CHAPTER **6**

Special Functions Related to the Euclidean Group in 3-Space

In this chapter *type A* and *B* operators forming a realization of $sl(2)$ will be extended to *type E* and *F* operators, respectively, forming a realization of the 6-dimensional Lie algebra $\mathscr{T}_6$. The *type E* and *F* operators yield recursion relations for hypergeometric and confluent hypergeometric functions which are of a different nature than the recursion relations derived so far. In particular it is much more difficult to compute the matrix elements of multiplier representations induced by these operators. For this reason the results presented here are incomplete, and much remains to be done to obtain all of the special function identities implied by the representation theory of $\mathscr{T}_6$.

The Lie algebra of E_6, the Euclidean group in 3-space, is a real form of $\mathscr{T}_6$. In Sections 6-4 and 6-5 this relationship will be used to compute matrix elements of the unitary irreducible representations of E_6. The matrix elements turn out to be spinor-valued solutions of the wave equation $(\nabla^2 + \omega^2)\,\Psi(\mathbf{r}) = 0$ and are of wide applicability in theoretical physics. These results were first computed by Vilenkin *et al.* [1]. See also Miller [2].

6-1 Representations of $\mathscr{T}_6$

$\mathscr{T}_6$ is the 6-dimensional complex Lie algebra with generators $\mathscr{P}^+, \mathscr{P}^-, \mathscr{P}^3, \mathscr{J}^+, \mathscr{J}^-, \mathscr{J}^3$ and commutation relations

$$\begin{gathered}
[\mathscr{J}^3, \mathscr{J}^\pm] = \pm\mathscr{J}^\pm, \qquad [\mathscr{J}^3, \mathscr{P}^\pm] = [\mathscr{P}^3, \mathscr{J}^\pm] = \pm\mathscr{P}^\pm, \\
[\mathscr{J}^+, \mathscr{P}^+] = [\mathscr{J}^-, \mathscr{P}^-] = [\mathscr{J}^3, \mathscr{P}^3] = \mathscr{O}, \\
[\mathscr{J}^+, \mathscr{J}^-] = 2\mathscr{J}^3, \qquad [\mathscr{J}^+, \mathscr{P}^-] = [\mathscr{P}^+, \mathscr{J}^-] = 2\mathscr{P}^3, \\
[\mathscr{P}^3, \mathscr{P}^\pm] = [\mathscr{P}^+, \mathscr{P}^-] = \mathscr{O}.
\end{gathered} \tag{6.1}$$

It is straightforward to verify that the Jacobi equality is satisfied, so the relations (6.1) actually define a Lie algebra. Clearly, the elements $\mathscr{J}^+, \mathscr{J}^-, \mathscr{J}^3$ generate a subalgebra of $\mathscr{T}_6$ isomorphic to $sl(2)$. Indeed we will identify this subalgebra with $sl(2)$. The elements $\mathscr{P}^+, \mathscr{P}^-, \mathscr{P}^3$ generate a 3-dimensional abelian subalgebra of $\mathscr{T}_6$ which is also an ideal.

Denote by T_6 the complex 6-parameter Lie group consisting of all elements $\{\mathbf{w}, g\}$,

$$\mathbf{w} = (\alpha, \beta, \gamma) \in \mathscr{C}^3, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2), \qquad ad - bc = 1,$$

with group multiplication

$$\{\mathbf{w}, g\}\{\mathbf{w}', g'\} = \{\mathbf{w} + g\mathbf{w}', gg'\} \tag{6.2}$$

where the plus sign $(+)$ denotes vector addition in $\mathscr{C}^3$ and

$$g\mathbf{w} = (a^2\alpha - b^2\beta + ab\gamma,\ -c^2\alpha + d^2\beta - cd\gamma,\ 2ac\alpha - 2bd\beta + (bc + ad)\gamma).$$

In particular the identity element of T_6 is $\{\mathbf{0}, \mathbf{e}\}$ where $\mathbf{0} = (0, 0, 0)$ and $\mathbf{e}$ is the identity element of $SL(2)$; and the inverse of an element $\{\mathbf{w}, g\}$ is given by

$$\{\mathbf{w}, g\}^{-1} = \{-g^{-1}\mathbf{w}, g^{-1}\}.$$

The associative law can be verified directly. The set of all elements of the form $\{\mathbf{0}, g\}$, $g \in SL(2)$, forms a subgroup of T_6 which can be identified with $SL(2)$. Similarly the set of all elements of the form $\{\mathbf{w}, \mathbf{e}\}$, $\mathbf{w} \in \mathscr{C}^3$, forms a subgroup of T_6 which can be identified with $\mathscr{C}^3$.

It is straightforward to show that $\mathscr{T}_6$ is the Lie algebra of T_6. Indeed the generators can be chosen so $\mathscr{J}^+, \mathscr{J}^-, \mathscr{J}^3$ generate the subgroup $SL(2)$ of T_6, while $\{\mathbf{w}, \mathbf{e}\} = \exp(\alpha\mathscr{P}^+ + \beta\mathscr{P}^- + \gamma\mathscr{P}^3)$ for $\mathbf{w} = (\alpha, \beta, \gamma) \in \mathscr{C}^3$. Considered as Lie derivatives on the group manifold of T_6, the generators are given by the expressions

$$\begin{aligned}
\mathscr{P}^+ &= a^2 \frac{\partial}{\partial\alpha} - c^2 \frac{\partial}{\partial\beta} + 2ac \frac{\partial}{\partial\gamma}, \\
\mathscr{P}^- &= -b^2 \frac{\partial}{\partial\alpha} + \left(\frac{1+bc}{a}\right)^2 \frac{\partial}{\partial\beta} - 2b \frac{(1+bc)}{a} \frac{\partial}{\partial\gamma}, \\
\mathscr{P}^3 &= ab \frac{\partial}{\partial\alpha} - c \frac{(1+bc)}{a} \frac{\partial}{\partial\beta} + (1+2bc) \frac{\partial}{\partial\gamma}, \qquad \mathscr{J}^+ = -a \frac{\partial}{\partial b}, \\
\mathscr{J}^- &= -b \frac{\partial}{\partial a} - \frac{(1+bc)}{a} \frac{\partial}{\partial c}, \qquad \mathscr{J}^3 = \frac{1}{2}\left(a \frac{\partial}{\partial a} - b \frac{\partial}{\partial b} + c \frac{\partial}{\partial c}\right)
\end{aligned} \tag{6.3}$$

(see Section 3-2). (It follows from the general results of Chapter 1 that as a local Lie group, T_6 is uniquely determined by the commutation relations (6.1). However, the global structure of the group is not uniquely determined and the global group (6.2) has been chosen for convenience).

Consider a complex vector space V and a representation ρ of $\mathscr{T}_6$ by linear operators on V. Set

$$\rho(\mathscr{P}^\pm) = \mathrm{P}^\pm, \qquad \rho(\mathscr{P}^3) = \mathrm{P}^3, \qquad \rho(\mathscr{J}^\pm) = \mathrm{J}^\pm, \qquad \rho(\mathscr{J}^3) = \mathrm{J}^3.$$

Then the linear operators $\mathrm{P}^\pm$, P^3, $\mathrm{J}^\pm$, J^3 satisfy commutation relations on V entirely analogous to (6.1), where now $[A, B] = AB - BA$ for operators A and B on V. We can define two operators on V which are of special importance for the representation theory of $\mathscr{T}_6$. They are

$$\mathbf{P}\cdot\mathbf{P} = -\mathrm{P}^+\mathrm{P}^- - \mathrm{P}^3\mathrm{P}^3, \qquad \mathbf{P}\cdot\mathbf{J} = -\tfrac{1}{2}(\mathrm{P}^+\mathrm{J}^- + \mathrm{P}^-\mathrm{J}^+) - \mathrm{P}^3\mathrm{J}^3. \tag{6.4}$$

It is easy to verify the relations

$$[\mathbf{P}\cdot\mathbf{P}, \rho(\alpha)] = 0, \qquad [\mathbf{P}\cdot\mathbf{J}, \rho(\alpha)] = 0$$

for all $\alpha \in \mathscr{T}_6$. Thus if ρ is an irreducible representation of $\mathscr{T}_6$ we would expect $\mathbf{P}\cdot\mathbf{P}$ and $\mathbf{P}\cdot\mathbf{J}$ to be multiples of the identity operator on V.

By restricting an irreducible representation ρ of $\mathscr{T}_6$ to the subalgebra $sl(2)$ we can obtain a representation $\rho/sl(2)$ (ρ restricted to $sl(2)$) of $sl(2)$. Suppose $\rho/sl(2)$ can be decomposed into a direct sum of irreducible representations of $sl(2)$ as classified in Chapter 5. In this case the operators P^+, P^-, P^3 may "mix up" the representations of $sl(2)$ and map basis vectors corresponding to one subspace of V, invariant under $sl(2)$, into other subspaces. By finding a functional realization of such a representation ρ it may then be possible to derive differential recursion relations relating special functions associated with **different** irreducible representations of $sl(2)$. It is this possibility which is a principle motivation for the consideration of $\mathscr{T}_6$. As will be shown, by choosing appropriate irreducible representations of $\mathscr{T}_6$ one can obtain recursion relations for hypergeometric and confluent hypergeometric functions which are distinct from those derived in Chapters 4 and 5.

To be specific we will classify all representations ρ of $\mathscr{T}_6$ on the vector space V with the properties:

(i) $V = \sum_{u\in Q} \oplus V_u$, where the summation ranges over a countable number of values of u, each value of u occurring at most once. $\rho/sl(2)$ acts irreducibly on each subspace V_u and coincides there with one of the irreducible representations $D(u, m_o)$, $\uparrow_u$, $\downarrow_u$, $D(2u)$ classified in Theorem 2.3.

(ii) $\rho/sl(2)$ has one of the possible forms:

$$(1)\quad \rho/sl(2) \cong \sum_{u\in Q} \oplus D(u, m_0),$$

$$(2)\quad \rho/sl(2) \cong \sum_{u\in Q} \oplus \uparrow_u,$$

$$(3)\quad \rho/sl(2) \cong \sum_{u\in Q} \oplus \downarrow_u,$$

$$(4)\quad \rho/sl(2) \cong \sum_{u\in Q} \oplus D(2u),$$

where in each case the summation extends over the same values of $u \in Q$ as in (i).

(iii) ρ is irreducible.

(iv) $\mathbf{P}\cdot\mathbf{P} \neq 0$ on V.

The results of this classification are as follows: The basis vectors of each subspace V_u of V are denoted by $f_m^{(u)}$ where $J^3 f_m^{(u)} = m f_m^{(u)}$ and the $\{f_m^{(u)}\}$ for a fixed u form a canonical basis for an irreducible representation of $sl(2)$ as listed in Theorem 2.3. $Q = \{u\colon V_u \subset V,\ V_u \neq 0\}$. Let ω and q be arbitrary complex constants such that $\omega \neq 0$.

Theorem 6.1 Every representation ρ of $\mathscr{T}_6$ satisfying conditions (i)–(iv) is isomorphic to a representation in the following list:

(a) $\uparrow_j(\omega, q)$, $1 \leqslant j \leqslant 4$. The spectrum $Q = \{-q + n\colon n \text{ an integer} \geqslant 0\}$. The $\uparrow_j(\omega, q)/sl(2)$ takes the form (j) given by condition (ii). For $j = 1$, $2q$ is not a nonnegative integer. If $j = 2$ or 3, $2q$ is not an integer. If $j = 4$ then $-2q$ is a nonnegative integer.

(b) $\downarrow_j(\omega, q), 1 \leqslant j \leqslant 3$. $2q$ is not a nonnegative integer. The spectrum $Q = \{q - n - 1: n \text{ an integer} \geqslant 0\}$. The $\downarrow_j(\omega, q)/sl(2)$ takes the form (j) given by condition (ii).

(c) $R_j(\omega, q, u_o), 1 \leqslant j \leqslant 3$. Here q and u_o are complex numbers such that $0 \leqslant \operatorname{Re} u_o < 1$ and none of $u_o \pm q$, or $2u_o$ is an integer. $Q = \{u_o + n: n \text{ an integer}\}$. The $R_j(\omega, q, u_o)/sl(2)$ takes the form (j) given by condition (ii).

(d) $R_1(\omega, 0, 0)$. $j = 1, q = u_o = 0,\quad Q = \{n: n \text{ an integer}\}$.

Corresponding to each of these representations there is a basis for V_u, $u \in Q$, of the form $\{f_m^{(u)}\}$ such that

$$\mathrm{J}^3 f_m^{(u)} = m f_m^{(u)}, \quad \mathrm{J}^+ f_m^{(u)} = (m - u) f_{m+1}^{(u)}, \quad \mathrm{J}^- f_m^{(u)} = -(m + u) f_{m-1}^{(u)}, \tag{6.5}$$

$$\mathrm{P}^3 f_m^{(u)} = \frac{\omega(u - q + 1)}{(2u + 1)(u + 1)} f_m^{(u+1)} + \frac{m\omega q}{u(u + 1)} f_m^{(u)} + \frac{\omega(u + q)(u + m)(u - m)}{u(2u + 1)} f_m^{(u-1)},$$

$$\mathrm{P}^+ f_m^{(u)} = \frac{\omega(u - q + 1)}{(2u + 1)(u + 1)} f_{m+1}^{(u+1)} - \frac{(u - m)\omega q}{u(u + 1)} f_{m+1}^{(u)} - \frac{\omega(u + q)(u - m)(u - m - 1)}{(2u + 1)u} f_{m+1}^{(u-1)},$$

$$\mathrm{P}^- f_m^{(u)} = -\frac{\omega(u - q + 1)}{(2u + 1)(u + 1)} f_{m-1}^{(u+1)} - \frac{(u + m)\omega q}{u(u + 1)} f_{m-1}^{(u)} + \frac{\omega(u + q)(u + m)(u + m - 1)}{(2u + 1)u} f_{m-1}^{(u-1)},$$

$$\mathbf{P} \cdot \mathbf{P} f_m^{(u)} = -\omega^2 f_m^{(u)}, \qquad \mathbf{P} \cdot \mathbf{J} f_m^{(u)} = -\omega q f_m^{(u)}.$$

The representations listed above are not all distinct. In fact we have the isomorphisms $\uparrow_1(\omega, q) \cong \downarrow_1(\omega, -q - 1)$ for $2q$ not an integer, and $R_1(\omega, q, u_o) \cong R_1(\omega, q, -u_o - 1)$.

The details of the proof of this theorem will not be given since they are somewhat tedious. The method of proof closely follows that of Naimark [2], Chapter 3. For our purposes it is enough to know that the listed representations in Theorem 6.1 actually are irreducible representations of $\mathscr{T}_6$ and this fact is easy to verify from the explicit expressions (6.5).

For each of the representations listed above we define operators $X(u, +)$ and $X(u, -)$ such that

$$X(u, +) = P^-J^+ + P^3J^3 + (u + 1)P^3 - \frac{\omega q}{u + 1} J^3 - \omega q I,$$
$$X(u, -) = -P^-J^+ - P^3J^3 + uP^3 - \frac{\omega q}{u} J^3 + \omega q I, \tag{6.6}$$

where I is the identity operator on V. If $f_m^{(u)}$ is a basis vector for V, then Eqs. (6.5) imply

$$X(u, +)f_m^{(u)} = \frac{\omega(u - q + 1)}{u + 1} f_m^{(u+1)},$$
$$X(u, -)f_m^{(u)} = \frac{\omega(u + m)(u - m)(u + q)}{u} f_m^{(u-1)}. \tag{6.7}$$

The range of values of u and m for which these expressions make sense depends on the particular representation of $\mathscr{T}_6$ with which we are concerned. It is clear from (6.7) that $X(u, +)$ and $X(u, -)$ can be considered as u-raising and u-lowering operators, respectively, in the same way that J^+ and J^- are m-raising and -lowering operators. However, there is a profound difference between these two sets of operators. In particular the $X(u, \pm)$ are functions of u and are not members of the Lie algebra generated by the elements $\rho(\alpha)$, $\alpha \in \mathscr{T}_6$. This is in distinction to $J^\pm$. Consequently, it is reasonable to expect that recursion relations for special functions obtained from (6.7) should be of a different nature from the recursion relations derived in Chapter 5.

For the classification of irreducible representations of $\mathscr{T}_6$ as given by Theorem 6.1 we have assumed $\mathbf{P} \cdot \mathbf{P} \not\equiv 0$ on V. It is possible to find additional irreducible representations of $\mathscr{T}_6$ which satisfy conditions (i)–(iii), but instead of (iv), satisfy

(iv)′ $\mathbf{P} \cdot \mathbf{P} \equiv 0, \quad \mathbf{P} \cdot \mathbf{J} \not\equiv 0$ on V.

Again, only the results of the classification will be given.

Theorem 6.2 Every representation ρ of $\mathscr{T}_6$ satisfying conditions (i)–(iv)′ is isomorphic to a representation of the form $R_j'(\zeta, u_o)$, $1 \leqslant j \leqslant 3$, where ζ and u_o are complex constants such that $\zeta \neq 0$, $0 \leqslant \operatorname{Re} u_o < 1$ and $2u_o$ is not an integer. The spectrum $Q = \{u_o + n\colon n \text{ an integer}\}$. $R_j'(\zeta, u_o)/sl(2)$ takes the form (j) given by condition (ii). Corresponding to each representation $R'_j(\zeta, u_o)$ there is a basis $\{ f^{(u)}_m\}$, $u \in Q$, for V such that

$$
\begin{aligned}
&\mathrm{J}^3 f^{(u)}_m = m f^{(u)}_m, \qquad \mathrm{J}^+ f^{(u)}_m = (m-u) f^{(u)}_{m+1}, \qquad \mathrm{J}^- f^{(u)}_m = -(m+u) f^{(u)}_{m-1},\\
&\mathrm{P}^3 f^{(u)}_m = \frac{-\zeta}{(2u+1)(u+1)} f^{(u+1)}_m + \frac{\zeta m}{u(u+1)} f^{(u)}_m\\
&\qquad + \frac{\zeta(u+m)(u-m)}{(2u+1)u} f^{(u-1)}_m,\\
&\mathrm{P}^+ f^{(u)}_m = \frac{-\zeta}{(2u+1)(u+1)} f^{(u+1)}_{m+1} - \frac{\zeta(u-m)}{u(u+1)} f^{(u)}_{m+1}\\
&\qquad - \frac{\zeta(u+m)(u-m-1)}{(2u+1)u} f^{(u-1)}_{m+1},\\
&\mathrm{P}^- f^{(u)}_m = \frac{\zeta}{(2u+1)(u+1)} f^{(u+1)}_{m-1} - \frac{\zeta(u+m)}{u(u+1)} f^{(u)}_{m-1}\\
&\qquad + \frac{\zeta(u+m)(u+m-1)}{(2u+1)u} f^{(u-1)}_{m-1},\\
&\mathbf{P}\cdot\mathbf{P} f^{(u)}_m = 0, \qquad \mathbf{P}\cdot\mathbf{J} f^{(u)}_m = -\zeta f^{(u)}_m.
\end{aligned}
\tag{6.8}
$$

With the exception of the isomorphism $R'_1(\zeta, u_o) \cong R'_1(\zeta, -u_o - 1)$, all of the above irreducible representations are distinct. (Equations (6.8) can be obtained formally from (6.5) by setting $q = -\zeta/\omega$ and passing to the limit as $\omega \to 0$.)

For each of the representations $R'_j(\zeta, u_o)$ listed above we define operators $\mathrm{X}'(u, +)$ and $X'(u, -)$ on V such that

$$
\begin{aligned}
\mathrm{X}'(u, +) &= \mathrm{P}^-\mathrm{J}^+ + \mathrm{P}^3\mathrm{J}^3 + (u+1)\mathrm{P}^3 - \frac{\zeta}{u+1}\mathrm{J}^3 - \zeta\mathrm{I},\\
\mathrm{X}'(u, -) &= -\mathrm{P}^-\mathrm{J}^+ - \mathrm{P}^3\mathrm{J}^3 + u\mathrm{P}^3 - \frac{\zeta}{u}\mathrm{J}^3 + \zeta\mathrm{I}.
\end{aligned}
\tag{6.9}
$$

As a consequence of relations (6.8), these operators satisfy

$$
\begin{aligned}
\mathrm{X}'(u, +) f^{(u)}_m &= \frac{-\zeta}{u+1} f^{(u+1)}_m,\\
\mathrm{X}'(u, -) f^{(u)}_m &= \frac{\zeta(u+m)(u-m)}{u} f^{(u-1)}_m,
\end{aligned}
\tag{6.10}
$$

where the range of values of u and m for which the expressions make sense depends on the representation $R'_j(\zeta, u_o)$. The $\mathrm{X}'(u, \pm)$ can be considered as raising and lowering operators for the index u. However, these operators differ greatly from the m-raising and -lowering operators J^+, J^- since they are functions of u and are not elements of the Lie algebra generated by the $\rho(\alpha)$, $\alpha \in \mathscr{T}_6$.

We could continue the classification scheme of representations of $\mathscr{T}_6$ given above and determine all representations satisfying conditions (i), (ii), (iii) and

$$\text{(iv)}'' \quad \mathbf{P}\cdot\mathbf{P} \equiv 0, \qquad \mathbf{P}\cdot\mathbf{J} \equiv 0 \quad \text{on} \quad V.$$

However, it turns out that for such representations we must have $P^+ \equiv P^- \equiv P^3 \equiv 0$. Thus these representations reduce to irreducible representations of $sl(2)$ which have already been classified in Chapter 5.

6-2 *Type E* Operators

We designate as *type E* the differential operators

$$\begin{gathered} J^3 = t\frac{\partial}{\partial t}, \qquad J^\pm = t^{\pm 1}\left(-(z^2-1)^{1/2}\frac{\partial}{\partial z} \pm \frac{z}{(z^2-1)^{1/2}}\,t\frac{\partial}{\partial t} \mp \frac{q}{(z^2-1)^{1/2}}\right), \\ P^3 = \omega z, \qquad P^\pm = \pm\omega t^{\pm 1}(z^2-1)^{1/2}, \qquad (6.11) \\ \mathbf{P}\cdot\mathbf{P} = -\omega^2, \qquad \mathbf{P}\cdot\mathbf{J} = -\omega q, \qquad \omega, q \in \mathscr{C}, \quad \omega \neq 0. \end{gathered}$$

Clearly these generalized Lie derivatives satisfy the commutation relations (6.1) and thus generate a Lie algebra isomorphic to $\mathscr{T}_6$. (In order to properly define the square roots occuring in (6.11) we assume z takes the values in the complex plane cut along the real axis from $-\infty$ to $+1$. If $z > 1$ we require $(z^2-1)^{1/2} > 0$. The operators (6.11) are defined for all $t \in \mathscr{C}$ except $t = 0$.) Here $J^\pm$, J^3 are just the *type A* operators (5.106). As will be shown, *type E* operators can be used to construct realizations of the representations of $\mathscr{T}_6$ listed in Theorem 6.1. However, they cannot be used to construct realizations of the representations given by Theorem 6.2, since if $\mathbf{P}\cdot\mathbf{P} = \omega^2 = 0$ then $P^+ = P^- = P^3 = 0$.

To construct the representations listed in Theorem 6.1 we must find analytic functions $f_m^{(u)}(z, t) = Z_m^{(u)}(z)\,t^m$ such that

$$\begin{aligned} \left[-(z^2-1)^{1/2}\frac{d}{dz} + \frac{(zm-q)}{(z^2-1)^{1/2}}\right] Z_m^{(u)}(z) &= (m-u)Z_{m+1}^{(u)}(z), \\ \left[-(z^2-1)^{1/2}\frac{d}{dz} - \frac{(mz-q)}{(z^2-1)^{1/2}}\right] Z_m^{(u)}(z) &= -(m+u)Z_{m-1}^{(u)}(z), \end{aligned} \qquad (6.12)$$

and

$$zZ_m^{(u)}(z) = \frac{(u-q+1)}{(2u+1)(u+1)} Z_m^{(u+1)}(z) + \frac{mq}{u(u+1)} Z_m^{(u)}(z)$$
$$+ \frac{(u+q)(u+m)(u-m)}{(2u+1)u} Z_m^{(u-1)}(z),$$
$$(z^2-1)^{1/2} Z_m^{(u)}(z) = \frac{(u-q+1)}{(2u+1)(u+1)} Z_{m+1}^{(u+1)}(z) - \frac{(u-m)q}{u(u+1)} Z_{m+1}^{(u)}(z)$$
$$- \frac{(u+q)(u-m)(u-m-1)}{(2u+1)u} Z_{m+1}^{(u-1)}(z), \tag{6.13}$$
$$-(z^2-1)^{1/2} Z_m^{(u)}(z) = -\frac{(u-q+1)}{(2u+1)(u+1)} Z_{m-1}^{(u+1)}(z) - \frac{(u+m)q}{u(u+1)} Z_{m-1}^{(u)}(z)$$
$$+ \frac{(u+q)(u+m)(u+m-1)}{(2u+1)u} Z_{m-1}^{(u-1)}(z),$$

valid for all $u \in Q$ (and for all m in the spectrum of the representation of $sl(2)$ corresponding to u) on the left-hand sides of these equations. Moreover, relations (6.7) imply

$$\left[(z^2-1)\frac{d}{dz} + (u+1)z - \frac{qm}{u+1}\right] Z_m^{(u)}(z) = \frac{(u-q+1)}{u+1} Z_m^{(u+1)}(z),$$
$$\left[-(z^2-1)\frac{d}{dz} + uz - \frac{qm}{u}\right] Z_m^{(u)}(z) = \frac{(u+m)(u-m)(u+q)}{u} Z_m^{(u-1)}(z). \tag{6.14}$$

Clearly, all of the above equations are independent of the nonzero parameter ω.

To find solutions $\{Z_m^{(u)}(z)\}$ for these equations it is enough to solve (6.12) and (6.14). Indeed the first of Eqs. (6.13) can be obtained by adding the two expressions (6.14). The second and third equations are simple consequences of the first and the relations $[\mathrm{P}^3, \mathrm{J}^\pm] = \pm\mathrm{P}^\pm$. With this simplification it is not difficult to find solutions. The functions

$$Z_m^{(u)}(z) = \Gamma(u+m+1)\, \mathfrak{B}_u^{-q,m}(z) \tag{6.15}$$

defined for all $u \in Q$ and all m in the spectrum of the irreducible representation of $sl(2)$ corresponding to u satisfy the above equations unless $j = 2$. (The fact that these functions satisfy Eqs. (6.12) follows directly from Chapter 5. Equations (6.14) can be verified through direct substitution of the infinite series expansions (A.9) for $\mathfrak{B}_u^{-q,m}(z)$. However for certain representations these functions are identically zero.) The special case $j = 2$ is left to the reader.

To be more precise we must specify the possible values of u, m, and q which can appear on the left-hand sides of (6.12)–(6.14), corresponding to each irreducible representation of $\mathscr{T}_6$. The results are

(a) $\uparrow_j(\omega, q)$ $\qquad 1 \leqslant j \leqslant 4; \quad \omega \neq 0$

$j = 1$: $2q$ not a nonnegative integer;
$u = -q, -q + 1, -q + 2, \ldots;$
$m = m_o, m_o \pm 1, m_o \pm 2, \ldots;$
where m_o is a complex number such that $q \pm m_o$ are not integers and $0 \leqslant \operatorname{Re} m_o < 1$.

$j = 2$: $2q$ not an integer;
$u = -q, -q + 1, -q + 2, \ldots;$
$m = -u, -u + 1, -u + 2, \ldots .$

$j = 3$: $2q$ not an integer;
$u = -q, -q + 1, -q + 2, \ldots;$
$m = u, u - 1, u - 2, \ldots .$

$j = 4$: $-2q$ a nonnegative integer;
$u = -q, -q + 1, \ldots;$
$m = -u, -u + 1, \ldots, +u.$

(b) $\downarrow_j(\omega, q)$ $\qquad 1 \leqslant j \leqslant 3; \quad \omega \neq 0; \quad 2q$ not a nonnegative integer;

$j = 1$: $u = q - 1, q - 2, q - 3, \ldots;$
$m = m_o, m_o \pm 1, m_o \pm 2, \ldots;$
where m_o is a complex number such that $q \pm m_o$ are not integers and $0 \leqslant \operatorname{Re} m_o < 1$.

$j = 2$: $u = q - 1, q - 2, \ldots;$
$m = -u, -u + 1, -u + 2, \ldots .$

$j = 3$: $u = q - 1, q - 2, \ldots; \quad m = u, u - 1, u - 2, \ldots .$

(c) $R_j(\omega, q, u_o)$, $\qquad 1 \leqslant j \leqslant 3; \quad \omega \neq 0; \quad q \in \mathscr{C}$,
$0 \leqslant \operatorname{Re} u_o < 1; \quad u_o \pm q$ and $2u_o$ are not integers;

$j = 1$: $u = u_o, u_o \pm 1, u_o \pm 2, \ldots; \quad m = m_o \pm 1, m_o \pm 2, \ldots;$
where m_o is a complex number such that $u_o \pm m_o$ are not integers and $0 \leqslant \operatorname{Re} m_o < 1$.

$j = 2$: $u = u_o, u_o \pm 1, u_o \pm 2, \ldots;$
$m = -u, -u + 1, -u + 2, \ldots .$

$j = 3$: $u = u_o, u_o \pm 1, u_o \pm 2, \ldots; \quad m = u, u - 1, u - 2, \ldots .$

(d) $R_1(\omega, 0, 0)$, $j = 1$; $\omega \neq 0$; $q = u_o = 0$.
$u = 0, \pm 1, \pm 2, \ldots$; $m = m_o, m_o \pm 1, m_o \pm 2, \ldots$;
where $m_o \in \mathcal{C}$ such that $0 < \operatorname{Re} m_o < 1$.

The functions (6.15) are, of course, not the only solutions of our set of recursion relations. The construction of a linearly independent set of solutions is left to the reader.

In conclusion, we have found realizations by *type E* operators of most of the representations listed in Theorem 6.1. The basis vectors for these realizations are the functions (6.15) obtained from Chapter 5 in connection with the realization of representations of $sl(2)$ by *type A* operators. Our results yield new recursion relations for these basis functions. In particular we have seen that the recursion relations (6.14) are of an entirely different nature from the recursion relations derived earlier in this book. Equations (6.14) do not correspond directly to a representation of $\mathcal{T}_6$ but rather to a representation of the universal enveloping algebra of $\mathcal{T}_6$ (Jacobson [1], Chapter 5). Thus, it is not possible to obtain addition theorems and generating functions from these relations using the simple procedure we have employed up to now.

In this connection it is interesting to examine Truesdell's procedure for deriving identities for special functions directly from their recursion relations (Truesdell [1]). A careful analysis of his method reveals that it is basically Lie algebraic in nature. Indeed, his method succeeds for all of the recursion relations derived in Chapters 3–5, but fails for the recursion relations (6.14). Our more detailed analysis provides an explanation for this failure.

The problem of extending the representations of $\mathcal{T}_6$ given above to local multiplier representations of T_6 is a rather difficult one and we will not solve it here. The principal difficulties involve the verification that the functions $f_m^{(u)}(z, t) = Z_m^{(u)}(z)\, t^m$ form an analytic basis for the representation space and the computation of matrix elements of the operators $\exp(x\mathrm{P}^+ + y\mathrm{P}^- + w\mathrm{P}^3)$. Among the addition theorems which can be obtained from such an analysis are identities involving spherical Bessel functions and Gegenbauer polynomials. In Section 6-4 we will solve a special case of this problem to compute the irreducible unitary representations of the group E_6 whose Lie algebra is a real form of $\mathcal{T}_6$.

Finally, any attempt to find realizations of representations of $\mathcal{T}_6$ by generalized Lie derivatives in one complex variable is doomed to failure. It will be proved in Chapter 8 that no such realization exists.

6-3 Type F Operators

The *type F* differential operators are

$$
\begin{gathered}
J^3 = t\frac{\partial}{\partial t}, \qquad J^{\pm} = t^{\pm 1}\left(z\frac{\partial}{\partial z} \pm t\frac{\partial}{\partial t} \mp \frac{z}{2}\right), \\
P^3 = 2\zeta z^{-1}, \qquad P^{\pm} = \pm 2\zeta t^{\pm 1} z^{-1}, \\
\mathbf{P}\cdot\mathbf{P} = 0, \qquad \mathbf{P}\cdot\mathbf{J} = -\zeta, \qquad 0 \neq \zeta \in \mathcal{C}.
\end{gathered}
\tag{6.16}
$$

These operators satisfy the commutation relations (6.1), and generate a Lie algebra isomorphic to $\mathcal{T}_6$. Here J^+, J^-, and J^3 are the *type B* operators (5.85), where we have set $t = e^{\tau}$, $q = \frac{1}{2}$. Since $\mathbf{P}\cdot\mathbf{P} = 0$ the generalized Lie derivatives (6.16) cannot be used to find realizations of the representations of $\mathcal{T}_6$ listed in Theorem 6.1. However, they are well suited for the realization of representations classified in Theorem 6.2.

To construct realizations of $R_j'(\zeta, u_o)$ using *type F* operators we must find nonzero analytic functions $f_m^{(u)}(z, t) = Z_m^{(u)}(z)\, t^m$ such that

$$
\begin{aligned}
\left[z\frac{d}{dz} + m - \frac{z}{2}\right] Z_m^{(u)}(z) &= (m-u) Z_{m+1}^{(u)}(z), \\
\left[z\frac{d}{dz} - m + \frac{z}{2}\right] Z_m^{(u)}(z) &= -(m+u) Z_{m-1}^{(u)}(z),
\end{aligned}
\tag{6.17}
$$

and

$$
\begin{aligned}
2z^{-1}Z_m^{(u)}(z) &= -\frac{Z_m^{(u+1)}(z)}{(2u+1)(u+1)} + \frac{m}{u(u+1)} Z_m^{(u)}(z) \\
&\quad + \frac{(u+m)(u-m)}{(2u+1)u} Z_m^{(u-1)}(z), \\
-2z^{-1}Z_m^{(u)}(z) &= \frac{Z_{m+1}^{(u+1)}(z)}{(2u+1)(u+1)} + \frac{(u-m)}{u(u+1)} Z_{m+1}^{(u)}(z) \\
&\quad + \frac{(u+m)(u-m-1)}{(2u+1)u} Z_{m+1}^{(u-1)}(z), \\
-2z^{-1}Z_m^{(u)}(z) &= \frac{Z_{m-1}^{(u+1)}(z)}{(2u+1)(u+1)} - \frac{(u+m)}{u(u+1)} Z_{m-1}^{(u)}(z) \\
&\quad + \frac{(u+m)(u+m-1)}{(2u+1)u} Z_{m-1}^{(u-1)}(z),
\end{aligned}
\tag{6.18}
$$

valid for all $u \in Q$ and all m in the spectrum of the representation of $sl(2)$ corresponding to u on the left-hand sides of these equations. From (6.10) we have the relations

$$\left[-2\frac{d}{dz} + 2\frac{u+1}{z} - \frac{m}{u+1}\right] Z_m^{(u)}(z) = \frac{-1}{u+1} Z_m^{(u+1)}(z),$$
$$\left[2\frac{d}{dz} + 2\frac{u}{z} - \frac{m}{u}\right] Z_m^{(u)}(z) = \frac{(u+m)(u-m)}{u} Z_m^{(u-1)}(z). \tag{6.19}$$

All of the above equations are independent of the parameter ζ. To solve these equations it is enough to find solutions for (6.17) and (6.19). This can be seen by noting that the first of Eqs. (6.18) is just the sum of the two equations (6.19). The second and third of Eqs. (6.18) can be obtained from the relations $[\mathrm{P}^3, \mathrm{J}^\pm] = \pm \mathrm{P}^\pm$.

It is now easy to find a solution. According to Section 5-8 on *type B* operators, the functions

$$Z_m^{(u)}(z) = (-z)^{u+1} e^{-z/2} L_{m-u-1}^{(2u+1)}(z), \qquad j = 1 \tag{6.20}$$

and the functions

$$Z_m^{(u)\prime}(z) = (-z)^{-u} e^{-z/2} \frac{\Gamma(m+u+1)}{\Gamma(m-u)} L_{m+u}^{(-2u-1)}(z)$$
$$= \frac{(-z)^{-u} e^{-z/2}}{\Gamma(-2u)} {}_1F_1(-m-u; -2u; z) \tag{6.21}$$

both satisfy Eqs. (6.17). Moreover, by using the power series expansion for the generalized Laguerre functions it is easy to verify that these functions also satisfy (6.19).

Corresponding to each irreducible representation $R_j'(\zeta, u_o)$ of $\mathscr{T}_6$ the following range of values for u and m can appear on the left-hand sides of Eqs. (6.17)–(6.19):

$R_j'(\zeta, u_o)$ $\qquad 1 \leqslant j \leqslant 3;\ \ \zeta \neq 0;\ \ 0 \leqslant \operatorname{Re} u_0 < 1;$ $2u_o$ not an integer.

$j = 1$: $\ u = u_o, u_o \pm 1, u_o \pm 2, \ldots;\ \ m = m_o, m_o \pm 1, m_o \pm 2, \ldots;$

where m_o is a complex number such that $m_o \pm u_o$ are not integers and $0 \leqslant \operatorname{Re} m_o < 1$.

$j = 2$: $\ u = u_o, u_o \pm 1, u_o \pm 2, \ldots;\ \ m = -u, -u+1, -u+2, \ldots$.

$j = 3$: $\ u = u_o, u_o \pm 1, u_o \pm 2, \ldots;\ \ m = u, u-1, u-2, \ldots$.

Thus, using *type F* operators we have constructed realizations of all representations of $\mathscr{T}_6$ listed in Theorem 6.2. Either (6.20) or (6.21), obtained from Section 5-8 in the study of *type B* operators, can be used as basis vectors for these realizations. Among the new relations derived for the basis functions are (6.19), which do not have the simple Lie algebraic interpretation enjoyed by the other recursion relations derived in this section. As might be expected from our corresponding discussion of *type E* operators, Truesdell's technique for obtaining special function identities from recursion relations fails when applied to (6.19).

Equations (6.17)–(6.18) can also be solved in terms of generalized Laguerre functions when $2u$ is an integer, even though this case is not included in the spectrum Q of any of the representations $R_j'(\zeta, u_o)$ listed above. Indeed, except for the singular values $u = -1, -\frac{1}{2}, 0$, we can always find solutions. However, since the equations become meaningless for the singular values of u, these solutions do not form a basis for a representation of $\mathcal{T}_6$.

The relatively difficult problem of computing matrix elements of the local multiplier representation of T_6 induced by the irreducible representation $R_j'(\zeta, u_o)$ of $\mathcal{T}_6$ will be omitted.

6-4 The Euclidean Group E_6

E_6 is the real 6-parameter Lie group consisting of all pairs $(\mathbf{r}, A)$ where $\mathbf{r} = (r_1, r_2, r_3)$ is a real column vector and A is an element of $SU(2)$. The group multiplication law is

$$(\mathbf{r}, A)(\mathbf{r}', A') = (\mathbf{r} + R(A)\mathbf{r}', AA') \tag{6.22}$$

where the plus sign $(+)$ means vector addition, AA' corresponds to multiplication in $SU(2)$, and $R(A)$ is a real 3×3 orthogonal matrix defined by (5.135). If

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad a\bar{a} + b\bar{b} = 1,$$

then $R(A)$ has the explicit expression

$$R(A) = \begin{pmatrix} \dfrac{a^2 - b^2 + \bar{a}^2 - \bar{b}^2}{2} & \dfrac{i(\bar{a}^2 + \bar{b}^2 - a^2 - b^2)}{2} & \bar{a}\bar{b} + ab \\ \dfrac{i(a^2 - b^2 - \bar{a}^2 + \bar{b}^2)}{2} & \dfrac{(\bar{a}^2 + \bar{b}^2 + a^2 + b^2)}{2} & i(-\bar{a}\bar{b} + ab) \\ -(\bar{a}b + a\bar{b}) & i(-\bar{a}b + a\bar{b}) & a\bar{a} - b\bar{b} \end{pmatrix}. \tag{6.23}$$

Since $SU(2)$ is simply connected, so is E_6. In fact E_6 is the simply connected covering group of the Euclidean group in 3-space: the group of all pairs $(\mathbf{r}, R)$, $\mathbf{r}$ a real 3-vector, $R \in O_3$, with group multiplication

$$(\mathbf{r}, R)(\mathbf{r}', R') = (\mathbf{r} + R\mathbf{r}', RR').$$

The two-to-one onto homomorphism relating the two groups is $(\mathbf{r}, \pm A) \to (\mathbf{r}, R(A))$, where $R(A) = R(-A)$ is defined by (5.135).

E_6 can be considered as a real subgroup of T_6. Indeed, as the reader can easily verify, the collection of all elements $\{\mathbf{w}, A\}$, where $\mathbf{w} = (\frac{1}{2}(-r_2 - ir_1), \frac{1}{2}(r_2 - ir_1), -ir_3)$, r_1, r_2, r_3 real, and $A \in SU(2)$, forms a subgroup of T_6 isomorphic to E_6. The isomorphism is given by $(\mathbf{r}, A) \leftrightarrow \{\mathbf{w}, A\}$, $\mathbf{r} = (r_1, r_2, r_3)$.

The real 6-dimensional Lie algebra $\mathscr{E}_6$ corresponding to E_6 is generated by elements $\mathscr{J}_k, \mathscr{P}_k$, $k = 1, 2, 3$, with commutation relations

$$[\mathscr{J}_j, \mathscr{J}_k] = \epsilon_{jkl}\mathscr{J}_l, \qquad [\mathscr{J}_j, \mathscr{P}_k] = \epsilon_{jkl}\mathscr{P}_l, \qquad [\mathscr{P}_j, \mathscr{P}_k] = \mathscr{O}, \qquad j, k, l = 1, 2, 3, \tag{6.24}$$

where ϵ_{jkl} is the completely antisymmetric tensor such that $\epsilon_{123} = +1$. The $\mathscr{J}_k$ form a basis for $su(2)$, a subalgebra of $\mathscr{E}_6$. We choose these generators so that they are related to the finite group elements by

$$(\mathbf{r}, A) = \exp(r_1\mathscr{P}_1 + r_2\mathscr{P}_2 + r_3\mathscr{P}_3) \exp \varphi_1\mathscr{J}_3 \exp \vartheta\mathscr{J}_1 \exp \varphi_2\mathscr{J}_3, \tag{6.25}$$

where $\mathbf{r} = (r_1, r_2, r_3)$ and $\varphi_1, \theta, \varphi_2$ are the Euler coordinates for A, (5.139). The formal elements $\mathscr{P}^\pm$, $\mathscr{P}^3$, $\mathscr{J}^\pm$, $\mathscr{J}^3$ defined in terms of the generators (6.24) by

$$\mathscr{P}^\pm = \mp\mathscr{P}_2 + i\mathscr{P}_1, \qquad \mathscr{P}^3 = i\mathscr{P}_3, \qquad \mathscr{J}^\pm = \pm\mathscr{J}_2 + i\mathscr{J}_1, \qquad \mathscr{J}^3 = i\mathscr{J}_3$$

can easily be shown to satisfy the commutation relations (6.1) for the complex Lie algebra $\mathscr{T}_6$. Thus, we have explicitly determined $\mathscr{E}_6$ as a real form of $\mathscr{T}_6$.

According to this result the irreducible representations of $\mathscr{T}_6$, classified in Section 6-1, induce irreducible representations of $\mathscr{E}_6$ by restriction. We shall determine which of these induced representations of $\mathscr{E}_6$ can be extended to an irreducible unitary representation of E_6 on a Hilbert space.

Following the usual procedure we let U be an irreducible unitary representation of E_6 on a Hilbert space $\mathscr{H}$ and define the infinitesimal operators $\mathrm{J}_k, \mathrm{P}_k$, $k = 1, 2, 3$, by

$$\begin{aligned} \mathrm{J}_k f &= \frac{d}{dt}\mathbf{U}(\exp t\mathscr{J}_k)f\Big|_{t=0}, \\ \mathrm{P}_k f &= \frac{d}{dt}\mathbf{U}(\exp t\mathscr{P}_k)f\Big|_{t=0}, \qquad k = 1, 2, 3, \end{aligned} \tag{6.26}$$

for all $f \in \mathscr{D}$. ($\mathscr{D}$ is a dense subspace of $\mathscr{H}$ satisfying properties (3.45), (3.46).) On $\mathscr{D}$ these infinitesimal operators satisfy the commutation relations

$$[\mathrm{J}_j, \mathrm{J}_k] = \epsilon_{jkl}\mathrm{J}_l, \qquad [\mathrm{J}_j, \mathrm{P}_k] = \epsilon_{jkl}\mathrm{P}_l, \qquad [\mathrm{P}_j, \mathrm{P}_k] = 0,$$
$$j, k, l = 1, 2, 3,$$

where ϵ_{jkl} is the completely antisymmetric tensor such that $\epsilon_{123} = +1$. The operators $\mathrm{J}^\pm$, J^3, $\mathrm{P}^\pm$, P^3 defined by

$$\mathrm{J}^\pm = \mp\mathrm{J}_2 + i\mathrm{J}_1, \qquad \mathrm{J}^3 = i\mathrm{J}_3, \qquad \mathrm{P}^\pm = \mp\mathrm{P}_2 + i\mathrm{P}_1, \qquad \mathrm{P}^3 = i\mathrm{P}_3 \tag{6.27}$$

satisfy commutation relations analogous to (6.1) and thus determine a representation ρ of the complex Lie algebra $\mathscr{T}_6$ on $\mathscr{D}$. We shall investigate the conditions under which ρ (restricted to some dense subspace $\mathscr{D}'$ of $\mathscr{D}$) is isomorphic to one of the irreducible representations of $\mathscr{T}_6$ listed in Theorems 6.1 and 6.2.

From the study of the representation theory of $SU(2)$ given in Section 5-16, all of the irreducible unitary representations of $SU(2)$ are finite-dimensional. Thus, the only irreducible representations of $\mathscr{T}_6$ which could possibly induce unitary representations of E_6 are those of the form $\uparrow_4(\omega, q)$, $\omega \neq 0$, $-2q$ a nonnegative integer. We shall determine under what conditions a unitary irreducible representation U of E_6 can induce the representation $\uparrow_4(\omega, q)$ of $\mathscr{T}_6$ on $\mathscr{D}'$. From Lemma 3.1,

$$\langle \mathrm{J}_k f, h\rangle = -\langle f, \mathrm{J}_k h\rangle, \qquad \langle \mathrm{P}_k f, h\rangle = -\langle f, \mathrm{P}_k h\rangle, \qquad k = 1, 2, 3,$$

for all $f, h \in \mathscr{D}$, where $\langle .\,, .\rangle$ is the inner product on $\mathscr{H}$. In terms of the operators (6.27) these conditions become

$$\begin{aligned} \langle \mathrm{J}^3 f, h\rangle &= \langle f, \mathrm{J}^3 h\rangle, \qquad \langle \mathrm{P}^3 f, h\rangle = \langle f, \mathrm{P}^3 h\rangle, \\ \langle \mathrm{J}^+ f, h\rangle &= \langle f, \mathrm{J}^- h\rangle, \qquad \langle \mathrm{P}^+ f, h\rangle = \langle f, \mathrm{P}^- h\rangle. \end{aligned} \tag{6.28}$$

The representation $\uparrow_4(\omega, q)$ is determined by Eqs. (6.5) where $-2q$ is a nonnegative integer, $u = -q, \; -q + 1, \; -q + 2, \ldots;$ $m = -u, -u + 1, \ldots, +u$. If we assume the basis vectors $f_m^{(u)}$ are in $\mathscr{D}$ and apply the requirements (6.28), the following restrictions emerge:

(1) ω is real,

(2) $\langle f_m^{(u)}, f_{m'}^{(u')}\rangle = 0$ unless $u = u'$ and $m = m'$,

(3) $|f_{m+1}^{(u)}|^2(u - m) = |f_m^{(u)}|^2(u + m + 1)$,

(4) $|f_m^{(u+1)}|^2 / |f_m^{(u)}|^2 = \dfrac{(u + q + 1)(u + m + 1)(u - m + 1)(2u + 1)}{(u - q + 1)(2u + 3)}$,

valid for all $u \in Q$ and $m = -u, -u+1, \ldots, +u$. Defining new basis vectors $\{p_m^{(u)}\}$, $u = -q, -q+1, \ldots$; $m = -u, -u+1, \ldots, +u$; for $\mathscr{D}$ by

$$p_m^{(u)} = (-1)^{u+m} \left[\frac{(2u+1)(u-q)!}{(u+m)!\,(u-m)!\,(u+q)!}\right]^{1/2} f_m^{(u)}, \qquad (6.29)$$

we obtain $| p_m^{(u)} | = | p_{m'}^{(u')} |$ for all m, m', u, u' in the spectrum of the representation. (The phase factor $(-1)^{u+m}$ has been chosen for convenience.) Without loss of generality the vectors $p_m^{(u)}$ can be assumed to be of length one. Hence, they form an orthonormal basis for $\mathscr{H}$. Expressing relations (6.5) in terms of the basis $p_m^{(u)}$ and setting $s = -q$, we find:

Theorem 6.3 The possible faithful irreducible representations of $\mathscr{E}_6$ induced by an irreducible unitary representation U of E_6 on a Hilbert space $\mathscr{H}$ are determined by the pair (ω, s), where ω is real and $2s$ is an integer. The representation (ω, s) of $\mathscr{E}_6$ acts on $\mathscr{H}$ as follows: $\mathscr{H}$ has an orthonormal basis $\{p_m^{(u)}\}$; $u = |s|, |s|+1, \ldots$; $m = -u, -u+1, \ldots, +u$. In terms of this basis the operators (6.27) are defined by

$$\mathrm{J}^3 p_m^{(u)} = m p_m^{(u)}, \qquad \mathrm{J}^+ p_m^{(u)} = [(u+m+1)(u-m)]^{1/2} p_{m+1}^{(u)},$$

$$\mathrm{J}^- p_m^{(u)} = [(u+m)(u-m+1)]^{1/2} p_{m-1}^{(u)},$$

$$\begin{aligned}
\mathrm{P}^3 p_m^{(u)} = &-\omega \left[\frac{(u+m+1)(u-m+1)(u+s+1)(u-s+1)}{(u+1)^2(2u+1)(2u+3)}\right]^{1/2} p_m^{(u+1)} \\
&- \frac{m\omega s}{u(u+1)} p_m^{(u)} - \omega \left[\frac{(u+m)(u-m)(u+s)(u-s)}{u^2(2u+1)(2u-1)}\right]^{1/2} p_m^{(u-1)}, \\
\mathrm{P}^+ p_m^{(u)} = &+\omega \left[\frac{(u+m+1)(u+m+2)(u+s+1)(u-s+1)}{(u+1)^2(2u+3)(2u+1)}\right]^{1/2} p_{m+1}^{(u+1)} \\
&- [(u+m+1)(u-m)]^{1/2} \frac{\omega s}{u(u+1)} p_{m+1}^{(u)} \\
&- \omega \left[\frac{(u-m)(u-m-1)(u+s)(u-s)}{u^2(2u+1)(2u-1)}\right]^{1/2} p_{m+1}^{(u-1)}, \qquad (6.30) \\
\mathrm{P}^- p_m^{(u)} = &-\omega \left[\frac{(u-m+2)(u-m+1)(u+s+1)(u-s+1)}{(u+1)^2(2u+3)(2u+1)}\right]^{1/2} p_{m-1}^{(u+1)} \\
&- [(u+m)(u-m+1)]^{1/2} \frac{\omega s}{u(u+1)} p_{m-1}^{(u)} \\
&+ \omega \left[\frac{(u+m)(u+m-1)(u+s)(u-s)}{u^2(2u+1)(2u-1)}\right]^{1/2} p_{m-1}^{(u-1)},
\end{aligned}$$

$$\mathbf{P} \cdot \mathbf{J} p_m^{(u)} = \sum_{j=1}^{3} \mathrm{P}_j \mathrm{J}_j p_m^{(u)} = \omega s p_m^{(u)},$$

$$\mathbf{P} \cdot \mathbf{P} p_m^{(u)} = \sum_{j=1}^{3} \mathrm{P}_j \mathrm{P}_j p_m^{(u)} = -\omega^2 p_m^{(u)}.$$

The representations (ω, s) and $(-\omega, -s)$ are isomorphic; otherwise all of the representations are distinct. (Note: The fact that the representation (ω, s) is defined for s negative follows from an examination of Eqs. (6.30). Even though the representation (ω, s) for $s < 0$ is isomorphic to $(-\omega, -s)$ it is convenient to consider (ω, s) in its own right.)

Theorem 6.3 lists the possible representations of $\mathscr{E}_6$ induced by unitary irreducible representations of E_6. We will now prove that each of these Lie algebra representations (ω, s) actually does originate from some unitary irreducible representation (also called (ω, s)) of E_6. This will be done by constructing the unitary representation.

According to expression (6.25) the finite group elements of E_6 are uniquely determined by the infinitesimal generators $\mathscr{P}_k$, $\mathscr{J}_k$, $k = 1, 2, 3$. Therefore, the unitary representation (ω, s) of E_6 on a Hilbert space $\mathscr{H}$ is uniquely determined by the infinitesimal operators P_k, J_k, $k = 1, 2, 3$, on $\mathscr{D}$; hence, by Eqs. (6.30). Indeed if $\mathbf{U}(\mathbf{r}, A)$, $(\mathbf{r}, A) \in E_6$, is a unitary operator on $\mathscr{H}$ corresponding to (ω, s) then

$$\mathbf{U}(\mathbf{r}, A) = \exp(r_1\mathrm{P}_1 + r_2\mathrm{P}_2 + r_3\mathrm{P}_3) \exp(\varphi_1\mathrm{J}_3) \exp(\theta\mathrm{J}_1) \exp(\varphi_2\mathrm{J}_3). \quad (6.31)$$

The construction of the irreducible unitary representations (ω, s) of E_6 is well known (Wightman [1]), and the results can be presented in an elegant coordinate free manner. Here, however, the construction will be carried out in a heavily coordinate-dependent manner to show the explicit connection between E_6 and the recursion relations (6.13) for special functions.

From the orthogonality relation (5.144) for the matrix elements $U^n_{nm}(A)$ on the group manifold $SU(2)$ it folows that for fixed s the functions

$$p^{(u)}_m(\gamma, \alpha) = (-1)^{u+m} \left[\frac{(2u+1)(u+s)!\,(u+m)!}{4\pi(u-s)!\,(u-m)!}\right]^{1/2} P^{s,m}_u(\cos\gamma) e^{im\alpha},$$

$$u = |s|,\ \ |s| + 1, \ldots;\quad m = -u, -u+1, \ldots, +u;$$

form an orthonormal set with respect to the measure $d\tau = \sin\gamma\, d\gamma\, d\alpha$, $0 \leqslant \gamma \leqslant \pi$, $0 \leqslant \alpha < 2\pi$, (mod 2π);

$$\langle p^{(u)}_m, p^{(u')}_{m'}\rangle = \int_0^\pi \int_0^{2\pi} \overline{p^{(u)}_m(\gamma, \alpha)} p^{(u')}_{m'}(\gamma, \alpha)\, d\tau = \delta_{u,u'}\, \delta_{m,m'} .$$

Furthermore, from (6.11), (6.15), and (6.29), the functions $p^{(u)}_m(\gamma, \alpha)$ satisfy Eqs. (6.30) with

$$\mathrm{J}^3 = -i\frac{\partial}{\partial\alpha},\qquad \mathrm{J}^\pm = e^{\pm i\alpha}\left(\pm\frac{\partial}{\partial\gamma} + i\cot\gamma\frac{\partial}{\partial\alpha} - \frac{s}{\sin\gamma}\right),$$
$$\mathrm{P}^\pm = \omega e^{\pm i\alpha}\sin\gamma,\qquad \mathrm{P}^3 = \omega\cos\gamma. \quad (6.32)$$

Here $z = \cos\gamma$, $t = -ie^{i\alpha}$, and $P_u^{s,m}(\cos\gamma) = e^{(-i\pi/2)(s+m)}\,\mathfrak{P}_u^{s,m}(\cos\gamma + i0)$.

From these results it seems reasonable to define a Hilbert space $\mathscr{H}$ of functions square integrable with respect to $d\tau$ such that the functions $p_m^{(u)}$ given above form an orthonormal basis for $\mathscr{H}$. The differential operators (6.32) can then be used to determine a multiplier representation of E_6 on $\mathscr{H}$ which should turn out to be the desired unitary representation (ω, s). This procedure works but it leads to a complication. For s an integer the $p_m^{(u)}(\gamma, \alpha)$ are functions of $e^{i\alpha}$, while if s is not an integer the basis vectors are functions of $e^{i\alpha/2}$. Because of this difference in behavior the two cases have to be treated separately.

To avoid these complications we adopt the device of multiplying each of the functions $p_m^{(u)}$ by $e^{is\alpha}$. Thus, we define new functions

$$\begin{aligned} h_m^{(u)}(\gamma, \alpha) &= e^{is\alpha} p_m^{(u)}(\gamma, \alpha) \\ &= (-1)^{u+m} \left[\frac{(2u+1)(u+s)!\,(u+m)!}{4\pi(u-s)!\,(u-m)!}\right]^{1/2} P_u^{s,m}(\cos\gamma) e^{i(m+s)\alpha} \\ &= Q_{sm}^u(\gamma) e^{i(m+s)\alpha}. \end{aligned}$$

Since $m + s$ is always an integer, whether or not s is an integer, this device allows us to treat both cases at once. It is obvious that the set $\{h_m^{(u)}\}$ is still orthonormal with respect to the inner product defined above:

$$\langle h_{m'}^{(u')}, h_m^{(u)} \rangle = \int_0^\pi \int_0^{2\pi} \overline{h_{m'}^{(u')}(\gamma, \alpha)} h_m^{(u)}(\gamma, \alpha)\, d\tau = \delta_{m',m}\delta_{u',u}.$$

Furthermore, from the results in Section 4-8 concerning transformations of differential operators under mappings of function spaces, it follows that the basis functions $h_m^{(u)}$ satisfy Eqs. (6.30) where now the differential operators are given by

$$\mathrm{J}^3 = -i\frac{\partial}{\partial\alpha} - s, \qquad \mathrm{J}^\pm = e^{\pm i\alpha}\left(\pm\frac{\partial}{\partial\gamma} + i\cot\gamma\frac{\partial}{\partial\alpha} + s\left(\frac{\cos\gamma - 1}{\sin\gamma}\right)\right).$$
$$\mathrm{P}^\pm = \omega e^{\pm i\alpha}\sin\gamma, \qquad \mathrm{P}^3 = \omega\cos\gamma.$$

Moreover, the infinitesimal operators P_k, J_k become

$$\begin{aligned} \mathrm{J}_1 &= \sin\alpha\frac{\partial}{\partial\gamma} + \cos\alpha\cot\gamma\frac{\partial}{\partial\alpha} + is\cos\alpha\left(\frac{1-\cos\gamma}{\sin\gamma}\right), \\ \mathrm{J}_2 &= -\cos\alpha\frac{\partial}{\partial\gamma} + \sin\alpha\cot\gamma\frac{\partial}{\partial\alpha} + is\sin\alpha\left(\frac{1-\cos\gamma}{\sin\gamma}\right), \\ \mathrm{J}_3 &= -\frac{\partial}{\partial\alpha} + is, \qquad \mathrm{P}_1 = -i\omega\cos\alpha\sin\gamma, \\ \mathrm{P}_2 &= -i\omega\sin\alpha\sin\gamma, \qquad \mathrm{P}_3 = -i\omega\cos\gamma. \end{aligned} \tag{6.33}$$

These operators will be used to construct the unitary irreducible representation (ω, s) of E_6 on $\mathscr{H}(\omega, s)$, where $\mathscr{H}(\omega, s)$ is the Hilbert space of all functions $f(\cos\gamma, e^{i\alpha})$ square integrable with respect to the measure $d\tau$. The inner product is given by

$$\langle f, g\rangle = \int_0^\pi \int_0^{2\pi} \overline{f(\cos\gamma, e^{i\alpha})}\, g(\cos\gamma, e^{i\alpha}) \sin\gamma \, d\gamma \, d\alpha, \qquad f, g \in \mathscr{H}(\omega, s).$$

From the remarks immediately following Eq. (5.144), concerning the completeness of the matrix elements $U^u_{nm}(A)$ on the group manifold $SU(2)$, we can conclude that the functions $h^{(u)}_m(\gamma, \alpha)$, $u = |s|, |s|+1, \ldots$; $m = -u, -u+1, \ldots, +u$; form an orthonormal basis for $\mathscr{H}(\omega, s)$.

It will now be shown that the operators $\mathbf{U}(\mathbf{r}, A)$, defined by (6.31) with infinitesimal generators (6.33), form a unitary irreducible representation of E_6 on $\mathscr{H}(\omega, s)$. The construction of these operators is a straightforward application of Theorem 1.10:

$$[\mathbf{U}(\mathbf{r}, A)f](\cos\gamma, e^{i\alpha}) = \exp(-i\omega\mathbf{r}\cdot\hat{\mathbf{x}}) \left[\frac{\bar{a}(\cos\gamma+1) + b\sin\gamma\, e^{i\alpha}}{a(\cos\gamma+1) + \bar{b}\sin\gamma\, e^{-i\alpha}}\right]^s$$
$$\cdot f(\cos\gamma', e^{i\alpha'}), \qquad f \in \mathscr{H}(\omega, s), \tag{6.34}$$

where

$$\mathbf{r}\cdot\hat{\mathbf{x}} = \sum_{j=1}^{3} r_j x_j\,; \qquad \hat{\mathbf{x}} = (x_1, x_2, x_3) = (\cos\alpha\sin\gamma, \sin\alpha\sin\gamma, \cos\gamma); \tag{6.35}$$

$$\cos\gamma' = \cos\gamma\,(1 - 2b\bar{b}) + (abe^{i\alpha} + \bar{a}\bar{b}e^{-i\alpha})\sin\gamma,$$
$$e^{i\alpha'} = \left[\frac{a^2\sin\gamma\, e^{i\alpha} - \bar{b}^2\sin\gamma\, e^{-i\alpha} - 2a\bar{b}\cos\gamma}{\bar{a}^2\sin\gamma\, e^{-i\alpha} - b^2\sin\gamma\, e^{i\alpha} - 2\bar{a}b\cos\gamma}\right]^{1/2}, \tag{6.36}$$
$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2).$$

Here, all bracketed quantities are of absolute value $+1$.

The action of the operators $\mathbf{U}(\mathbf{r}, A)$ can be written in a more transparent form by identifying the elements of $\mathscr{H}(\omega, s)$ with functions square integrable with respect to Lebesgue measure on the unit 2-sphere. Thus, we make the identification $f(\cos\gamma, e^{i\alpha}) \equiv f(\hat{\mathbf{x}})$ where the unit vector $\hat{\mathbf{x}}$ is given by (6.35). Integration with respect to the measure $d\tau$ can now be interpreted as integration over the surface of the 2-sphere. In terms of this notation the action of the operators $\mathbf{U}(\mathbf{r}, A)$ on $\mathscr{H}(\omega, s)$ is

$$[\mathbf{U}(\mathbf{r}, A)f](\hat{\mathbf{x}}) = \exp(-i\omega\mathbf{r}\cdot\hat{\mathbf{x}}) \left[\frac{\bar{a}(x_3+1) + b(x_1+ix_2)}{a(x_3+1) + \bar{b}(x_1-ix_2)}\right]^s f(R(A^{-1})\hat{\mathbf{x}}), \tag{6.37}$$

where the orthogonal matrix $R(A)$ is defined by (6.23). In this form it is obvious that these operators are defined and unitary for every $(\mathbf{r}, A) \in E_6$ (the orthogonal transformation $R(A^{-1})$ preserves the scalar product while the multiplier has absolute value 1). The group property

$$\mathbf{U}(\mathbf{r}, A)\,\mathbf{U}(\mathbf{r}', A') = \mathbf{U}(\mathbf{r} + R(A)\mathbf{r}', AA')$$

is true by construction, but can also be verified directly. Thus we have defined a unitary representation of E_6 on $\mathscr{H}(\omega, s)$ which induces the Lie algebra representation (ω, s) of $\mathscr{E}_6$ classified in Theorem 6.3.

The group representation can be shown to be irreducible since its induced Lie algebra representation is irreducible. Furthermore, the group representations (ω, s) and $(-\omega, -s)$ are unitary equivalent since their induced Lie algebra representations are isomorphic. We shall not give direct proofs of these facts since the proofs are somewhat tedious. See, however, Naimark [2], Chapter 3.

6-5 The Matrix Elements of (ω, s)

We turn now to the problem of determining the matrix elements of the operators $\mathbf{U}(\mathbf{r}, A)$ with respect to the orthonormal basis $\{h_m^{(u)}\}$ of $\mathscr{H}(\omega, s)$. For fixed u the functions $h_m^{(u)}(\hat{\mathbf{x}})$, m $= -u, -u+1, \ldots, +u$, form a canonical basis for the irreducible representation D_u of $SU(2)$. Thus we have

$$[\mathbf{U}(\mathbf{0}, A)h_m^{(u)}](\hat{\mathbf{x}}) = \sum_{k=-u}^{u} U_{km}^u(A)h_k^{(u)}(\hat{\mathbf{x}})$$

where the matrix elements $U_{km}^u(A)$ are given in terms of Jacobi polynomials by (5.143). Clearly the matrix elements of the operator $\mathbf{U}(\mathbf{0}, A)$ are

$$\langle h_n^{(v)}, \mathbf{U}(\mathbf{0}, A)h_m^{(u)}\rangle = U_{nm}^u(A)\,\delta_{v,u}\,.$$

The computation of matrix elements of the operator $\mathbf{U}(\mathbf{r}, I)$, I the identity element of $SU(2)$, is somewhat more difficult. Before performing the computation it is necessary to introduce some supplementary material from Section 5-16. If $A \in SU(2)$ has coordinates $(\varphi_1, \theta, \varphi_2)$ we can write the matrix elements $U_{nm}^u(A)$ in the form

$$U_{nm}^u(\varphi_1, \theta, \varphi_2) = e^{-in\varphi_1}T_{n,m}^u(\theta)e^{-in\varphi_2}$$

where

$$T_{n,m}^u(\theta) = (i)^{n-m}\left[\frac{(u+m)!\,(u-n)!}{(u+n)!\,(u-m)!}\right]^{1/2} P_u^{-n,m}(\cos\theta).$$

The functions $T^u_{n,m}(\theta)$ satisfy the relations

$$T^u_{n,m}(\theta) = T^u_{m,n}(\theta) = T^u_{-n,-m}(\theta),$$

$$T^u_{n,m}(\theta)T^{u'}_{n',m'}(\theta) = \sum_l \mathrm{C}(u, n; u', n' \mid l, n+n') \cdot \mathrm{C}(u, m; u', m' \mid l, m+m')T^l_{n+n',m+m'}(\theta), \tag{6.38}$$

where the $\mathrm{C}(\,\cdot\,)$ are Clebsch–Gordan coefficients for $SU(2)$, (5.53). The basis elements $h^{(u)}_m(\gamma, \alpha) = Q^u_{s,m}(\gamma)\, e^{i(m+s)\alpha}$ and the T functions are related by the simple formula

$$T^u_{nm}(\theta) = e^{(i\pi/2)(m+n+2u)} \left[\frac{4\pi}{2u+1}\right]^{1/2} Q^u_{-n,m}(\theta).$$

Furthermore, the spherical harmonics $Y^m_l(\theta, \varphi)$ are given by

$$Y^m_l(\theta, \varphi) = (-1)^l Q_{0,m}(\theta)e^{im\varphi}.$$

By definition the matrix elements of $\mathbf{U}(\mathbf{r}, I)$ are

$$\begin{aligned}\langle h^{(v)}_n, \mathbf{U}(\mathbf{r}, I)h^{(u)}_m\rangle &= [v, n \mid \omega, s \mid u, m](\mathbf{r})\\ &= \int_0^\pi \int_0^{2\pi} \overline{Q^v_{s,n}(\gamma)e^{i(n+s)\alpha}} \exp(-i\omega\mathbf{r}\cdot\hat{\mathbf{x}})Q^u_{s,m}(\gamma)e^{i(m+s)\alpha}\\ &\quad \cdot \sin\gamma\, d\gamma\, d\alpha.\end{aligned} \tag{6.39}$$

Writing $\mathbf{r}$ in spherical coordinates

$$\mathbf{r} = (r\sin\theta_r\cos\varphi_r\,,\, r\sin\theta_r\sin\varphi_r\,,\, r\cos\theta_r),$$

we make use of the well-known formula

$$\exp(i\mathbf{p}\cdot\mathbf{r}) = 4\pi \sum_{l=0}^{\infty} \sum_{k=-l}^{l} i^l j_l(pr)Y^k_l(\theta_r\,, \varphi_r)\overline{Y^k_l(\theta_p\,, \varphi_p)} \tag{6.40}$$

where the j_l are spherical Bessel functions, (A.24) (Kurşunoğlu [1], Chapter 4). Substituting this identity into (6.39), applying (6.38), and simplifying, we obtain

$$\begin{aligned}[v, n \mid \omega, s \mid u, m](\mathbf{r}) &= (4\pi)^{1/2} \sum_{l=0}^{\infty} \left[\frac{(2l+1)(2v+1)}{2u+1}\right]^{1/2}\\ &\quad \cdot i^l j_l(\omega r)Y^{m-n}_l(\theta_r\,, \varphi_r)\, \mathrm{C}(l, 0; v, s \mid u, s)\\ &\quad \cdot \mathrm{C}(l, m-n; v, n \mid u, m).\end{aligned} \tag{6.41}$$

The fact that the **U** operators form a unitary representation of E_6 allows us to derive a number of relations satisfied by the functions (6.41). Thus, unitarity implies

$$[v, n \mid \omega, s \mid u, m](-\mathbf{r}) = \overline{[u, m \mid \omega, s \mid v, n](\mathbf{r})}. \tag{6.42}$$

Furthermore, the group property $\mathbf{U}(\mathbf{r}_1, I)\,\mathbf{U}(\mathbf{r}_2, I) = \mathbf{U}(\mathbf{r}_1 + \mathbf{r}_2, I)$ leads to

$$[v, n \mid \omega, s \mid u, m](\mathbf{r}_1 + \mathbf{r}_2) = \sum_{h=|s|}^{\infty} \sum_{q=-h}^{h} [v, n \mid \omega, s \mid h, q](\mathbf{r}_1) \cdot [h, q \mid \omega, s \mid u, m](\mathbf{r}_2). \tag{6.43}$$

When $s = v = n = 0$, this identity reduces to the addition theorem for spherical waves

$$j_u(\omega r) Y_u^m(\theta_r, \varphi_r) = \sum_{h=0}^{\infty} \sum_{q=-h}^{h} \sum_{l=0}^{\infty} (4\pi)^{1/2} (i)^{h+l-u} \left[\frac{(2l+1)(2h+1)}{(2u+1)}\right]^{1/2} \cdot j_h(\omega r_1) j_l(\omega r_2) Y_h^q(\theta_{r_1}, \varphi_{r_1}) Y_l^{m-q}(\theta_{r_2}, \varphi_{r_2}) \cdot \mathrm{C}(l, 0; h, 0 \mid u, 0) \mathrm{C}(l, m-q; h, q \mid u, m) \tag{6.44}$$

which was first derived by Friedman and Russek [1]. Here, $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$.

The group property $\mathbf{U}(\mathbf{r}, A) = \mathbf{U}(\mathbf{0}, A)\,\mathbf{U}(A^{-1}\mathbf{r}, I) = \mathbf{U}(\mathbf{r}, I)\,\mathbf{U}(\mathbf{0}, A)$ implies

$$\sum_{n'=-v}^{v} U_{nn'}^{v}(A)[v, n' \mid \omega, s \mid u, m](A^{-1}\mathbf{r}) = \sum_{m'=-u}^{u} [v, n \mid \omega, s \mid u, m'](\mathbf{r}) U_{m'm}^{u}(A). \tag{6.45}$$

Fix v and consider the $2v + 1$ component quantity

$$\chi_{v;um}^{(\omega,s)}(\mathbf{r}) = ([v, n \mid \omega, s \mid u, m](\mathbf{r})), \qquad n = -v, -v+1, \ldots, +v, \tag{6.46}$$

for some u, m. Define the action W of E_6 on $\chi(\mathbf{r})$ by (in matrix notation)

$$[\mathbf{W}(\mathbf{a}, A)\chi_{v;um}^{(\omega,s)}](\mathbf{r}) = U^v(A)\chi_{v;um}^{(\omega,s)}(R(A^{-1})(\mathbf{r} + \mathbf{a})). \tag{6.47}$$

The identity (6.45) shows that $\chi(\mathbf{r})$ transforms like a spinor field of weight v; in fact, under the action of $SU(2)$ it transforms like the eigenvector $h_m^{(u)}$ of the irreducible representation D_u. Furthermore, (6.43) shows

$$[\mathbf{W}(\mathbf{a}, I)\chi_{v;um}^{(\omega,s)}](r) = \sum_{h=|s|}^{\infty} \sum_{q=-h}^{h} [h, q \mid \omega, s \mid u, m](\mathbf{a}) \chi_{v;hq}^{(\omega,s)}(\mathbf{r}). \tag{6.48}$$

Let $M(\omega, s, v)$ be the complex vector space generated by all finite linear combinations of the v-spinor functions $\chi^{(\omega,s)}_{v;um}$; $u = |s|, |s|+1, \ldots$; $m = -u, -u+1, \ldots, +u$. We can uniquely define an inner product $\langle .,. \rangle^*$ on $M(\omega, s, v)$, linear in the second argument, conjugate linear in the first, by requiring

$$\langle \chi^{(\omega,s)}_{v;um}, \chi^{(\omega,s)}_{v;u'm'} \rangle^* = \delta_{u,u'}\,\delta_{m,m'}$$

for all admissible u, u', m, m'. Completing $M(\omega, s, v)$ with respect to this inner product we obtain an abstract Hilbert space $\mathscr{H}(\omega, s, v)$. Denote again by W the action of E_6 on $\mathscr{H}(\omega, s, v)$ induced by (6.47). Then, W is a unitary irreducible representation of E_6 on $\mathscr{H}(\omega, s, v)$, unitary equivalent to the representation (ω, s). In fact the unitary equivalence maps the basis vector $h^{(u)}_m$ into the v-spinor $\chi^{(\omega,s)}_{v;um}$. Clearly, we can construct a representation of E_6 unitary equivalent to (ω, s) for each value of $v = |s|, |s|+1, \ldots$.

Fix v again and observe that the action of E_6 on $\mathscr{H}(\omega, s, v)$ induces an irreducible representation of $\mathscr{E}_6$ isomorphic to (ω, s). A tedious computation using (6.47) shows that the infinitesimal operators corresponding to this representation are

$$\mathrm{J}^{\pm} = e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\varphi}\right) + \mathrm{S}^{\pm}, \qquad \mathrm{J}^3 = -i\frac{\partial}{\partial\varphi} + \mathrm{S}^3,$$

$$\mathrm{P}^{\pm} = e^{\pm i\varphi}\left(i\sin\theta\,\frac{\partial}{\partial r} + i\frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \mp \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right), \tag{6.49}$$

$$\mathrm{P}^3 = i\cos\theta\,\frac{\partial}{\partial r} - i\frac{\sin\theta}{r}\frac{\partial}{\partial\theta},$$

where $\mathbf{r} = (r\sin\theta\cos\varphi,\ r\sin\theta\sin\varphi,\ r\cos\theta)$ and

$$\begin{aligned} \mathrm{S}^{\pm}[v, n \mid \omega, s \mid u, m] &= [(v \pm n)(v \mp n + 1)]^{1/2}[v, n \mp 1 \mid \omega, s \mid u, m], \\ \mathrm{S}^3[v, n \mid \omega, s \mid u, m] &= n[v, n \mid \omega, s \mid u, m]. \end{aligned} \tag{6.50}$$

By replacing the eigenvectors $p^{(u)}_m$ with the v-spinors $\chi^{(\omega,s)}_{v;um}$ and substituting expressions (6.49) into Eqs. (6.30) the reader can easily obtain a series of recursion relations for the matrix elements $[v, n \mid \omega, s \mid u, m](\mathbf{r})$. Note also that the last of Eqs. (6.30) yields the identity

$$(\nabla^2 + \omega^2)[v, n \mid \omega, s \mid u, m](\mathbf{r}) = 0, \tag{6.51}$$

where ∇^2 is the Laplacian. Thus the functions $\chi^{(\omega,s)}_{v;um}(\mathbf{r})$ are spinor-valued solutions of the wave equation.

The v-spinor functions defined above are of special importance in mathematical physics. It is known that a v-spinor $\Psi(\mathbf{r})$ satisfying the wave equation $(\nabla^2 + \omega^2)\,\Psi(\mathbf{r}) = 0$ can be expanded as a countable linear combination of the spinors $\chi^{(\omega,s)}_{v:um}(\mathbf{r})$ where $m = -u, -u+1,\ldots, +u$; $u = |s|, |s|+1,\ldots$; $s = -v, -v+1,\ldots, +v$. Such an expansion is useful because of the simple transformation properties of the χ functions under the action of E_6. Thus, our results yield recursion relations and addition theorems for spinor-valued solutions of the wave equation. The spinors χ also have simple completeness and orthogonality properties. For these see Miller [2].

This concludes our analysis of the representation theory of E_6. We have seen that the matrix elements of the translation subgroup of E_6 are the so-called spherical wave solutions of the wave equation. The matrix elements have been chosen with respect to a basis indexed by the irreducible representations of $SU(2)$. By computing the matrix elements with respect to a basis indexed by the irreducible representations of the subgroup $\{(\mathbf{r}, I)\}$ or $\{(\mathbf{r}, \exp \alpha \mathscr{J}_3)\}$ (isomorphic to a 2-sheeted covering group of E_3), we could derive identities relating spherical wave solutions of plane wave or cylindrical wave solutions of the wave equation. This computation is not difficult but will not be attempted here.

The representations (ω, s) (ω real, $2s$ an integer, $(\omega, s) \cong (-\omega, -s)$) derived above constitute all faithful irreducible unitary representations of E_6. The only remaining unitary irreducible representations of E_6 not unitary equivalent to a representation (ω, s) are those which map the translation subgroup $\{(\mathbf{r}, I)\}$ into the identity operator. Thus, on the subgroup $SU(2)$ they must be unitary equivalent to a representation D_u of $SU(2)$. For proofs of these statements see Wightman [1] and Mackey [1].