y) = g_m(x)e^{my}$ the dependence on y can be factored out and the relations simplify to a system of second order ordinary differential equations for the functions $g_m(x)$. The $g_m(x)$ will turn out to be special functions and the action of ρ will yield recursion relations, generating functions and addition theorems for these functions.

To determine the possible functions $g_m(x)$ which can arise in this way we recall

$$C_{a,b} = J^+J^- + a^2J^3J^3 - a^2J^3 - bJ^3E$$

and evaluate (2.31) for each of the operator types. The following differential equations are obtained for $g_m(x)$.

$\mathscr{G}(1, 0)$ For *type A* operators Eq. (2.31) becomes

$$-\frac{1}{\sin(x+p)}\left[\frac{d}{dx}\sin(x+p)\frac{d}{dx}g_m(x)\right] + \left[\frac{m^2+q^2-2qm\cos(x+p)}{\sin^2(x+p)}\right]g_m(x) = \lambda g_m(x).$$

According to Theorem 2.3, $\lambda = u(u+1)$. One solution of this equation is

$$g_m(x) = w^{(m-q)/2}(1+w)^{-u}F(m-u, -q-u; m-q+1; -w)$$

where $w = \tan^2[(x+p)/2]$. If $m - q$ is not a positive integer there is a linearly independent solution of the form

$$g_m(x) = w^{(q-m)/2}(1+w)^{-u}F(q-u, -m-u; q-m+1; -w).$$

These results show the connection between *type A* operator realizations of $\mathscr{G}(1, 0)$ and the hypergeometric functions.

For *type B* operators we obtain

$$-e^{-ix}\frac{d}{dx}\left[e^{ix}\frac{d}{dx}g_m(x)\right] + [-q^2e^{-2ix} + 2mqe^{-ix}]\,g_m(x) = u(u+1)\,g_m(x)$$

where $u(u+1) = \lambda$. If $2u$ is not an integer this equation has the linearly independent solutions

$$g_m(x) = (2qz)^{u+1}e^{-qz}\,{}_1F_1(u-m+1; 2u+2; 2qz)$$

and

$$g_m(x) = (2qz)^{-u}\,e^{-qz}\,{}_1F_1(-u-m; -2u; 2qz)$$

where $z = -ie^{-ix}$. This shows the relation between *type B* operator representations of $\mathscr{G}(1, 0)$ and the confluent hypergeometric functions.

$\mathscr{G}(0, 1)$ For *type C'* operators Eq. (2.31) becomes

$$-\frac{1}{x}\frac{d}{dx}\left[x\frac{d}{dx}g_m(x)\right] + \left[\frac{(m-q)^2}{x^2} - \frac{\mu(m+q+1)}{2} + \frac{\mu^2x^2}{16}\right]g_m(x)$$
$$= \mu\omega g_m(x), \qquad \lambda = \mu\omega.$$

If $m - q$ is not an integer this equation has the linearly independent solutions

$$g_m(x) = \left(\frac{x^2}{4}\right)^{\xi} \exp\left(-\frac{x^2}{8}\right) {}_1F_1(\xi - \eta + \tfrac{1}{2}\,; 2\xi + 1; x^2/4)$$

and

$$g_m(x) = \left(\frac{x^2}{4}\right)^{-\xi} \exp\left(-\frac{x^2}{8}\right) {}_1F_1(-\xi - \eta + \tfrac{1}{2}\,; -2\xi + 1; x^2/4)$$

where $\xi = (m - q)/2$ and $\eta = (\mu/2)(m + q + 2\omega + 1)$. These solutions are "functions of the paraboloid of revolution," and are closely related to the confluent hypergeometric functions.

Type D' operators yield the equation

$$-\frac{d^2}{dx^2}g_m(x) + \left(\frac{\mu^2x^2}{4} - \frac{\mu}{2} - m\mu\right)g_m(x) = \mu\omega g_m(x), \qquad \lambda = \mu\omega.$$

This is the parabolic cylinder equation which has as linearly independent solutions the parabolic cylinder functions $D_{m+\omega}(\sqrt{\mu}x)$, $D_{m+\omega}(-\sqrt{\mu}x)$.

$\mathscr{G}(0, 0)$ The functions $g_m(x)$ corresponding to *type C''* operators satisfy

$$-\frac{1}{x}\frac{d}{dx}\left[x\frac{d}{dx}g_m(x)\right] + \frac{m^2}{x^2}g_m(x) = \omega^2 g_m(x), \qquad \lambda = \omega^2.$$

This is Bessel's equation and its solutions are cylindrical functions. In particular, the Bessel functions $J_m(\omega x)$ and $J_{-m}(\omega x)$ satisfy this equation.

For *type D''* operators we obtain

$$-\frac{d^2}{dx^2}g_m(x) = \omega^2 g_m(x), \qquad \lambda = \omega^2.$$

The solutions $e^{\pm i\omega x}$ are special functions of a very simple kind.

This brief survey shows the deep relationship between functions of hypergeometric type and the Lie algebras $\mathscr{G}(a, b)$. In the next three chapters this relationship will be examined in detail.

2-8 Realizations of $\mathscr{G}(a, b)$ in One Variable

In the last section we found differential operators J^+, J^-, J^3, E in two complex variables, satisfying the commutation relations:

$$[J^3, J^+] = J^+, \quad [J^3, J^-] = -J^-, \quad [J^+, J^-] = 2a^2 J^3 - bE,$$
$$[J^3, E] = [J^+, E] = [J^-, E] = 0. \tag{2.32}$$

An analogous problem, which will be solved now, is to find realizations of (2.32) in terms of linear differential operators in one complex variable z. In particular we look for all nonzero differential operators of the form

$$J^3 = \lambda + z\frac{d}{dz}, \qquad E = \mu$$
$$J^+ = j_1(z) + j_2(z)\frac{d}{dz}, \qquad J^- = k_1(z) + k_2(z)\frac{d}{dz} \tag{2.33}$$

such that the commutation relations (2.32) are satisfied. Here, λ and μ are complex constants and j_1, j_2, k_1, k_2 are functions of z to be determined. J^3 has been chosen so that its eigenfunctions f_m will be of the form $z^{m-\lambda}$, i.e., powers of z. Although this choice of J^3 appears special it will be shown in Chapter 8 that every realization of $\mathscr{G}(a, b)$ by differential operators in one complex variable is equivalent to a realization of the form (2.33).

The operators (2.33) satisfy the commutation relations (2.32) if and only if

$$j_1(z) = c_1 z, \qquad j_2(z) = c_2 z^2, \qquad k_1(z) = \frac{c_3}{z}, \qquad k_2(z) = c_4$$

where the constants $c_1, \ldots, c_4$ satisfy the equations

$$c_2 c_4 = -a^2, \qquad c_2 c_3 + c_1 c_4 = -2a^2\lambda + b\mu. \tag{2.34}$$

It is enough to solve these equations for the three cases $\mathscr{G}(1, 0)$, $\mathscr{G}(0, 1)$, $\mathscr{G}(0, 0)$.

$\mathscr{G}(1, 0)$ Conditions (2.34) become $c_2 c_4 = -1$, $c_2 c_3 + c_1 c_4 = -2\lambda$. By a change of variable z if necessary, we can assume $c_2 = -c_4 = 1$. This leaves only the condition $c_3 - c_1 = -2\lambda$. Thus, the general solution is

$$J^3 = \lambda + z\frac{d}{dz}, \qquad E = \mu$$
$$J^+ = (2\lambda + c_3)z + z^2\frac{d}{dz}, \qquad J^- = \frac{c_3}{z} - \frac{d}{dz} \tag{2.35}$$

where c_3 is a constant.

$\mathcal{G}(0, 1)$ Conditions (2.34) become $c_2c_4 = 0$, $c_2c_3 + c_1c_4 = \mu$. We assume $\mu \neq 0$. If $c_2 = 0$ then $c_4 \neq 0$ and $c_1c_4 = \mu$. By a change of the variable z if necessary, we can assume $c_1 = \mu$, $c_4 = 1$. Thus, we obtain the solution

$$\begin{aligned} J^3 &= \lambda + z\frac{d}{dz}, \qquad E = \mu, \\ J^+ &= \mu z, \qquad J^- = \frac{c_3}{z} + \frac{d}{dz}, \qquad c_3 \in \mathcal{C}. \end{aligned} \tag{2.36}$$

If, however, $c_4 = 0$ then $c_2 \neq 0$ and $c_2c_3 = \mu$. We can assume $c_3 = \mu$, $c_2 = 1$ to obtain the solution

$$\begin{aligned} J^3 &= \lambda + z\frac{d}{dz}, \qquad E = \mu, \\ J^+ &= c_1 z + z^2\frac{d}{dz}, \qquad J^- = \frac{\mu}{z}, \qquad c_1 \in \mathcal{C}. \end{aligned} \tag{2.37}$$

$\mathcal{G}(0, 0)$ Conditions (2.34) are $c_2c_4 = 0$, $c_2c_3 + c_1c_4 = 0$. In order that $J^+ \not\equiv 0$, $J^- \not\equiv 0$ we must have $c_2 = c_4 = 0$. By a change of the variable z if necessary, we can assume $c_1 = c_3 \neq 0$ and obtain the solution

$$J^3 = \lambda + z\frac{d}{dz}, \qquad E = \mu, \qquad J^+ = c_1 z, \qquad J^- = \frac{c_1}{z}, \qquad c_1 \in \mathcal{C}. \tag{2.38}$$

In the next three chapters the differential operators (2.35)-(2.38) will be used to construct realizations of the abstract representations of $\mathcal{G}\ (a, b)$ classified in Section 2-6.